A Π-C-system defined by a Π-universe in a locally Cartesian closed category[1]

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Abstract

This is the fourth paper in a series started in [5].

1 Introduction

The concept of a C-system was introduced in [5].

Notation: For morphisms $f : X \to Y$ and $g : Y \to Z$ we denote their composition as $f \circ g$. For functors $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}''$ we use the standard notation $G \circ F$ for their composition.

2 Π-C-systems

The notion of a Π-C-system is equivalent to the notion of a contextual category with products of families of types from [2]. We use the name Π-C-systems to emphasize the fact that we are dealing here with an additional structure on a C-system rather than with a property of such an object.

Let us recall first the following definitions.

Definition 2.1 *[**inthom**] Let $\mathcal{C}$ be a category with direct products. For objects $Y, Y' \in \mathcal{C}$ the internal Hom-object from Y to Y' is a pair $(\underline{Hom}(Y,Y'), ev_{Y,Y'} : \underline{Hom}(Y,Y') \times Y \to Y')$ such that for any Z the mapping $Hom(Z, \underline{Hom}(Y,Y')) \to Hom(Z \times Y, Y')$ given by $f \mapsto (f \times Id_Y) \circ ev_{Y,Y'}$ is a bijection.*

Definition 2.2 *[**2009.11.24.def2**] Let $\mathcal{C}$ be a category. Let $g : Z \to Y$, $f : Y \to X$ be a pair of morphisms such that for any $U \to X$ a fiber product $U \times_X Y$ exists. A pair*

$$(w : W \to X, h : W \times_X Y \to Z)$$

such that $h \circ g = pr_2$ is called a universal pair for (g, f) if for any $U \to X$ the map

$$Hom_X(U, W) \to Hom_Y(U \times_X Y, Z)$$

of the form $u \mapsto (u \times Id_Y) \circ h$ is a bijection.

If a universal pair exists then it is easily seen to be unique up to a unique isomorphism. We denote such a pair by $(\Pi(g,f), e_{g,f} : \Pi(g,f) \times_X Y \to Z)$. Note that if $f' : Y \to X$ and $pr : Y' \times_X Y \to Y$ is the projection then

$$(\Pi(pr, f), pr' \circ e_{pr,f} : \Pi(g,f) \times_X Y \to Y') = (\underline{Hom}_X(Y,Y'), ev : \underline{Hom}_X(Y,Y') \times_X Y \to Y')$$

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so that relative internal Hom-objects are particular cases of universal pairs. The converse is also true and given all relative internal Hom-objects one can construct universal pairs.

Definition 2.3 *[**2009.11.24.def1**] A* Π*-structure on a C-system* CC *is a collection of data of the form*

1. *for each* $Y \in Ob(CC)_{\geq 2}$ *an object* $\Pi(Y) \in Ob(CC)$ *such that* $ft(\Pi(Y)) = ft^2(Y)$,
2. *for each* $Y \in Ob(CC)_{\geq 2}$ *a morphism* $eval : p^*_{ft(Y)}(\Pi(Y)) \to Y$ *over* $ft(Y)$,

such that

(i) for any $f : Z \to ft^2(Y)$ *one has* $f^*(\Pi(Y)) = \Pi(f^*(Y))$ *and* $f^*(eval_Y) = eval_{f^*(Y)}$,

(ii) $(\Pi(Y), eval_Y))$ *is a universal pair for* $(p_Y, p_{ft(Y)})$.

A Π*-C-system is a C-system with a* Π*-structure.*

Remark 2.4 [**2014.10.30.rm1**] The type of Π-C-systems is easily seen to be constructively equivalent to the type of "contextual categories with families of types" defined in [2, Def. 1.13, p.83].

3 Π-structures on the C-systems of the form $CC(\mathcal{C}, p)$.

For every category $\mathcal{C}$ and a morphism $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ in $\mathcal{C}$ such that the pull-backs of p along all morphisms to $\mathcal{U}$ (merely) exist we have constructed in [3] a C-system $CC(\mathcal{C}, p)$. In this section we will show how to construct a Π-structure on $CC(\mathcal{C}, p)$ from a certain structure on p when $\mathcal{C}$ is a locally cartesian closed category.

Recall that a category $\mathcal{C}$ is called a lcc (locally cartesian closed) category if it has fiber products and all the over-categories $\mathcal{C}/X$ have internal Hom-objects. If C is an lcc category then for $Y \to X$, $Y' \to X$ one writes $\underline{Hom}_X(Y, Y')$ for the internal Hom-object from Y to Y' in $\mathcal{C}/X$.

Definition 3.1 *[**2009.10.27.def1**] Let* $\mathcal{C}$ *be an lcc category and let* $p_i : \widetilde{U}_i \to U_i$, $i = 1, 2, 3$ *be three morphisms in* $\mathcal{C}$. *A* Π*-structure on* (p_1, p_2, p_3) *is a Cartesian square of the form*

$$\textbf{[Pisq1]} \qquad \begin{array}{ccc} \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times \widetilde{U_2}) & \xrightarrow{\widetilde{P}} & \widetilde{U}_3 \\ {\scriptstyle p_2'}\downarrow & & \downarrow{\scriptstyle p_3} \\ \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) & \xrightarrow{P} & U_3 \end{array} \qquad (1)$$

such that p_2' *is the natural morphism defined by* p_2. *A* Π*-structure on* $p : \widetilde{U} \to U$ *is a* Π*-structure on* (p, p, p).

