The (Π, λ)-structures on the C-systems defined by universe categories[1]

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Abstract

We then define the notion of a $(P, \widetilde{P})$-structure on a universe in a locally cartesian closed category category and construct a (Π, λ)-structure on the C-systems $CC(\mathcal{C}, p)$ from a $(P, \widetilde{P})$-structure on p.

In the last section we define homomorphisms of C-systems with (Π, λ)-structures and functors of universe categories with $(P, \widetilde{P})$-structures and show that the construction of the previous section is functorial relative to these definitions.

Contents

1 Introduction

The concept of a C-system in its present form was introduced in [**?**]. The type of the C-systems is constructively equivalent to the type of contextual categories defined by Cartmell in [**?**] and [**?**] but the definition of a C-system is slightly different from the Cartmell's foundational definition.

In this paper we consider what might be the most important structure on C-systems - the structure that corresponds, for the syntactic C-systems, to the operations of dependent product, λ-abstraction and application. A C-system formulation of this structure was introduced by John Cartmell in [**?**, pp. 3.37 and 3.41] as a part of what he called a strong M.L. structure. It was studied further by Thomas Streicher in [**?**, p.71] who called a C-system (contextual category) together with such a structure a "contextual category with products of families of types".

The constructions and proofs of the main part of the paper require knowing many facts about C-systems. These facts are established in Section **??**. Many of these facts are new, some have been stated by Cartmell [**?**] and Streicher [**?**], but without proper mathematical

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proofs. Among notable new facts we can mention Lemma **??** that shows that the canonical direct product in a C-system is strictly associative.

In Section **??** we construct on any C-system presheaves $\mathcal{O}b_n$ and $\widetilde{\mathcal{O}b}_n$. These presheaves play a major role in our approach to the C-system formulation of systems of operations that correspond to systems of inference rules. The main result here is Construction **??** for Problem **??**. It is likely that constructions for various other variants of this problem involving morphisms between presheaves $\mathcal{O}b_*$ and $\widetilde{\mathcal{O}b}_*$ can be given. The full generality of this result should involve as the source fiber products of $\mathcal{O}b_*$ and $\widetilde{\mathcal{O}b}_*$ relative to morphisms satisfying certain properties and as the target $\mathcal{O}b_*$ or $\widetilde{\mathcal{O}b}_*$. We limit ourselves to Construction **??** here because it is the only case that will be required later in the paper.

In Section **??** we first remind the definition of the product of families of types structure on a C-system. Then, in Definition**??**, we give the first of the two main definitions of this paper, the definition of a (Π, λ)-structure. In the rest of this section we work on constructing a bijection between the sets of structures of products of families of types and (Π, λ)-structures on a given C-system. This is probably the most technical part of the paper which is not surprising considering how different Definitions **??** and **??** are.

This construction uses most of the results of Section **??**.

The (Π, λ)-structures correspond to the $(\Pi, \lambda, app, \beta, \eta)$-system of inference rules. In Remark **??** we outline the definitions of classes of structures that correspond to the similar systems but without the β- or η-rules. Such structures appear as natural variations of the (Π, λ)-structures.

In Section 3 we consider the case of C-systems of the form $CC(\mathcal{C}, p)$ introduced in [**?**]. They are defined, in a functorial way, by a category $\mathcal{C}$ with a final object and a morphism $p : \widetilde{U} \to U$ together with the choice of pullbacks of p along all morphisms in $\mathcal{C}$. A morphism with such choices is called a universe in $\mathcal{C}$. As a corollary of general functoriality we also obtain a construction of an isomorphism that connects the C-systems $CC(\mathcal{C}, p)$ corresponding to different choices of pullbacks and different choices of final objects. It makes it possible to say that $CC(\mathcal{C}, p)$ is defined by $\mathcal{C}$ and p.

We provide several intermediate results about $CC(\mathcal{C}, p)$ when $\mathcal{C}$ is a locally cartesian closed category leading to the main result of this paper - Construction 2.4 that produces a (Π, λ)-structure on $CC(\mathcal{C}, p)$ from a simple pullback[3] based on p. This construction was first announced in [**?**]. It and the ideas that it is based on are among the most important ingredients of the construction of the univalent model of the Martin-Lof type theory.

In the following sections we study the behavior of our construction with respect to universe category functors and prove that it is functorial with respect to functors equipped with an additional structure that reflects compatibility with the choice of the generating pullback.

