Products of families of types in C-systems defined by a universe category[1]

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Abstract

We introduce the notion of a (Π, λ)-structure on a C-system and construct a bijection, for a given C-system, between the sets of (Π, λ)-structures and structures of products of families of types introduced previously by Cartmell and Streicher.

We then define the notion of a P-structure on a universe in a locally cartesian closed category category and construct a (Π, λ)-structure on the C-systems $CC(\mathcal{C}, p)$ from a P-structure on p.

In the last section we define homomorphisms of C-systems with (Π, λ)-structures and functors of universe categories with P-structures and show that the construction of the previous section is functorial relative to these definitions.

Contents

1 Introduction

The concept of a C-system in its present form was introduced in [10]. The type of the C-systems is constructively equivalent to the type of contextual categories defined by Cart-

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mell in [1] and [2] but the definition of a C-system is slightly different from the Cartmell's foundational definition.

In this paper we consider what might be the most important structure on C-systems - the structure that corresponds, for the syntactic C-systems, to the operations of dependent product, λ-abstraction and application. A C-system formulation of this structure was introduced by John Cartmell in [1, pp. 3.37 and 3.41] as a part of what he called a strong M.L. structure. It was studied further by Thomas Streicher in [7, p.71] who called a C-system (contextual category) together with such a structure a "contextual category with products of families of types".

The constructions and proofs of the main part of the paper require knowing many facts about C-systems. These facts are established in Section **??**. Many of these facts are new, some have been stated by Cartmell [1] and Streicher [7], but without proper mathematical proofs. Among notable new facts we can mention Lemma **??** that shows that the canonical direct product in a C-system is strictly associative.

In Section **??** we construct on any C-system presheaves $\mathcal{O}b_n$ and $\widetilde{\mathcal{O}b}_n$. These presheaves play a major role in our approach to the C-system formulation of systems of operations that correspond to systems of inference rules. The main result here is Construction **??** for Problem **??**. It is likely that constructions for various other variants of this problem involving morphisms between presheaves $\mathcal{O}b_*$ and $\widetilde{\mathcal{O}b}_*$ can be given. The full generality of this result should involve as the source fiber products of $\mathcal{O}b_*$ and $\widetilde{\mathcal{O}b}_*$ relative to morphisms satisfying certain properties and as the target $\mathcal{O}b_*$ or $\widetilde{\mathcal{O}b}_*$. We limit ourselves to Construction **??** here because it is the only case that will be required later in the paper.

In Section **??** we first remind the definition of the product of families of types structure on a C-system. Then, in Definition**??**, we give the first of the two main definitions of this paper, the definition of a (Π, λ)-structure. In the rest of this section we work on constructing a bijection between the sets of structures of products of families of types and (Π, λ)-structures on a given C-system. This is probably the most technical part of the paper which is not surprising considering how different Definitions **??** and **??** are.

This construction uses most of the results of Section **??**.

The (Π, λ)-structures correspond to the $(\Pi, \lambda, app, \beta, \eta)$-system of inference rules. In Remark **??** we outline the definitions of classes of structures that correspond to the similar systems but without the β- or η-rules. Such structures appear as natural variations of the (Π, λ)-structures.

In Section 2 we consider the case of C-systems of the form $CC(\mathcal{C}, p)$ introduced in [9]. They are defined, in a functorial way, by a category $\mathcal{C}$ with a final object and a morphism $p : \widetilde{U} \to U$ together with the choice of pullbacks of p along all morphisms in $\mathcal{C}$. A morphism with such choices is called a universe in $\mathcal{C}$. As a corollary of general functoriality we also obtain a construction of an isomorphism that connects the C-systems $CC(\mathcal{C}, p)$ corresponding to different choices of pullbacks and different choices of final objects. It makes it possible to say that $CC(\mathcal{C}, p)$ is defined by $\mathcal{C}$ and p.

We provide several intermediate results about $CC(\mathcal{C}, p)$ when $\mathcal{C}$ is a locally cartesian closed category leading to the main result of this paper - Construction 2.2.3 that produces a (Π, λ)-structure on $CC(\mathcal{C}, p)$ from a simple pullback square based on p. This construction was first announced in [8]. It and the ideas that it is based on are among the most important ingredients of the construction of the univalent model of the Martin-Lof type theory.

In the following sections we study the behavior of our construction with respect to universe category functors and prove that it is functorial with respect to functors equipped with an additional structure that reflects compatibility with the choice of the generating pullback squares.

One may wonder how the construction of this paper relates to the earlier ideas of Seely [6] and their refinement by Clairambault and Dybjer [3]. This question requires further study.