Problem 3.2 *[**2014.10.30.prob1.fromold**] Let* $\mathcal{C}$ *be as above,* $p : \widetilde{U} \to U$ *and let* $(\widetilde{P}, P)$ *be a* Π*-structure on* (p, p, p). *To construct a structure of* Π*-C-system on* $CC = CC(\mathcal{C}, p)$.

Construction 3.3 [**2014.10.30.contr1.fromold**] We start by recalling some constructions in $\mathcal{C}$.

Lemma 3.4 *[**2009.11.24.l5**] Consider a pair of pull back squares*

[**2009.11.24.eq3**]
$$\begin{array}{ccc} I_2 & \xrightarrow{\widetilde{F}_1} & \widetilde{U}_1 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ I_1 & \xrightarrow{F_1} & U_1 \end{array} \qquad \begin{array}{ccc} I_3 & \xrightarrow{\widetilde{F}_2} & \widetilde{U}_2 \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ I_2 & \xrightarrow{F_2} & U_2 \end{array} \tag{2}$$

Then there exists a unique morphism $f_{F_1,F_2} : I_1 \to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2)$ such that its composition with the canonical morphism to U_1 is F_1 and the composition of its adjoint

$$(f_{F_1,F_2} \times_{U_1} \widetilde{U}_1) \circ ev : I_2 = I_1 \times_{U_1} \widetilde{U}_1 \to U_1 \times U_2$$

with the projection to U_2 is F_2.

Proof: Follows immediately from the definition of internal Hom-objects.

Lemma 3.5 *[**2009.11.24.l3**] In the notation of Lemma 3.4 let*

$$\begin{array}{ccc} J_2 & \xrightarrow{\phi_2} & I_2 \\ \downarrow & & \downarrow{\scriptstyle q_1} \\ J_1 & \xrightarrow{\phi_1} & I_1 \end{array} \qquad \begin{array}{ccc} J_3 & \xrightarrow{\phi_3} & I_3 \\ \downarrow & & \downarrow{\scriptstyle q_2} \\ J_2 & \xrightarrow{\phi_2} & I_2 \end{array}$$

be two pull-back squares. Then $f_{F_1\phi_1,F_2\phi_2} = \phi_1 \circ f_{F_1,F_2}$.

Proof: Straightforward.

Let $p_1 : \widetilde{U}_1 \to U_1$, $p_2 : \widetilde{U}_2 \to U_2$ be a pair of morphisms in an lcc $\mathcal{C}$. Consider a pull-back square of the form

[**2009.11.24.eq4**]
$$\begin{array}{ccc} Fam_2(p_1,p_2) & \longrightarrow & \widetilde{U}_2 \\ {\scriptstyle p_{12}}\downarrow & & \downarrow{\scriptstyle p_2} \\ \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} & \xrightarrow{ev \circ pr_2} & U_2 \end{array} \tag{3}$$

where

$$ev : \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} \to U_1 \times U_2$$

is the canonical morphism.

Then for any two pull-back squares as in Lemma 3.4, the morphism f_{F_1,F_2} defines factorizations of the pull-back squares (2) of the form

$$\begin{array}{ccccc} I_2 & \xrightarrow{f_{F_1,F_2} \times_{U_1} \widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} & \xrightarrow{pr} & \widetilde{U}_1 \\ {\scriptstyle q_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_1} \\ I_1 & \xrightarrow{f_{F_1,F_2}} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) & \longrightarrow & U_1 \end{array}$$

and

$$\begin{array}{ccccc}
I_3 & \longrightarrow & Fam_2(p_1,p_2) & \longrightarrow & \widetilde{U}_2 \\
{\scriptstyle q_2}\big\downarrow & & \big\downarrow{\scriptstyle p_{12}} & & \big\downarrow{\scriptstyle p_2} \\
I_2 & \xrightarrow{f_{F_1,F_2}\times_{U_1}\widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U_1} & \xrightarrow{ev\circ pr_2} & U_2
\end{array}$$

respectively and joining the left hand side squares of these diagrams we get a diagram with pull-back squares of the form

$$\begin{array}{ccc}
I_3 & \longrightarrow & Fam_2(p_1,p_2) \\
{\scriptstyle q_2}\big\downarrow & & \big\downarrow{\scriptstyle p_{12}} \\
I_2 & \xrightarrow{f_{F_1,F_2}\times_{U_1}\widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U_1} \\
{\scriptstyle q_1}\big\downarrow & & \big\downarrow{\scriptstyle pr_1} \\
I_1 & \xrightarrow{f_{F_1,F_2}} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)
\end{array}$$

Let

$$g:\underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times \widetilde{U}_2)\times_{U_1}\widetilde{U_1}\to Fam_2(p_1,p_2)$$

be the morphism over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U_1}$ whose composition with the projection $Fam_2(p_1,p_2)\to \widetilde{U}_2$ equals $\widetilde{ev}\circ pr_2$ where

$$\widetilde{ev}:\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times \widetilde{U}_2)\times_{U_1}\widetilde{U_1}\to U_1\times\widetilde{U}_2$$

is the canonical morphism.