One may wonder how the construction of this paper relates to the earlier ideas of Seely [**?**] and their refinement by Clairambault and Dybjer [**?**]. This question requires further study.

The methods of this paper are fully constructive.

[3]We say "a pullback" instead of "a pullback square".

The paper is written in the formalization-ready style that is in such a way that no long arguments are hidden even when they are required only to substantiate an assertion that may feel obvious to readers who are closely associated with a particular tradition of mathematical thought.

As a result, a number of lemmas, especially in the appendices, may be well know to many readers. Their proofs are nevertheless included to comply with the requirements of the formalization ready style.

On the other hand, not all preliminary lemmas are included or a reference to a complete proof is given. There are some, but very much fewer than is usual in today's papers, exceptions.

The main result of this paper is not a theorem but a construction and so are many of the intermediate results. Because of the importance of constructions for this paper we use a special pair of names Problem-Construction for the specification of the goal of a construction and the description of the particular solution.

In the case of a Theorem-Proof pair one usually refers (by name or number) to the theorem when using the proof of this theorem. This is acceptable in the case of theorems because the future use of their proofs is such that only the fact that there is a proof but not the particulars of the proof matter.

In the case of a Problem-Construction pair the content of the construction often matters in the future use. Because of this we have to refer to the construction and not to the problem and we assign in this paper numbers both to Problems and to Constructions.

We use below the concept of a universe. In the Zermelo-Fraenkel set theory, the main intended formalization base for this paper, a universe is simply a set U that is usually assumed to satisfy some properties such as, for example, that it is closed under formation of pairs - if two sets A and B are elements of U then the set representing the pair (A, B) is an element of U. We do not provide a precise set of such conditions that we assume. To assume the universes mentioned in the paper to be Grothendieck universes would certainly suffice but in most cases we need a much weaker set of conditions. It is likely that the conditions that we need are weak enough to be able to prove the existence of such universes inside the "canonical" Zermelo-Fraenkel theory without any large cardinal axioms.

In this paper we continue to use the diagrammatic order of writing composition of morphisms, i.e., for $f : X \to Y$ and $g : Y \to Z$ the composition of f and g is denoted by $f \circ g$.

We denote by Φ° the functor $PreShv(C') \to PreShv(C)$ given by the pre-composition with a functor $\Phi^{op} : C^{op} \to (C')^{op}$, that is,

$$\Phi^\circ(F)(X) = F(\Phi(X))$$

In the literature this functor is denoted both by Φ^* and Φ_* and we decided to use a new unambiguous notation instead.

Acknowledgements are at the end of the paper.

While abbreviated notations may be helpful for getting a general impression from a brief scroll through the paper, long notations become indispensable when one seeks true understanding.

In view of Lemma 2.6, Construction **??** can be used not only to construct the product of families of types structures on C-systems, but also to prove that such structures do not exist. This applies also to structures corresponding to other systems of inference rules in type theory. For example, a similar technique may be used not only to construct a model of a particular kind of higher inductive types, but also to show that for a given universe p no such model on $CC(\mathcal{C}, p)$ exists.

That construction for Problem 2.7, without the part that concerns the bijection, exists was originally stated in [**?**, Proposition 2] with a sketch of a proof given in the 2009 version of [**?**].

2 Construction of (Π, λ)-structures from $(P, \widetilde{P})$-structures

In this section we describe a method of constructing (Π, λ)-structures on C-systems of the form $CC(\mathcal{C}, p)$ where $\mathcal{C}$ is a locally cartesian closed universe category $(\mathcal{C}, p)$ with a binary product structure.

Let us recall the following definition from [**?**]:

Definition 2.1 *[**2015.03.09.def1**] Let CC be a C-system. A pre-(Π, λ)-structure on CC is a pair of morphisms of presheaves*

$$\Pi : \mathcal{O}b_2 \to \mathcal{O}b_1$$
$$\lambda : \widetilde{\mathcal{O}b}_2 \to \widetilde{\mathcal{O}b}_1$$

such that the square

$$[\mathbf{2015.03.09.eq1}]\quad \begin{array}{ccc} \widetilde{\mathcal{O}b}_2 & \xrightarrow{\lambda} & \widetilde{\mathcal{O}b}_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ \mathcal{O}b_2 & \xrightarrow{\Pi} & \mathcal{O}b_1 \end{array} \tag{1}$$

commutes.

A pre-(Π, λ)-structure is called a (Π, λ)-structure if the square (1) is a pullback.

The functors I_p were defined in [**?**, Sec. 2.6].