The methods of this paper are fully constructive. It is also written in the formalization-ready style that is in such a way that no long arguments are hidden even when they are required only to substantiate an assertion that may feel obvious to readers who are closely associated with a particular tradition of mathematical thought.

The main result of this paper is not a theorem but a construction and so are many of the intermediate results. Because of the importance of constructions for this paper we use a special pair of names Problem-Construction for the specification of the goal of a construction and the description of the particular solution.

In the case of a Theorem-Proof pair one usually refers (by name or number) to the theorem when using the proof of this theorem. This is acceptable in the case of theorems because the future use of their proofs is such that only the fact that there is a proof but not the particulars of the proof matter.

In the case of a Problem-Construction pair the content of the construction often matters in the future use. Because of this we have to refer to the construction and not to the problem and we assign in this paper numbers both to Problems and to Constructions.

We use below the concept of a universe. In the Zermelo-Fraenkel set theory, the main intended formalization base for this paper, a universe is simply a set U that is usually assumed to satisfy some properties such as, for example, that it is closed under formation of pairs - if two sets A and B are elements of U then the set representing the pair (A, B) is an element of U. We do not provide a precise set of such conditions that we assume. To assume the universes mentioned in the paper to be Grothendieck universes would certainly suffice but in most cases we need a much weaker set of conditions. It is likely that the conditions that we need are weak enough to be able to prove the existence of such universes inside the "canonical" Zermelo-Fraenkel theory without any large cardinal axioms.

In this paper we continue to use the diagrammatic order of writing composition of morphisms, i.e., for $f : X \to Y$ and $g : Y \to Z$ the composition of f and g is denoted by $f \circ g$.

We denote by Φ° the functor $PreShv(C') \to PreShv(C)$ given by the pre-composition with a functor $\Phi^{op} : C^{op} \to (C')^{op}$, that is,

$$\Phi^\circ(F)(X) = F(\Phi(X))$$

In the literature this functor is denoted both by Φ^* and Φ_* and we decided to use a new unambiguous notation instead.

Acknowledgements are at the end of the paper.

While abbreviated notations may be helpful for getting a general impression from a brief scroll through the paper, long notations become indispensable when one seeks true understanding.

2 P-structures on universes and (Π, λ)-structures

2.1 More on the C-systems of the form $CC(\mathcal{C}, p)$

Let us start by considering a general category $\mathcal{C}$. Let $p : \widetilde{U} \to U$ be a morphism in $\mathcal{C}$. Recall from [9] that a universe structure on p is a choice of pullback squares of the form

$$\begin{array}{ccc} (X;F) & \xrightarrow{Q(F)} & \widetilde{U} \\ {\scriptstyle p_{X,F}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{F} & U \end{array}$$

for all X and all morphisms $F : X \to U$. A universe in $\mathcal{C}$ is a morphism with a universe structure on it and a universe category is a category with a universe and a choice of a final object pt.

We may use the notation $(X; F_1, \dots, F_n)$ for $(\dots(X; F_1); \dots F_n)$.

For $f : W \to X$ and $g : W \to \widetilde{U}$ such that $f \circ F = g \circ p$ we will denote by $f * g$ the unique morphism such that

$$(f * g) \circ p_{X,F} = f$$

$$(f * g) \circ Q(F) = g$$

For $X' \stackrel{f}{\to} X \stackrel{F}{\to} U$ we let $Q(f, F)$ denote the morphism

$$(p_{X', f\circ F} \circ f) * Q(f \circ F) : (X'; f \circ F) \to (X; F)$$

Observe that one has

$$[\mathbf{2016.08.24.eq4}] Q(f \circ F) = Q(f, F) \circ Q(F) \tag{2.1.1}$$

$$[\mathbf{2016.08.26.eq2}] Q(Id_X, F) = Id_{(X;F)} \tag{2.1.2}$$

$$[\mathbf{2016.08.26.eq3}] Q(f' \circ f, F) = Q(f', f \circ F) \circ Q(f, F) \tag{2.1.3}$$

Let S be a pullback square of the form

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X \end{array} \qquad [\textbf{2016.08.24.eq1}] \qquad (2.1.4)$$

For $F : X' \to X$ and $p : Y \to X$ in a category $\mathcal{C}$ let $T(F,p)$ be the set of morphisms $h : X' \to Y$ such that $h \circ p = F$. The proof of the following lemma is omitted because it belongs to general category theory.