Lemma 3.6 *[**2009.11.24.l2**] The pair*

$$(\underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times \widetilde{U}_2)\to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2), g)$$

is universal for (p_{12}, pr_1).

Proof: For a given $w: Z\to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)$, a morphism $Z\to \underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times\widetilde{U}_2)$ over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)$ is the same as a morphism $Z\times_{U_1}\widetilde{U}_1\to\widetilde{U}_2$ such that the adjoint of its composition with $p_2:\widetilde{U}_2\to U_2$ is w.

A morphism from Z to the universal pair for p_{12} over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)$ is a morphism $Z\times_{U_1}\widetilde{U_1}\to\widetilde{U}_2$ whose composition with p_2 is $(w\times_{U_1} Id_{\widetilde{U_1}})\circ(ev\circ pr)$ which coincides with the condition that the composition of its adjoint with p_2 is w. This can be also seen from the diagram

$$\begin{array}{ccccc}
 & & Fam_2(p_1,p_2) & \longrightarrow & \widetilde{U}_2 \\
 & & {\scriptstyle p_{12}}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\
\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2)\times_{U_1}\widetilde{U}_1 & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U}_1 & \xrightarrow{ev\circ pr} & U_2 \\
\big\downarrow & & \big\downarrow{\scriptstyle pr} & & \\
\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2) & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times U_2) & &
\end{array}$$

Lemma 3.7 *[**2009.11.24.l4**] For two pull back squares as in (2), consider a pull-back square of the form*

$$\begin{array}{ccc} R(F_1,F_2) & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2) \\ \downarrow & & \downarrow \\ I_1 & \stackrel{f_{F_1,F_2}}{\longrightarrow} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2) \end{array}$$

and the morphism

$$g_{F_1,F_2} : R(F_1,F_2)\times_{I_1} I_2 \to I_3$$

whose composition with the morphism $I_3\to\widetilde{U}_2$ coincides with the composition

$$R(F_1,F_2)\times_{I_1} I_2 = R(F_1,F_2)\times_{U_1}\widetilde{U}_1 \to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2)\times_{U_1}\widetilde{U}_1 \stackrel{ev\circ pr}{\to} \widetilde{U}_2$$

Then $(R(F_1,F_2), g_{F_1,F_2})$ is a universal pair for (q_2,q_1).

Proof: It follows from Lemma 3.6 and the fact that in a lcc a pull-back of a universal pair is a universal pair.

Let us now construct a Π-structure on $CC = CC(\mathcal{C}, p)$. Let $n\geq 2$ and $(F_1,\ldots,F_n)\in CC$. Denote $(pt, F_1,\ldots,F_{n-2})$ by I. Then we have two morphisms $F_{n-1}: I\to U$ and $F_n : (I, F_{n-1})\to U$.

Applying Lemma 3.4 to the corresponding pull-back squares we get a morphism

$$f_{F_{n-1},F_n} : I \to \underline{Hom}_U(\widetilde{U}, U\times U)$$

Set $\Pi(F_1,\ldots,F_n) = (I, P\circ f_{F_{n-1},F_n}) = (F_1,\ldots,F_{n-2}, P\circ f_{F_{n-1},F_n})$. Since the square (1) is a pull-back square there is a unique morphism $\Pi(F_1,\ldots,F_n)\to\underline{Hom}_U(\widetilde{U}, U\times\widetilde{U})$ such that the diagram

$$\begin{array}{ccccc} \Pi(F_1,\ldots,F_n) & \longrightarrow & \underline{Hom}_U(\widetilde{U}, U\times\widetilde{U}) & \stackrel{\widetilde{P}}{\longrightarrow} & \widetilde{U} \\ \downarrow & & \downarrow & & \downarrow \\ I & \stackrel{f_{F_{n-1},F_n}}{\longrightarrow} & \underline{Hom}_U(\widetilde{U}, U\times U) & \stackrel{P}{\longrightarrow} & U \end{array}$$

commutes and the composition of the two upper arrows is $Q(f_{F_{n-1},F_n})$. The left hand side square in this diagram is automatically a pull-back square. Applying to this square Lemma 3.7 we obtain a morphism

$$eval_{(F_1,\ldots,F_n)} : (I, F_{n-1}, (P\circ f_{F_{n-1},F_n})\circ pr) \to (I, F_{n-1}, F_n)$$

over (I, F_{n-1}) (where $pr : (I, F_{n-1})\to I$ is the projection).

The fact that this construction satisfies the first condition of Definition 2.3 follows from Lemma 3.5. The fact that it satisfies the second condition of this definition follows from Lemma 3.7.

4 A reformulation of the Π-structure.

The definition of the Π-structure given in Definition 2.3 is convenient for the construction of instances of Π-structure on the C-systems such as $CC(\mathcal{C}, p)$ but much less convenient for the construction of such instances on C-systems that arise from the syntactic data such as the C-systems

$CC(R, LM)$ from [4]. It is also inconvenient for studying the Π-structures on the regular quotients of the C-systems.