Definition 2.2 *[**2015.03.29.def1**] Let $\mathcal{C}$ be a locally cartesian closed category with a binary product structure and $p : \widetilde{U} \to U$ a universe in $\mathcal{C}$. A pre-$(P, \widetilde{P})$-structure on p is a pair of morphisms*

$$\widetilde{P} : I_p(\widetilde{U}) \to \widetilde{U}$$
$$P : I_p(U) \to U$$

such that the square

$$[\mathbf{2009.prod.square}]\quad \begin{array}{ccc} I_p(\widetilde{U}) & \xrightarrow{\widetilde{P}} & \widetilde{U} \\ {\scriptstyle I_p(p)}\downarrow & & \downarrow{\scriptstyle p} \\ I_p(U) & \xrightarrow{P} & U \end{array} \tag{2}$$

commutes.

A pre-$(P,\widetilde{P})$-structure is called a $(P,\widetilde{P})$-structure if the square (2) is a pullback.

We will often say pre-P-structure (resp. P-structure) instead of pre-$(P,\widetilde{P})$-structure (resp. $(P,\widetilde{P})$-structure).

Problem 2.3 *[***2015.03.17.prob0***] Let $(\mathcal{C},p)$ be a locally cartesian closed universe category with a binary product structure. Let $(P,\widetilde{P})$ be a pre-$(P,\widetilde{P})$-structure on p. To construct a pre-(Π,λ)-structure on $CC(\mathcal{C},p)$.*

Construction 2.4 *[***2015.03.17.constr3***]*Consider the diagram:

$$[\mathbf{2016.12.09.eq1}]\quad \begin{array}{ccccccc} \widetilde{\mathcal{O}b}_2 & \xrightarrow{\widetilde{\mu}_2} & int^{\circ}(Yo(I_p(\widetilde{U}))) & \xrightarrow{int^{\circ}(Yo(\widetilde{P}))} & int^{\circ}(Yo(\widetilde{U})) & \xrightarrow{\mu_1^{-1}} & \widetilde{\mathcal{O}b}_1 \\ \downarrow{\partial} & & \downarrow{int^{\circ}(Yo(I_p(p)))} & & \downarrow{int^{\circ}(Yo(p))} & & \downarrow{\partial} \\ \mathcal{O}b_2 & \xrightarrow{\mu_2} & int^{\circ}(Yo(I_p(U))) & \xrightarrow{int^{\circ}(Yo(P))} & int^{\circ}(Yo(U)) & \xrightarrow{\mu_1^{-1}} & \mathcal{O}b_1 \end{array} \tag{3}$$

where μ_n and $\widetilde{\mu}_n$ are isomorphisms defined in [**?**, Sec. 2.6]. The left hand side and the right hand side squares of this diagram commute because the squares in [**?**, Problem 2.42] commute. The middle square commutes because the square (2) commutes and both Yo and int° are functors. Therefore, the outside rectangle commutes and we conclude that the pair of morphisms

$$[\mathbf{2016.12.09.eq3}]\begin{array}{l} \lambda = \widetilde{\mu}_2 \circ int^{\circ}(Yo(\widetilde{P})) \circ \widetilde{\mu}_1^{-1} \\ \Pi = \mu_2 \circ int^{\circ}(Yo(P)) \circ \mu_1^{-1} \end{array} \tag{4}$$

is a pre-(Π,λ)-structure on $CC(\mathcal{C},p)$.

Lemma 2.5 *[***2017.01.07.l4***] In the context of Construction 2.4, if $(P,\widetilde{P})$ is a $(P,\widetilde{P})$-structure then the pre-(Π,λ)-structure constructed there is a (Π,λ)-structure.*

Proof: We need to show that the external square of the diagram (3) is a pullback.

Horizontal composition of pullbacks is a pullback. The left hand side square is a pullback because it is a commutative square with two parallel sides being isomorphisms. The right hand side square is a pullback for the same reason.

It remains to show that the middle square is pullback. This square is obtained by applying first the functor Yo and then the functor int° to the pullback square (2).

Our claim follows now from two facts:

1. the Yoneda functor $Yo : \mathcal{C} \to PreShv(\mathcal{C})$ takes pullbacks to pullbacks,
2. for any functor $F : \mathcal{C}' \to \mathcal{C}$, the functor

$$F^{\circ} : PreShv(\mathcal{C}) \to PreShv(\mathcal{C}')$$

   of pre-composition with F^{op}, takes pullbacks to pullbacks.

We assume that these two facts are known.