Lemma 2.1.1 *[**2016.08.24.l2**] For a pullback square S of the form (2.1.4) the formula $o' \mapsto o' \circ g$ defines a bijection $StM_S : sec(p') \to T(f,p)$.*

Our next result is a corollary of this lemma in the case when p is a universe. For $F : X \to U$ let $S(F)$ be the canonical pullback square based on F. By the previous lemma it defines a bijection

$$StM_{S(F)} : sec(p_F) \to T(F,p)$$

For $H : X \to \widetilde{U}$ let $S_H \in sec(p_{H\circ p})$ be given by the formula

$$S_H = StM^{-1}_{S(H\circ p)}(H)$$

Note that we have

$$S_H = Id_X * H$$

Lemma 2.1.2 *[**2016.08.26.l1**] For a universe p in $\mathcal{C}$ and $X \in \mathcal{C}$, the function*

$$\amalg_{F\in Mor_{\mathcal{C}}(X,U)} sec(p_F) \to Mor(X, \widetilde{U})$$

given by the formula $(F,s) \mapsto s \circ Q(F)$ is a bijection with the inverse given by the formula $H \mapsto (H \circ p, S_H)$.

Proof: Let us denote the first function by Φ and second one by Ψ. We have

$$\Phi(\Psi(H)) = StM^{-1}_{S(H\circ p)}(H) \circ Q(H \circ p) = StM_{S(H\circ p)}(StM^{-1}_{S(H\circ p)}(H)) = H$$

and

$$\Psi(\Phi(F,s)) = \Psi(s \circ Q(F)) = ((s \circ Q(F)) \circ p, StM^{-1}_{S((s\circ Q(F))\circ p)}(s \circ Q(F))) =$$
$$(F, StM^{-1}_{S(F)}(s \circ Q(F))) = (F,s)$$

where the third equation follows from the commutativity of $S(F)$ and the fourth one from the definition of $StM_{S(F)}$. This completes the proof of the lemma.

For an $o \in sec(p)$ and S as above, define $f^*_S(o)$ as the unique element of $sec(p')$ such that

$$[\textbf{2016.08.24.eq2}] f^*_S(o) \circ g = f \circ o \qquad (2.1.5)$$

Such a morphism exists because $Id_{X'} \circ f = (f \circ o) \circ p$.

The proofs of the following two lemmas are omitted because they belongs to general category theory.

Lemma 2.1.3 *[**2016.08.26.l4**] Let $p : Y \to X$ be a morphism in $\mathcal{C}$ and $\Phi : \mathcal{C} \to \mathcal{C}'$ a fully faithful functor. Then for $o \in sec(p)$ one has $\Phi(o) \in sec(\Phi(p))$ and the resulting function*

$$s(\Phi)_p : sec(p) \to sec(\Phi(p))$$

is a bijection.

Lemma 2.1.4 *[**2016.08.24.l1**] Let $\Phi : \mathcal{C} \to \mathcal{C}'$ be a fully faithful functor, S a pullback square of the form (2.1.4) in $\mathcal{C}$ and $o \in sec(p)$.*

Then, $\Phi(S)$ is a pullback square in $\mathcal{C}'$, $\Phi(s) \in sec(\Phi(p))$ and

$$\Phi(f^*_S(o)) = \Phi(f)^*_{\Phi(S)}(\Phi(o))$$

The construction of the C-system $CC(\mathcal{C}, p)$ presented in [9] can be described as follows. One defines first, by induction on n, pairs $(Ob_n, int_n : Ob_n \to \mathcal{C})$ where $Ob_n = Ob_n(\mathcal{C}, p)$ is a set and int_n is a function from Ob_n to objects of $\mathcal{C}$. The definition is as follows:

1. Ob_0 is the standard one point set *unit* whose element we denote by tt. The function int_0 maps tt to the final object pt of the universe category structure on $\mathcal{C}$,

2. $Ob_{n+1} = \amalg_{A \in Ob_n} Hom(int(A), U)$ and $int_{n+1}(A, F) = (int(A); F)$.

We then define $Ob(CC(\mathcal{C}, p))$ as $\amalg_{n \geq 0} Ob_n$ such that elements of $Ob(CC(\mathcal{C}, p))$ are pairs $\Gamma = (n, A)$ where $A \in Ob_n(\mathcal{C}, p)$. We define the function $int : Ob(CC(\mathcal{C}, p)) \to \mathcal{C}$ as the sum of functions int_n.