In this section we will give another description of Π-structures that is better suited for these purposes. In order to do this we reformulate the Π-structure in terms of a structure on the sets $B_n = Ob(CC)_n$ and $\widetilde{B}_n = \widetilde{Ob}(CC)_n$ that satisfies conditions that only refer to operations $\partial, ft, T, \widetilde{T}, S, \widetilde{S}$ and δ on these sets that were introduced in [5] and [6]. In other words we will describe the Π-structure on a C-system in terms of a structure on the corresponding (unital) B0-system. This is analogous of how we have described the sub-systems and regular quotients of C-systems in [5].

Definition 4.1 *[2014.11.03.def2] An non-unital ap-Π-structure on a non-unital B0-system* $\mathbf{B} = (B_n, \widetilde{B}_{n+1}, \partial, ft, T, \widetilde{T}, S, \widetilde{S})$ *is given by three families of operations of the form:*

1. $\Pi : B_{n+2} \to B_{n+1}$
2. $\lambda : \widetilde{B}_{n+2} \to \widetilde{B}_{n+1}$
3. $ap : (\widetilde{B}_{n+1})_{\partial} \times_{\Pi} (B_{n+2})_{ft} \times_{\partial} (\widetilde{B}_{n+1}) \to \widetilde{B}_{n+1}$

(where $n \geq 0$) that satisfy the following conditions:

1. *for $Y \in B_{n+2}$ one has*
   (a) $ft\,\Pi(Y) = ft^2(Y)$,
   (b) *for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}(Y)$,*
   $$T(Z, \Pi(Y)) = \Pi(T(Z, Y)),$$
   (c) *for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^{i+1}(Y)$,*
   $$S(t, \Pi(Y)) = \Pi(S(t, Y)),$$
2. *for $s \in \widetilde{B}_{n+2}$ one has*
   (a) $\partial\,\lambda(s) = \Pi\,\partial(s)$,
   (b) *for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}\,\partial(s)$,*
   $$\widetilde{T}(Z, \lambda(s)) = \lambda(\widetilde{T}(Z, s)),$$
   (c) *for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^{i+1}\,\partial(s)$,*
   $$\widetilde{S}(t, \lambda(s)) = \lambda(\widetilde{S}(t, s)),$$
3. *for $r \in \widetilde{B}_{n+1}$, $Y \in B_{n+2}$ and $f \in \widetilde{B}_{n+1}$ such that $\partial(r) = ft(Y)$ and $\partial(f) = \Pi(Y)$ one has*
   (a) $\partial(ap(f, Y, r)) = S(r, Y)$,
   (b) *for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}(Y)$,*
   $$\widetilde{T}(Z, ap(f, Y, r)) = ap(\widetilde{T}(Z, f), T(Z, Y), \widetilde{T}(Z, r)),$$

(c) for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^{i+1}(Y)$,

$$\widetilde{S}(t, ap(f, Y, r)) = ap(\widetilde{S}(t, f), S(t, Y), \widetilde{S}(t, r)),$$

4. *for $r \in \widetilde{B}_{n+1}$, $s \in \widetilde{B}_{n+2}$ such that $ft(\partial(s)) = \partial(r)$*

$$ap(\lambda(s), \partial\, s, r) = \widetilde{S}(r, s)$$

(β-reduction).

Definition 4.2 *[**2014.11.03.def3**] Let $\mathbf{B}$ be a unital B0-system. Then a non-unital ap-Π-structure is called a unital ap-Π-structure if for $n \geq 0$, $Y \in B_{n+2}$, $f \in \widetilde{B}_{n+1}$ such that $\partial(f) = \Pi(Y)$,*

$$[\mathbf{2009.11.30.oldeq1}]\lambda(ap(\widetilde{T}(ft(Y), f)), T(ft(Y), Y), \delta_{ft(Y)}) = f \tag{4}$$

(η-reduction).

Problem 4.3 *[**2014.11.03.prob2**] To construct a bijection between Π-structures on a C-system CC and unital ap-Π-structures on the corresponding B0-system $\mathbf{B}$.*

Construction 4.4 [**2014.11.03.constr1**] Suppose that we are given a Π-structure on CC. Then we have maps

1. $\Pi : B_{n+2} \to B_{n+1}$, $n \geq 0$,
2. $\lambda : \widetilde{B}_{n+2} \to \widetilde{B}_{n+1}$, $n \geq 0$,
3. $ap : (\widetilde{B}_{n+1})_{\partial} \times_{\Pi} (B_{n+2})_{ft} \times_{\partial} (\widetilde{B}_{n+1}) \to \widetilde{B}_{n+1}$, $n \geq 0$

defined as follows. The map Π is the map from Definition 2.3. Since $(\Pi(Y), eval_Y)$ is a universal pair for $(p_Y, p_{ft(Y)})$ the mapping

$$\phi_Y : \{f \in \widetilde{B}_{n+1} \,|\, \partial(f) = \Pi(Y)\} \to \{s \in \widetilde{B}_{n+2} \,|\, \partial(s) = Y\}$$

given by the formula

$$\phi_Y(f) = \widetilde{T}(ft(Y), f) \circ eval_Y$$

is a bijection. One defines λ_Y as the inverse to this bijection.