There is an important class of cases when the function from $(P, \widetilde{P})$-structures on p to (Π, λ)-structures on $CC(\mathcal{C}, p)$ defined by Construction 2.4 is a bijection.

Lemma 2.6 *[**2016.09.09.l1**] Let $(\mathcal{C}, p)$ be a universe category such that the functor*

$$Yo \circ int^{\circ} : \mathcal{C} \to PreShv(CC(\mathcal{C}, p))$$

is fully faithful. Then the function from the pre-$(P, \widetilde{P})$-structures on p to the pre-(Π, λ)-structures on $CC(\mathcal{C}, p)$ defined by Construction 2.4 is a bijection.

Moreover, the restriction of this function to the function from $(P, \widetilde{P})$-structures to (Π, λ)-structures, which is defined in view of Lemma 2.5, is a bijection as well.

Proof: Let

$$\widetilde{\alpha} : Mor_{PreShv(CC(\mathcal{C},p))}(int^{\circ}(Yo(I_p(\widetilde{U}))), int^{\circ}(Yo(\widetilde{U}))) \to Mor_{\mathcal{C}}(I_p(\widetilde{U}), \widetilde{U})$$

$$\alpha : Mor_{PreShv(CC(\mathcal{C},p))}(int^{\circ}(Yo(I_p(U))), int^{\circ}(Yo(U))) \to Mor_{\mathcal{C}}(I_p(U), U)$$

be the inverses to $(Yo \circ int^{\circ})_{I_p(\widetilde{U}),\widetilde{U}}$ and $(Yo \circ int^{\circ})_{I_p(U),U}$ respectively.

Given a pre-(Π, λ)-structure (Π, λ) let

$$[\mathbf{2016.09.09.eq1}] \begin{array}{l} \widetilde{P} = \widetilde{\alpha}(\widetilde{\mu}_2^{-1} \circ \lambda \circ \widetilde{\mu}_1) \\ P = \alpha(\mu_2^{-1} \circ \Pi \circ \mu_1) \end{array} \quad (5)$$

Then $\widetilde{P} : I_p(\widetilde{U}) \to \widetilde{U}$ and $P : I_p(U) \to U$. Let S be the square that $\widetilde{P}$ and P form with $I_p(p)$ and p. Then the square $(Yo \circ int^{\circ})(S)$ is of the form

$$[\mathbf{2017.01.07.eq7}] \begin{array}{ccc} int^{\circ}(Yo(I_p(\widetilde{U}))) & \xrightarrow{\widetilde{\mu}_2^{-1} \circ \lambda \circ \widetilde{\mu}_1} & int^{\circ}(Yo(\widetilde{U})) \\ {\scriptstyle int^{\circ}(Yo(I_p(p)))} \downarrow & & \downarrow {\scriptstyle int^{\circ}(Yo(p))} \\ int^{\circ}(Yo(I_p(U))) & \xrightarrow{\mu_2^{-1} \circ \Pi \circ \mu_1} & int^{\circ}(Yo(U)) \end{array} \quad (6)$$

Since the left and right squares of (3) commute and their horizontal arrows are isomorphisms, the square $(Yo \circ int^{\circ})(S)$ is isomorphic to the original square formed by Π and λ and as a square isomorphic to a commutative square is commutative. Since $Yo \circ int^{\circ}$ is faithful, that is, injective on morphisms between a given pair of objects we conclude that S is commutative, that is, $(P, \widetilde{P})$ defined in (5) is a pre-$(P, \widetilde{P})$-structure.

One verifies immediately that the function from pre-(Π, λ)-structures to pre-$(P, \widetilde{P})$-structures that this construction defines is both left and right inverse to the function defined by Construction 2.4.

Assume now that we started with a (Π, λ)-structure. Then the square $(Yo \circ int^{\circ})(S)$ is isomorphic to a pullback and therefore is a pullback. By our assumption, the functor $Yo \circ int^{\circ}$

is fully-faithful. Fully-faithful functors reflect pullbacks, that is, if the image of a square under a fully-faithful functor is a pullback than the original square is a pullback. We conclude that both the direct and the inverse bijections map the subsets of $(P,\widetilde{P})$-structures and (Π,λ)-structures to each other. Therefore, e.g. by [**?**, Lemma 5.1], the restrictions of the total bijections to these subsets are bijections as well.

The lemma is proved.