The morphisms in $CC(\mathcal{C}, p)$ are defined by

$$Mor_{CC(\mathcal{C},p)} = \amalg_{\Gamma, \Gamma' \in Ob(CC)} Hom_{\mathcal{C}}(int(\Gamma), int(\Gamma'))$$

and the function int on morphisms maps a triple $((\Gamma, \Gamma'), a)$ to a. Note that the subset in Mor that consists of f such that $dom(f) = \Gamma$ and $codom(f) = \Gamma'$ is not equal to the set $Hom_{\mathcal{C}}(int(\Gamma), int(\Gamma'))$ but instead to the set of triples of the form $f = ((\Gamma, \Gamma'), a)$ where $a \in Hom_{\mathcal{C}}(int(\Gamma), int(\Gamma'))$.

The length function is defined by $l((n, A)) = n$.

One defines pt as $pt = (0, tt)$. It is the only object of length 0.

If $\Gamma = (n, A)$ where $n > 0$ then, by construction, $A = (B, F)$ where $F : int(B) \to U$. The ft function is defined on such Γ by $ft(\Gamma) = (n - 1, B)$ and on pt by $ft(pt) = pt$.

Lemma 2.1.5 *[**2016.08.22.l1**] For $\Gamma = (n, A)$ and $T = (n', B) \in Ob(CC(\mathcal{C}, p))$ one has $T \in \mathcal{O}b_1(\Gamma)$ if and only if $n' = n + 1$ and $B = (A, F)$.*

Proof: By definition, $T \in \mathcal{O}b_1(\Gamma)$ if and only if $l(T) = l(\Gamma) + 1$ and $ft(T) = \Gamma$.

To prove the "if" part observe that if $T = (n + 1, (A, F))$ then $l(T) = n + 1 = l(\Gamma) + 1$, in particular, $l(T) > 0$ and therefore $ft(T) = (n, A) = \Gamma$.

To prove the "only if" part assume that $T \in \mathcal{O}b_1(\Gamma)$. Then $n' = l(\Gamma) + 1 = n + 1$. Since $n' > 0$, B is a pair of the form (A', F). Finally $ft(T) = (n, A') = (n, A)$ and therefore $A' = A$.

The p-morphism for $\Gamma = (n, A)$ where $n > 0$ and $A = (B, F)$ is given by $((\Gamma, ft(\Gamma)), p_{B,F})$ where $p_{B,F}$ is a part of the universe structure on p.

For $f : (n, A') \to (n, A)$ and T such that $l(T) = l(\Gamma) + 1$ and $ft(T) = \Gamma$ one has, by Lemma 2.1.5, $T = (n + 1, (A, F))$ and one defines

$$[\mathbf{2016.08.22.eq2}] f^*(T) = (n + 1, (A', int(f) \circ F)) \tag{2.1.6}$$

and

$$[\mathbf{2016.08.22.eq3}] q(f, T) = ((f^*(T), T), Q(int(f), F)) \tag{2.1.7}$$

The axioms of a C-system are verified in [9].

Let us denote by

$$int^\circ : PreShv(\mathcal{C}) \to PreShv(CC(\mathcal{C}, p))$$

the functor of pre-composition with int^{op} and by

$$Yo : \mathcal{C} \to PreShv(\mathcal{C})$$

the Yoneda embedding of $\mathcal{C}$.

Problem 2.1.6 *[**2015.04.30.prob1a**] To construct an isomorphism of presheaves*

$$u_1 : \mathcal{O}b_1 \to int^\circ(Yo(U))$$

such that for $\Gamma = (n, A) \in Ob(CC(\mathcal{C}, p))$ and $T = (n + 1, (A, F)) \in \mathcal{O}b_1(\Gamma)$ one has

$$[\mathbf{2015.04.30.eq3a}] u_{1,\Gamma}(T) = F \tag{2.1.8}$$

Construction 2.1.7 [**2016.08.22.constr1**] By definition of int° and Yo we need to construct a family of functions of the form

$$u_{1,\Gamma} : \mathcal{O}b_1(\Gamma) \to Mor_{\mathcal{C}}(int(\Gamma), U)$$

parametrized by $\Gamma \in Ob(CC(\mathcal{C}, p))$ such that such that (2.1.8) holds, for any $f : \Gamma' \to \Gamma$ and any $T \in \mathcal{O}b_1(\Gamma)$ one has

$$[\mathbf{2015.04.30.eq1a}] u_{1,\Gamma'}(f^*(T)) = int(f) \circ u_{1,\Gamma}(T) \tag{2.1.9}$$

and for any Γ the function $u_{1,\Gamma}$ is a bijection.

By Lemma 2.1.5, the conditions (2.1.8) define our family completely and it remains to verify (2.1.9) and the bijectivity condition.