The map ap sends a triple (f, Y, r) such that $\partial(r) = ft(Y)$ and $\partial(f) = \Pi(Y)$ to

$$ap(f, Y, r) = \widetilde{S}(r, \widetilde{T}(ft(Y), f) \circ eval)$$

as partially illustrated by the following diagram:

$$\begin{array}{ccccc}
 & & Y & \longleftarrow & S(r, Y) \\
 & & {\scriptstyle p_Y}\downarrow & & \downarrow \\
p^*_{ft(Y)}(\Pi(Y)) & \longrightarrow & ft(Y) & \underset{r}{\longleftarrow} & ft^2(Y) \\
\downarrow & & \downarrow & & \\
\Pi(Y) & \xrightarrow{p_{\Pi(Y)}} & ft^2(Y) & &
\end{array}$$

Lemma 4.5 *[2009.11.30.l1] Let $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$ and $f \in \widetilde{B}(\Pi(Y))$. Then one has*

$$g^*(\phi_Y(f), i+2) = \phi_{g^*(Y,i+2)}(g^*(f, i+1))$$

Proof: Let $h_1 = q(g, ft(Y), i+1)$, $h_2 = q(g, Y, i+2)$. Then one has

$$g^*(\phi_Y(f), i+2) = h_1^*(\phi_Y(f)) = h_1^*(\widetilde{T}(ft(Y), f) \circ eval_Y) = h_1^*(\widetilde{T}(ft(Y), f)) \circ h_1^*(eval_Y)$$

$$= p^*_{g^*(ft(Y),i+1)}(h_2^*(f)) \circ eval_{h_1^*(Y)} = \phi_{h_1^*(Y)}(h_2^*(f)) = \phi_{g^*(Y,i+2)}(g^*(f, i+1)).$$

As an immediate corollary of Lemma 4.5 we have:

Lemma 4.6 *[2009.11.30.l2] Let $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$ and $r \in \widetilde{B}(Y)$. Then one has*

$$g^*(\lambda(r), i+1) = \lambda(g^*(r, i+2)).$$

Lemma 4.7 *[2009.11.30.l3] Let $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$, $r \in \widetilde{B}(ft(Y))$ and $f \in \widetilde{B}(\Pi(Y))$. Then one has*

$$g^*(ap(f, Y, r), i+1) = ap(g^*(f, i+1), g^*(Y, i+2), g^*(r, i+2))$$

Proof: Let $h_1 = q(g, ft(Y), i+1)$, $h_2 = q(g, Y, i+2)$. Then one has:

$$g^*(ap(f, Y, r), i+1) = h_2^*(\widetilde{S}(r, \widetilde{T}(ft(Y), f)) \circ eval) = h_2^*(r^*(\widetilde{T}(ft(Y), f) \circ eval)) =$$

$$= (h_2^*(r))^* h_1^*(\widetilde{T}(ft(Y), f) \circ eval)) = (h_2^*(r))^*(h_1^* p^*_{ft(Y)}(f) \circ h_1^*(eval)) =$$

$$= (g^*(r, i+2))^*(p^*_{g^*(ft(Y),i+1)}(h_2^*(f)) \circ eval) = ap(g^*(f, i+1), g^*(Y, i+2), g^*(r, i+2)).$$

Proposition 4.8 *[2009.11.29.prop1] Let CC be a C-system. Let further $(\Pi, eval)$ be a Π-structure on CC. Then the maps Π, λ, ap constructed above form an ap-Π-structure.*

Proof: Consider the conditions of Definition 4.1:

(1a) Follows from Definition 2.3(1). (1b) Follows from Definition 2.3(i) applied to $f = q(p_Z, ft^2(Y), i-1)$. (1c) Follows from Definition 2.3(i) applied to $f = q(t, ft^2(Y), i-1)$.

(2a) Follows from the definition of λ. (2b) Follows from Lemma 4.6 applied to p_Z. (2c) Follows from Lemma 4.6 applied to t.

(3a) Follows from the definition of ap. (3b) Follows from Lemma 4.7 applied to p_Z. (3c) Follows from Lemma 4.7 applied to t.

(4) One has

$$ap(\lambda(s), \partial s, r) = r^*((p^*_{ft(Y)}(\lambda(s)) \circ eval)) = r^*(\phi_Y(s)) = r^*(s) = \widetilde{S}(r, s).$$

Consider the condition of Definition 4.2. Let $T_1 = T(ft(Y), ft(Y))$ and $T_2 = T(ft(Y), Y)$. Then

$$ap(\widetilde{T}(ft(Y), f), T(ft(Y), Y), \delta_{ft(Y)}) = \delta^*_{ft(Y)}(p^*_{T_1}(p^*_{ft(Y)}(f)) \circ eval_{T_2}) =$$

$$= \delta^*_{ft(Y)} p^*_{T_1} p^*_{ft(Y)}(f) \circ \delta^*_{ft(Y)}(eval_{T_2}) = p^*_{ft(Y)}(f) \circ eval_{\delta^*_{ft(Y)}(T_2)} = p^*_{ft(Y)}(f) \circ eval_Y = \phi_Y(f)$$

which implies (4) by definition of λ.