Problem 2.7 *[***2016.12.09.prob2***] Let $(\mathcal{C},p)$ be a universe category.*

To construct a function from the set of $(P,\widetilde{P})$-structures on p to the set of structures of products of families of types on $CC(\mathcal{C},p)$.

To show that if the functor $Yo\circ int^{\circ}$ is fully faithful than this function is a bijection.

Construction 2.8 [**2016.12.09.constr2**] The required function is the composition of the function of Construction 2.4 with the construction for [**?**, Problem 4.5] described in that paper.

Remark 2.9 [**2017.01.07.rem1**] One can define a mixed $(P,\widetilde{P})$-structure (or pre-$(P,\widetilde{P})$-structure) as follows:

Definition 2.10 *[***2009.10.27.def1***] Let $\mathcal{C}$ be an lcc category and let $p_i:\widetilde{U}_i\to U_i$, $i=1,2,3$ be three morphisms in $\mathcal{C}$. A $(P,\widetilde{P})$-structure on (p_1,p_2,p_3) is a pullback of the form*

$$\begin{array}{ccc} I_{p_1}(\widetilde{U}_2) & \xrightarrow{\widetilde{P}} & \widetilde{U}_3 \\ {\scriptstyle I_{p_1}(p_2)}\downarrow & & \downarrow{\scriptstyle p_3} \\ I_{p_1}(U_2) & \xrightarrow{P} & U_3 \end{array} \tag{7}$$

[**Pisq1**]

Then a $(P,\widetilde{P})$-structure on p is a $(P,\widetilde{P})$-structure on (p,p,p). This concept can be used to construct universes in C-systems that participate in impredicative (Π,λ)-structures.

3 Functoriality properties of the (Π,λ)-structures constructed from $(P,\widetilde{P})$-structures

Recall that in [**?**, pp. 1067-68] we have constructed, for any homomorphism $H:CC\to CC'$ of C-systems, and any $n\geq 0$, natural transformations

$$H\mathcal{O}b_n:\mathcal{O}b_i\to H^{\circ}(\mathcal{O}b_i)$$

where for $\Gamma\in CC$ and $T\in\mathcal{O}b_i(\Gamma)$ one has

$$H\mathcal{O}b_n(T)=H_{Ob}(T)$$

and

$$H\widetilde{\mathcal{O}b}_n : \widetilde{\mathcal{O}b}_i \to H^\circ(\widetilde{\mathcal{O}b}_i)$$

where for $\Gamma \in CC$ and $o \in \widetilde{\mathcal{O}b}_n(\Gamma)$ one has

$$H\widetilde{\mathcal{O}b}_n(o) = H_{Mor}(o)$$

Definition 3.1 *[***2016.09.13.def1***] Let $H : CC \to CC'$ be a homomorphism of C-systems. Let (Π, λ) and (Π', λ') be pre-(Π, λ)-structures on CC and CC' respectively.*

Then H is called a (Π, λ)-homomorphism if the following two squares commute

$$\begin{array}{ccc} \mathcal{O}b_2 & \xrightarrow{\Pi} & \mathcal{O}b_1 \\ {\scriptstyle H\mathcal{O}b_2}\downarrow & & \downarrow{\scriptstyle H\mathcal{O}b_1} \\ H^\circ(\mathcal{O}b_2) & \xrightarrow{H^\circ(\Pi')} & H^\circ(\mathcal{O}b_1) \end{array} \qquad \begin{array}{ccc} \widetilde{\mathcal{O}b}_2 & \xrightarrow{\lambda} & \widetilde{\mathcal{O}b}_1 \\ {\scriptstyle H\widetilde{\mathcal{O}b}_2}\downarrow & & \downarrow{\scriptstyle H\widetilde{\mathcal{O}b}_1} \\ H^\circ(\widetilde{\mathcal{O}b}_2) & \xrightarrow{H^\circ(\lambda')} & H^\circ(\widetilde{\mathcal{O}b}_1) \end{array}$$

If (Π, λ) and (Π', λ') are (Π, λ)-structures then H is called a (Π, λ)-homomorphism if it is a (Π, λ)-homomorphism with respect to the corresponding pre-(Π, λ)-structures.