For $\Gamma = (n, A)$, $T = (n+1, (A, F))$, $\Gamma' = (n', A')$ and $f : \Gamma' \to \Gamma$ we have, by (2.1.6),

$$f^*(T) = (n'+1, (A', int(f) \circ F))$$

Therefore,

$$u_{1,\Gamma'}(f^*(T)) = u_{1,\Gamma'}((n'+1, (A', int(f) \circ F))) = int(f) \circ F = int(f) \circ u_{1,\Gamma}(T)$$

which proves (2.1.9).

The bijectivity condition also follows from Lemma 2.1.5 which implies that for $\Gamma = (n, A)$ the function $F \mapsto (n+1, (A, F))$ is a well defined inverse to $u_{1,\Gamma}$.

This completes Construction 2.1.7.

Observe that by (2.1.8) and Lemma 2.1.5, for any $\Gamma \in Ob(CC(\mathcal{C}, p))$ and $T \in \mathcal{O}b_1(\Gamma)$ we have

$$[\mathbf{2015.05.02.eq1a}] int(T) = (int(\Gamma); u_{1,\Gamma}(T)) \tag{2.1.10}$$

and

$$[\mathbf{2016.08.24.eq3}] int(p_T) = p_{u_{1,\Gamma}(T)} \tag{2.1.11}$$

We also have, for $f : \Gamma' \to \Gamma$ and T as above

$$[\mathbf{2016.08.30.eq3}] int(q(f, T)) = Q(int(f), u_{1,\Gamma}(T)) \tag{2.1.12}$$

Lemma 2.1.8 [**2016.08.22.l2**] *For $\Gamma = (n, A) \in Ob(CC(\mathcal{C}, p))$ and $o \in \widetilde{\mathcal{O}b_1}(\Gamma)$ one has*

$$[\mathbf{2016.08.22.eq1}] codom(int(o)) = (int(\Gamma); u_{1,\Gamma}(\partial(o))) \tag{2.1.13}$$

Proof: We have $codom(o) = \partial(o) \in \mathcal{O}b_1(\Gamma)$. Therefore (2.1.13) follows from the equality $codom(int(f)) = int(codom(f))$ and (2.1.10).

Problem 2.1.9 [**2015.04.30.prob1b**] *To construct an isomorphism of presheaves*

$$\widetilde{u}_1 : \widetilde{\mathcal{O}b}_1 \to int^{\circ}(Yo(\widetilde{U}))$$

such that for $o \in \widetilde{\mathcal{O}b}_1(\Gamma)$ one has

$$[\mathbf{2015.04.30.eq4a}] \widetilde{u}_{1,\Gamma}(o) = int(o) \circ Q(u_{1,\Gamma}(\partial(o))) \tag{2.1.14}$$

where the right hand side is defined by (2.1.13) and the equality $dom(Q(F)) = (dom(F); F)$.

Construction 2.1.10 [**2016.08.22.constr2**] By definition of int° and Yo we need to construct a family of functions of the form

$$\widetilde{u}_{1,\Gamma} : \widetilde{\mathcal{O}b}_1(\Gamma) \to Mor_{\mathcal{C}}(int(\Gamma), \widetilde{U})$$

parametrized by $\Gamma \in Ob(CC(\mathcal{C}, p))$ such that (2.1.14) holds, for any $f : \Gamma' \to \Gamma$ and any $o \in \widetilde{\mathcal{O}b}_1(\Gamma)$ one has

$$[\mathbf{2015.04.30.eq1b}]\widetilde{u}_{1,\Gamma'}(f^*(o)) = int(f) \circ \widetilde{u}_{1,\Gamma}(o) \tag{2.1.15}$$

and for any Γ the function $\widetilde{u}_{1,\Gamma}$ is a bijection.

The equalities (2.1.14) define our double family completely and it remains only to prove (2.1.15) and the bijectivity condition.

To prove (2.1.15) let S be the pullback square

$$[\mathbf{2016.08.24.eq2b}] \quad \begin{array}{ccc} f^*(\partial(o)) & \xrightarrow{q(f,\partial(o))} & \partial(o) \\ {\scriptstyle p_{f^*(\partial(o))}}\downarrow & & \downarrow{\scriptstyle p_{\partial(o)}} \\ \Gamma' & \xrightarrow{f} & \Gamma \end{array} \tag{2.1.16}$$

in $CC(\mathcal{C}, p)$. By (2.1.5), (**??**) and the definition of a morphism over Γ' we have

$$[\mathbf{2016.08.24.eq5}]f^*_S(o) = f^*(o) \tag{2.1.17}$$

where on the left is the morphism defined above in the context of all categories and on the right is the morphism defined in Lemma **??** in the context of C-systems.