To construct the inverse bijection consider a C-system CC and let

1. $\Pi : B_{n+2} \to B_{n+1}$, $n \geq 0$,
2. $\lambda : \widetilde{B}_{n+2} \to \widetilde{B}_{n+1}$, $n \geq 0$,
3. $ap : (\widetilde{B}_{n+1})_\partial \times_\Pi (B_{n+2})_{ft} \times_\partial (\widetilde{B}_{n+1}) \to \widetilde{B}_{n+1}$, $n \geq 0$

be maps satisfying the conditions of Definitions 4.1 and 4.2. For each $Y \in \widetilde{B}_{n+2}$ define a morphism

$$eval_Y : T(ft(Y), \Pi(Y)) \to Y$$

by the formula

$$eval_Y = ap(\delta_Z, T_2(Z, Y), p^*_Z(\delta_{ft(Y)})) \circ q(p_Z, Y)$$

where $Z = p^*_{ft(Y)}(\Pi(Y))$.

Proposition 4.9 *[***2009.11.30.prop2***] Under the assumption made above the morphisms $eval_Y$ are well defined and $(\Pi, eval)$ is a Π-structure on CC.*

Proof: Let us show that $eval_Y$ is well defined. This requires us to check the following conditions:

1. $ft^2(Y) = ft(\Pi(Y))$, therefore Z is defined,
2. $ft(Z) = ft\partial(\delta_{ft(Y)})$ since $ft(Z) = ft(Y)$, therefore $p^*_Z(\delta_{ft(Y)})$ is defined,
3. $ft^2(Z) = ft^2(Y)$, therefore $T_2(Z, Y)$ is defined,
4. $\partial(p^*_Z(\delta_{ft(Y)}))) = p^*_Z p^*_{ft(Y)}(ft(Y))$, $ft(T_2(Z, Y)) = T_2(Z, ft(Y)) = p^*_Z p^*_{ft(Y)}(ft(Y))$,
5. $\partial(\delta_Z) = p^*_Z(Z) = p^*_Z p^*_{ft(Y)}(\Pi(Y)) = \Pi_{T_2(Z,Y)}$, therefore $ap = ap(\delta_Z, T_2(Z, Y), p^*_Z(\delta_{ft(Y)}))$ is defined,
6. $$\partial(ap) = (p^*_Z(\delta_{ft(Y)}))^*(T_2(Z, Y)) = (p^*_Z(\delta_{ft(Y)}))^* T(Z, T(ft(Y), Y)) =$$
$$= (p^*_Z(\delta_{ft(Y)}))^*(p_Z)^*((p_{ft(Y)})^*(Y, 2), 2) = (p^*_Z(\delta_{ft(Y)}))^* q(p_Z, p^*_Y(ft(Y)))^*(p_{ft(Y)})^*(Y, 2) =$$
$$= (p^*_Z(\delta_{ft(Y)}))^* q(p_Z, p^*_Y(ft(Y)))^* q(p_{ft(Y)}, ft(Y))^*(Y) =$$
$$= (q(p_{ft(Y)}, ft(Y)) q(p_Z, p^*_Y(ft(Y))) p^*_Z(\delta_{ft(Y)}))^*(Y) = p^*_Z(Y)$$
and $q(p_Z, Y) : p^*_Z(Y) \to Y$. Therefore $eval_Y$ is defined and is a morphism from Z to Y as required by Definition 2.3(2).

We leave the verification of the conditions (i) of (ii) of Definition 2.3 for the later, more mechanized version of this paper.

5 Another reformulation of the Π-structure.

Problem 5.1 *[***2014.11.03.prob1***] To construct a bijection between pairs of families of operations on a unital B-system* $\mathbf{B}$ *of the form*

$$\Pi : B_{n+2} \to B_{n+1}$$

$$ap : (\widetilde{B}_{n+1})_{\partial} \times_{\Pi} (B_{n+2})_{ft} \times_{\partial} (\widetilde{B}_{n+1}) \to \widetilde{B}_{n+1}$$

satisfying the conditions 1(a)-(c) and 3(a)-(c) of Definition 4.1 and pairs of families of operations

$$\Pi : B_{n+2} \to B_{n+1}$$

$$ap1 : (\widetilde{B}_{n+1})_{\partial} \times_{\Pi} (B_{n+2}) \to \widetilde{B}_{n+2}$$

such that Π *again satisfies the conditions 1(a)-(c) of Definition 4.1 and* $ap1$ *satisfies the following conditions.*

For $n \geq 0$, $Y \in B_{n+2}$ *and* $f \in \widetilde{B}_{n+1}$ *such that* $\partial(f) = \Pi(Y)$ *one has:*

1. $\partial(ap1(f, Y)) = Y$,
2. *for* $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ *such that* $ft(Z) = ft^{i+1}(Y)$ *one has*
$$\widetilde{T}(Z, ap1(f, Y)) = ap1(\widetilde{T}(Z, f), T(Z, Y))$$
3. *for* $n+1 \geq i \geq 1$, $t \in B_{n+1-i}$ *such that* $\partial(t) = ft^{i+1}(Y)$ *one has*
$$\widetilde{S}(t, ap1(f, Y)) = ap1(\widetilde{S}(t, f), S(t, Y))$$
4. $\widetilde{S}(\delta(ft(Y)), ap1(\widetilde{T}(ft(Y), f), T(ft(Y), Y))) = ap1(f, Y)$