Unfolding the definition of $H\mathcal{O}b_i$ and $H\widetilde{\mathcal{O}b}_i$ we see that H is a (Π, λ)-homomorphism if and only if for all $\Gamma \in CC$ one has

1. for all $T \in \mathcal{O}b_2(\Gamma)$ one has

$$[\mathbf{2016.09.13.eq1}] H(\Pi_\Gamma(T)) = \Pi'_{H(\Gamma)}(H(T)) \tag{8}$$

2. for all $o \in \widetilde{\mathcal{O}b}_2(\Gamma)$ one has

$$[\mathbf{2016.09.13.eq2}] H(\lambda_\Gamma(o)) = \lambda'_{H(\Gamma)}(H(o)) \tag{9}$$

The morphisms ξ and $\widetilde{\xi}$ used in the following theorem are defined in [**?**, Sec. 3.4].

Theorem 3.2 *[***2015.03.21.th1***] Let $(\mathcal{C}, p)$ and $(\mathcal{C}', p')$ be universe categories with locally cartesian closed and binary product structures. Let $\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi})$ be a universe category functor and let $(P, \widetilde{P})$, $(P', \widetilde{P}')$ be pre-$(P, \widetilde{P})$-structures on p and p' respectively.*

Assume that the squares

$$[\mathbf{2015.03.23.sq1}] \quad \begin{array}{ccc} \Phi(I_p(U)) & \xrightarrow{\Phi(P)} & \Phi(U) \\ {\scriptstyle \xi_{\mathbf{\Phi}}}\downarrow & & \downarrow{\scriptstyle \phi} \\ I_{p'}(U') & \xrightarrow{P'} & U' \end{array} \qquad \begin{array}{ccc} \Phi(I_p(\widetilde{U})) & \xrightarrow{\Phi(\widetilde{P})} & \Phi(\widetilde{U}) \\ {\scriptstyle \widetilde{\xi}_{\mathbf{\Phi}}}\downarrow & & \downarrow{\scriptstyle \widetilde{\phi}} \\ I_{p'}(\widetilde{U}') & \xrightarrow{\widetilde{P}'} & \widetilde{U}' \end{array} \tag{10}$$

commute. Then the homomorphism

$$H(\Phi, \phi, \widetilde{\phi}) : CC(\mathcal{C}, p) \to CC(\mathcal{C}', p')$$

is a homomorphism of C-systems with pre-(Π, λ)*-structures relative to the pre-*(Π, λ)*-structures obtained from* $(P, \widetilde{P})$ *and* $(P', \widetilde{P}')$ *by Construction 2.4.*

Proof: We have to show that for all $\Gamma \in Ob(CC(\mathcal{C}, p))$, $T \in Ob_2(\Gamma)$ and $o \in \widetilde{Ob}_2(\Gamma)$ the equalities (8) and (9) hold. We will prove the first equality. The proof of the second is strictly parallel to the proof of the first.

From [**?**, Eq. 2.76], using the fact that μ_n is a bijection, for $F \in Mor_{\mathcal{C}}(int(\Gamma), I_p^{n-1}(U))$, we have

$$[\mathbf{2017.01.13.eq1}] H(\mu_n^{-1}(F)) = \mu_n^{-1}(\psi(\Gamma) \circ \Phi(F) \circ \xi_{n-1}) \tag{11}$$

Therefore,

$$H(\Pi(T)) = H(\mu_1^{-1}(\mu_2(T) \circ P)) = H(u_1^{-1}(\eta_1(u_2(T)) \circ P)) = (u_1)^{-1}(\psi(\Gamma) \circ \Phi(\eta_1(u_2(T)) \circ P) \circ \phi) =$$

$$(u_1)^{-1}(\psi(\Gamma) \circ \Phi(\eta(u_2(T))) \circ \Phi(P) \circ \phi)$$

where the first equality holds by the definition of Π, the second one by the definition of μ_n given in [**?**, Eq. 2.76] and by the fact the $\eta_0 = Id$ (cf. [**?**, Construction 2.40]), the third equality holds by [Lemma 3.15]presheavesOb and the third equality by the composition axiom of functor Φ. Next

$$\Pi'(H(T)) = (u_1)^{-1}(\mu_2(H(T)) \circ P') = (u_1)^{-1}(\eta'(u_2(H(T))) \circ P')$$

Let us show that

$$\eta'(u_2(H(T))) \circ P' = \psi(\Gamma) \circ \Phi(\eta(u_2(T))) \circ \Phi(P) \circ \phi$$

By Lemma **??**(1) we have

$$\eta'(u_2(H(T))) \circ P' = \psi(\Gamma) \circ \Phi(\eta(u_2(T))) \circ \xi_\Phi \circ P'$$

It remains to show that

$$\xi_\Phi \circ P' = \Phi(P) \circ \phi$$

which is our assumption about the commutativity of the square first square in (10).

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