We now have, where we write u instead of $u_{1,\Gamma}$ and $u_{1,\Gamma'}$ and $\widetilde{u}$ instead of $\widetilde{u}_{1,\Gamma}$ and $\widetilde{u}_{1,\Gamma'}$,

$$\widetilde{u}(f^*(o)) = int(f^*(o)) \circ Q(u(\partial(f^*(o)))) = int(f^*(o)) \circ Q(u(f^*(\partial(o)))) =$$

$$int(f^*(o)) \circ Q(int(f) \circ u(\partial(o))) = int(f^*(o)) \circ Q(int(f), u(\partial(o))) \circ Q(u(\partial(o))) =$$

$$int(f^*(o)) \circ int(q(f, \partial(o))) \circ Q(u(\partial(o))) = int(f^*_S(o)) \circ int(q(f, \partial(o))) \circ Q(u(\partial(o))) =$$

$$int(f)^*_{int(S)}(int(o)) \circ int(q(f, \partial(o))) \circ Q(u(\partial(o))) = int(f) \circ int(o) \circ Q(u(\partial(o))) = int(f) \circ \widetilde{u}(o)$$

where the first equality is by (2.1.14), second is by definition of $f^*(o)$, the third is by (2.1.9), the fourth is by (2.1.1), the fifth is by (2.1.7) since (2.1.10) holds for $T = \partial(o)$, the sixth is by (2.1.12), the seventh is by Lemma 2.1.4, the eighth is by (2.1.5) and the ninth again by (2.1.14). This completes the proof of (2.1.15).

To prove that the function $\widetilde{u}$ is a bijection we will represent it as the composition of functions that we can show to be bijections. The functions are of the form

$$\widetilde{\mathcal{O}b}_1(\Gamma) \to \amalg_{T \in \mathcal{O}b_1(\Gamma)} \partial^{-1}(T) \to \amalg_{F : int(\Gamma) \to U} sec(p_F) \to Mor(int(\Gamma), \widetilde{U})$$

and are given by the formulas

$$o \mapsto (\partial(o), o) \qquad (T, o) \mapsto (u(T), int(o)) \qquad (F, s) \mapsto s \circ Q(F)$$

The first function is the function $X \to \amalg_{y\in Y} f^{-1}(y)$, which is defined and is a bijection for any function of sets $f : X \to Y$. The second one is the total function of the function u and the family of functions $s(int)_{p_T}$ of Lemma 2.1.3, since u and the functions $s(int)_{p_T}$ are bijections so is the total function. The third function is the bijection of Lemma 2.1.2.

Let us show that the composition of these bijections equals $\widetilde{u}$. Indeed, for $o \in \widetilde{\mathcal{O}b}_1(\Gamma)$ we have

$$o \mapsto (\partial(o), o) \mapsto (u(\partial(o)), int(o)) \mapsto int(o) \circ Q(u(\partial(o))) = \widetilde{u}(o)$$

This completes Construction 2.1.10.

Remark 2.1.11 [**2016.08.26.rem1**] The inverse to $\widetilde{u}_{1,\Gamma}$ can be defined by the formula

$$\widetilde{u}_{1,\Gamma}^{-1}(H) = int^{-1}_{\Gamma, u_{1,\Gamma}^{-1}(H\circ p)}(S_H)$$

Note that while we can omit explicitly mentioning $dom(f)$ and $codom(f)$ when we write $int(f)$ we must specify them when we write $int^{-1}(f)$ because int is bijective only on the subsets of morphisms with fixed domain and codomain. This makes the expression for $\widetilde{u}_{1,\Gamma}^{-1}$ longer than one would prefer.

It is easy to see from (2.1.8) and (2.1.14) that the square of morphisms of presheaves

$$[\mathbf{2016.08.20.eq1}] \quad \begin{array}{ccc} \widetilde{\mathcal{O}b}_1 & \xrightarrow{\widetilde{u}_1} & int^{\circ}(Yo(\widetilde{U})) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle int^{\circ}(Yo(p))} \\ \mathcal{O}b_1 & \xrightarrow{u_1} & int^{\circ}(Yo(U)) \end{array} \qquad (2.1.18)$$

commutes.

We will now construct isomorphisms $u_{2,\Gamma}$ and $\widetilde{u}_{2,\Gamma}$ similar to the isomorphisms $u_{1,\Gamma}$ and $\widetilde{u}_{1,\Gamma}$ but having as sources the presheaves $\mathcal{O}b_2$ and $\widetilde{\mathcal{O}b}_2$.