Construction 5.2 [**2014.11.03.constr2**]Given Π and $ap1$ define ap as follows:

$$ap(f, Y, r) = \widetilde{S}(r, ap1(f, Y))$$

For the properties of ap we have:

$$\partial(ap(f, Y, r)) = \partial(\widetilde{S}(r, ap1(f, Y))) = S(r, \partial(ap1(f, Y))) = S(r, Y)$$

$$\widetilde{T}(Z, ap(f, Y, r)) = \widetilde{T}(Z, \widetilde{S}(r, ap1(f, Y))) = \widetilde{S}(\widetilde{T}(Z, r), \widetilde{T}(Z, ap1(f, Y))) =$$
$$\widetilde{S}(\widetilde{T}(Z, r), ap1(\widetilde{T}(Z, f), T(Z, Y))) = ap(\widetilde{T}(Z, r), T(Z, Y), \widetilde{T}(Z, f))$$

(using the TS-condition of B-systems).

$$\widetilde{S}(t, ap(f, Y, r)) = \widetilde{S}(t, \widetilde{S}(r, ap1(f, Y))) = \widetilde{S}(\widetilde{S}(t, r), \widetilde{S}(t, ap1(f, Y))) =$$

$$\widetilde{S}(\widetilde{S}(t, r), ap1(\widetilde{S}(t, f), S(t, Y))) = ap(\widetilde{S}(t, f), S(t, Y), \widetilde{S}(t, r))$$

(using the SS-condition of B-systems).

Given Π and ap define $ap1$ as follows:

$$ap1(f, Y) = ap(\widetilde{T}(ft(Y), f), T(ft(Y), Y), \delta(ft(Y)))$$

For the properties of $ap1$ we have:

$$\partial(ap1(f,Y)) = \partial(ap(\widetilde{T}(ft(Y),f),T(ft(Y),Y),\delta(ft(Y)))) =$$
$$S(\delta(ft(Y)),T(ft(Y),Y)) = Y$$

(using SδT-condition of B-system).

$$\widetilde{T}(Z,ap1(f,Y)) = \widetilde{T}(Z,ap(\widetilde{T}(ft(Y),f),T(ft(Y),Y),\delta(ft(Y)))) =$$
$$ap(\widetilde{T}(Z,\widetilde{T}(ft(Y),f)),T(Z,T(ft(Y),Y)),\widetilde{T}(Z,\delta(ft(Y)))) =$$
$$ap(\widetilde{T}(T(Z,ft(Y)),\widetilde{T}(Z,f)),T(T(Z,ft(Y)),T(Z,Y)),\delta(T(Z,ft(Y)))) =$$
$$ap(\widetilde{T}(ft(T(Z,Y)),\widetilde{T}(Z,f)),T(ft(T(Z,Y)),T(Z,Y)),\delta(ft(T(Z,Y)))) =$$
$$ap1(\widetilde{T}(Z,f),T(Z,Y))$$

(using the TT- and Tδ-conditions of B-systems).

$$\widetilde{S}(t,ap1(f,Y)) = \widetilde{S}(t,ap(\widetilde{T}(ft(Y),f),T(ft(Y),Y),\delta(ft(Y)))) =$$
$$ap(\widetilde{S}(t,\widetilde{T}(ft(Y),f)),S(t,T(ft(Y),Y)),\widetilde{S}(t,\delta(ft(Y)))) =$$
$$ap(\widetilde{T}(S(t,ft(Y)),\widetilde{S}(t,f)),T(S(t,ft(Y)),S(t,Y)),\delta(S(t,ft(Y)))) =$$
$$ap(\widetilde{T}(ft(S(t,Y)),\widetilde{S}(t,f)),T(ft(S(t,Y)),S(t,Y)),\delta(ft(S(t,Y)))) =$$
$$ap1(\widetilde{S}(t,f),S(t,Y))$$

(using the ST- and Sδ-conditions of B-systems).

$$\widetilde{S}(\delta(ft(Y)),ap1(\widetilde{T}(ft(Y),f),T(ft(Y),Y))) =$$
$$\widetilde{S}(\delta(ft(Y)),ap(\widetilde{T}(ft(T(ft(Y),Y)),\widetilde{T}(ft(Y),f)),$$
$$T(ft(T(ft(Y),Y)),T(ft(Y),Y)),\delta(ft(T(ft(Y),Y))))) =$$
$$ap(\widetilde{S}(\delta(ft(Y)),\widetilde{T}(ft(T(ft(Y),Y)),\widetilde{T}(ft(Y),f))),$$
$$S(\delta(ft(Y)),T(ft(T(ft(Y),Y)),T(ft(Y),Y))),\widetilde{S}(\delta(ft(Y)),\delta(ft(T(ft(Y),Y))))) =$$
$$ap(\widetilde{S}(\delta(ft(Y)),\widetilde{T}(ft(Y),\widetilde{T}(ft(Y),f))),$$
$$S(\delta(ft(Y)),T(ft(Y),T(ft(Y),Y))),\widetilde{S}(\delta(ft(Y)),\delta(T(ft(Y),ft(Y))))) =$$
$$ap(\widetilde{T}(ft(Y),f),T(ft(Y),Y),\delta(ft(Y))) = ap1(f,Y)$$

(using the TT- SδT- δT- conditions of B-systems).