For any $V \in \mathcal{C}$ we define functor data $D_p(-, V)$ given on objects by

$$D_p(X, V) := \amalg_{F:X\to U} Hom((X; F), V)$$

and on morphisms by

$$D_p(f, V) : (F_1, F_2) \mapsto (f \circ F_1, Q(f, F_1) \circ F_2)$$

Lemma 2.1.12 *[***2016.09.07.l1***] The functor data $D_p(-, V)$ specified above is a functor, i.e., one has*

1. *for $X \in \mathcal{C}$ we have $D_p(Id_X, V) = Id_{D_p(X,V)}$,*
2. *for $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$ one has*

$$D_p(f \circ g, V) = D_p(g, V) \circ D_p(f, V)$$

Proof: For the first property we have

$$D_p(Id_X, V)((F_1, F_2)) = (Id_X \circ F_1, Q(Id_X, F_1) \circ F_2) = (F_1, F_2)$$

where the second equality is by (2.1.2).

For the second one we have

$$D_p(f\circ g, V)(F_1, F_2) = (f\circ g\circ F_1, Q(f\circ g, F_1)\circ F_2) = (f\circ(g\circ F_1), Q(f, g\circ F_1)\circ(Q(g, F_1)\circ F_2)) =$$

$$D_p(f, V)(D_p(g, V)(F_1, F_2)) = (D_p(g, V) \circ D_p(f, V))(F_1, F_2)$$

where the second equality is by (2.1.3).

The sets $D_p(X, V)$ are also functorial in V according to the formula

$$D_p(X, r)(F_1, F_2) = (F_1, F_2 \circ r)$$

The fact that $D_p(X, Id_V) = Id_{D_p(X,V)}$ and $D_p(X, r \circ r') = D_p(X, r) \circ D_p(X, r')$ are obvious. It is also immediate from the definitions that for and for $f : X \to X'$, $g : V \to V'$ we have

$$D_p(f, V) \circ D_p(X, g) = D_p(X', g) \circ D_p(f, V')$$

The latter equality shows that $D_p(-, r)$ are morphisms of presheaves and the former that our construction defines a functor

$$D_p : \mathcal{C} \to PreShv(\mathcal{C})$$

Let Γ be as above and $T \in \mathcal{O}b_2(\Gamma)$. Then, by (2.1.10), we have $int(ft(T)) = (int(\Gamma); u_{1,\Gamma}(ft(T)))$ and therefore the pair

$$u_{2,\Gamma}(T) = (u_{1,\Gamma}(ft(T)), u_{1,ft(T)}(T))$$

is an element of $D_p(int(\Gamma), U)$. This defines a family of functions

$$u_{2,\Gamma} : \mathcal{O}b_2(\Gamma) \to D_p(int(\Gamma), U)$$

parametrized by $\Gamma \in Ob(CC(\mathcal{C}, p))$.

Lemma 2.1.13 *[***2015.05.02.prob2a***] [***2015.05.02.constr2a***] The family $u_{2,\Gamma}$ is an isomorphism of presheaves*

$$u_2 : \mathcal{O}b_2 \to int^{\circ}(D_p(-, U))$$

Proof: We can write $u_{2,\Gamma}$ as a composition of the bijection

$$Ob_2(\Gamma) \to \amalg_{\Gamma' \in Ob_1(\Gamma)} Ob_1(\Gamma')$$

that sends T to $(ft(T), T)$ with the function

$$\amalg_{\Gamma' \in Ob_1(\Gamma)} Ob_1(\Gamma') \to \amalg_{F \in Hom(int(\Gamma), U)} Hom((int(\Gamma); F), U)$$

that is the total function of the function $u_{1,\Gamma}$ and the family of functions $u_{1,\Gamma'}$ given for all $\Gamma' \in Ob_1(\Gamma)$. Since $u_{1,\Gamma}$ is a bijection and for each Γ', $u_{1,\Gamma'}$ is a bijection, the total function is a bijection. This proves that $u_{2,\Gamma}$ are bijections.

It remains to prove that $u_{2,\Gamma}$ form a morphism of presheaves, that is, that for any $f : \Gamma' \to \Gamma$ and $T \in \mathcal{O}b_2(\Gamma)$ we have

$$u_{2,\Gamma'}(f^*(T)) = (int(f) \circ u_{1,\Gamma}(ft(T)), Q(int(f), u_{1,\Gamma}(ft(T))) \circ u_{1,ft(T)}(T))$$

We have

$$u_2(f^*(T)) = (u_{1,\Gamma'}(ft(f^*(T))), u_{1,ft(f^*(T))}(f^*(T)))$$

Since $ft(f^*(T)) = f^*(ft(T))$ we have

$$u_{1,\Gamma'}(ft(f^*(T))) = int(f) \circ u_{1,\Gamma}(ft(T))$$

by (2.1.15).