To check that these constructions are mutually inverse:

Starting with ap we have

$$ap'(f,Y,r) = \widetilde{S}(r,ap1(f,Y)) = \widetilde{S}(r,ap(\widetilde{T}(ft(Y),f),T(ft(Y),Y),\delta(ft(Y)))) =$$
$$ap(\widetilde{S}(r,\widetilde{T}(ft(Y),f)),S(r,T(ft(Y),Y)),\widetilde{S}(r,\delta(ft(Y)))) =$$
$$ap(f,Y,r)$$

(using the STid- and δSid- conditions of B-systems).

Starting with $ap1$ we have

$$ap1'(f,Y) = ap(\widetilde{T}(ft(Y),f),T(ft(Y),Y),\delta(ft(Y))) =$$
$$\widetilde{S}(\delta(ft(Y)),ap1(\widetilde{T}(ft(Y),f),T(ft(Y),Y))) = ap1(f,Y)$$

6 $\amalg$-structure on C-systems of the form $CC(R, LM)$ and their regular subquotients.

In a remarkable paper [1] A. Hirschowitz and M. Maggesi introduced the notion of an exponential monad ([1, p.559]). Let $\mathcal{C}$ be a category with finite coproducts $\amalg$ and a final object pt. Let $Maybe : \mathcal{C} \to \mathcal{C}$ be the functor of the form $X \mapsto X \amalg pt$. For a monad $R = (R, \rho, eta)$ on $\mathcal{C}$ there is a natural transformation

$$\gamma : Maybe \circ R \to R \circ Maybe$$

given by $R(i_X : X \to X \amalg pt)$ on $R(X)$ and by the restriction of $\eta_{X \amalg pt} : X \amalg pt \to R(X \amalg pt)$ to the pt on the pt.

The authors of [1] observe that for a left module $LM = (LM, \mu)$ over R, the functor LM' of the form $LM' = LM \circ Maybe$ together with the natural transformation

$$\mu' : LM \circ Maybe \circ R \stackrel{LM \circ \gamma}{\longrightarrow} LM \circ R \circ Maybe \stackrel{\mu \circ Maybe}{\longrightarrow} LM \circ Maybe$$

is again a left R-module. When $\mathcal{C}$ is the category of sets and R is the monad of λ-expressions modulo α-equivalence the λ-abstraction is an R-linear homomorphism of left R-modules

$$abs : R' \to R$$

and the same applies to the monads and modules obtained from general signatures with bindings. When, in addition to the α-equivalence, the λ-expressions are considered modulo β- and η-equivalences the resulting monads of expressions have the property that abs becomes an isomorphism. This leads to the following definition ([1]).

Definition 6.1 *[***HM2010.p559***] An exponential structure on a monad R on Sets is an R-linear isomorphism of left R-modules:*

$$abs : R' \to R$$

It is further shown that there are two other equivalent ways of presenting an exponential structure. One is by specifying an explicit inverse isomorphism which is denoted

$$ap1 : R \to R'$$

and another one by specifying in addition to abs an R-linear morphism

$$app : R \times R \to R$$

that, together with abs, satisfies two equations expressing the analogs of the β- and η-equivalences if $abs(E(x_1, \dots, x_n, y))$ is interpreted as the λ-abstraction $\lambda y.E(x_1, \dots, x_n, y)$ and $app(E, F)$ as application $E\,F$.

Remark 6.2 [**2014.11.03.rem1**] Note that $R' \to R$ is a morphism of *left* R-modules. Both R' and R also have natural structures of right R-modules that are given by the R-algebra structures on $R(X)$ and $R(X \amalg pt)$. If $R' \to R$ where an isomorphism of right R-modules i.e. an isomorphism of R-algebras this would be equivalent to having an isomorphism $X \amalg pt \to X$ in the Kleisli category of R. The isomorphism of left R-modules does not have such an interpretation.

Problem 6.3 *[***2014.11.03.prob3***] To construct for a pair (R, LM) where R is a monad on sets and LM a left R-module with values in $Sets$ a function from the pairs $(Prod, abs)$ where*

$$Prod : LM \times LM' \to LM$$

is a homomorphism of left R-modules and

$$abs : R' \to R$$

an exponential structure on R to the ap-Π-structures on $B(R, LM)$.

Construction 6.4 [**2014.11.03.constr2**]We first construct for any triple $(Prod, abs, app)$ an ap-Π-structure on $B(R, LM)$. We define the operations as follows.

$$\Pi(E_1, \ldots, E_n, A, B) = (E_1, \ldots, E_n, Prod(A, B))$$

$$\lambda(E_1, \ldots, E_n, A, B, r) = (E_1, \ldots, E_n, Prod(A, B), abs(a))$$

$$ap((E_1, \ldots, E_n, Prod(A, B), f), (E_1, \ldots, E_n, A, B), (E_1, \ldots, E_n, A, r)) = (E_1, \ldots, E_n, ???app(f, r))$$

??? Write the explicit formulas for the operations on B(R,LM) using the $\eta, bind, \rho$.

Theorem about Π-structures on the regular quotients!

Remark 6.5 A Π-structure on (p_1, p_2, p_3) corresponds to the rule

$$\frac{\Gamma, X : U_1, f : X \to U_2}{\Gamma, X : U_1, f : X \to U_2 \vdash \prod x : X.ev(f, x) : U_3}$$

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