Next we have $f^*(T) = q(f, ft(T))^*(T)$ where $q(f, ft(T)) : f^*(ft(T)) \to ft(T)$ by (**??**) and therefore

$$u_{1,ft(f^*(T))}(f^*(T)) = u_{1,f^*(ft(T))}(q(f, ft(T))^*(T)) = int(q(f, ft(T)) \circ u_{1,ft(T)}(T) =$$

$$Q(int(f), u_{1,\Gamma}(ft(T))) \circ u_{1,ft(T)}(T)$$

where the first equality is by (**??**), the second by (2.1.15) and the third by (2.1.7). This completes the proof of Lemma 2.1.13.

Let Γ be as above and $o \in \widetilde{\mathcal{O}b}_2(\Gamma)$. Then, by (2.1.10), we have

$$int(ft(\partial(o))) = (int(\Gamma); u_{1,\Gamma}(ft(\partial(o))))$$

and therefore the pair

$$\widetilde{u}_{2,\Gamma}(o) = (u_{1,\Gamma}(ft(\partial(o))), \widetilde{u}_{1,ft(\partial(o))}(o))$$

is an element of $D_p(int(\Gamma), \widetilde{U})$. This defines a family of functions

$$\widetilde{u}_{2,\Gamma} : \widetilde{\mathcal{O}b}_2(\Gamma) \to D_p(int(\Gamma), \widetilde{U})$$

parametrized by $\Gamma \in Ob(CC(\mathcal{C}, p))$.

Lemma 2.1.14 *[***2015.05.02.prob2b***] [***2015.05.02.constr2b***] The family $\widetilde{u}_{2,\Gamma}$ is an isomorphism of presheaves*

$$\widetilde{u}_2 : \widetilde{\mathcal{O}b}_2 \to int^{\circ}(D_p(-, \widetilde{U}))$$

Proof: We can write $\widetilde{u}_{2,\Gamma}$ as a composition of the bijection

$$\widetilde{\mathcal{O}b}_2(\Gamma) \to \amalg_{\Gamma' \in \mathcal{O}b_1(\Gamma)} \widetilde{\mathcal{O}b}_1(\Gamma')$$

that sends o to $(ft(\partial(o)), o)$ with the function

$$\amalg_{\Gamma' \in Ob_1(\Gamma)} \widetilde{\mathcal{O}b}_1(\Gamma') \to \amalg_{F \in Hom(int(\Gamma), U)} Hom((int(\Gamma); F), \widetilde{U})$$

that is the total function of the function $u_{1,\Gamma}$ and the family of functions $\widetilde{u}_{1,\Gamma'}$ given for all $\Gamma' \in \mathcal{O}b_1(\Gamma)$. Since $u_{1,\Gamma}$ is a bijection and for each Γ', $\widetilde{u}_{1,\Gamma'}$ is a bijection, the total function is a bijection.

It remains to prove that $u_{2,\Gamma}$ form a morphism of presheaves, that is, that for any $f : \Gamma' \to \Gamma$ and $T \in \mathcal{O}b_2(\Gamma)$ we have

$$u_{2,\Gamma'}(f^*(T)) = (int(f) \circ u_{1,\Gamma}(ft(T)), Q(int(f), u_{1,\Gamma}(ft(T))) \circ u_{1,ft(T)}(T))$$

We have

$$u_2(f^*(T)) = (u_{1,\Gamma'}(ft(f^*(T))), u_{1,ft(f^*(T))}(f^*(T)))$$

Since $ft(f^*(T)) = f^*(ft(T))$ we have

$$u_{1,\Gamma'}(ft(f^*(T))) = int(f) \circ u_{1,\Gamma}(ft(T))$$

by (2.1.15).

Next we have $f^*(T) = q(f, ft(T))^*(T)$ where $q(f, ft(T)) : f^*(ft(T)) \to ft(T)$ by (**??**) and therefore

$$u_{1,ft(f^*(T))}(f^*(T)) = u_{1,f^*(ft(T))}(q(f, ft(T))^*(T)) = int(q(f, ft(T)) \circ u_{1,ft(T)}(T) =$$

$$Q(int(f), u_{1,\Gamma}(ft(T))) \circ u_{1,ft(T)}(T)$$

where the first equality is by (**??**), the second by (2.1.15) and the third by (2.1.7). This completes the proof of Lemma 2.1.13.

Lemma 2.1.15 *[***2016.08.26.l3***] The square of morphisms presheaves*

$$\begin{array}{ccc} \widetilde{\mathcal{O}b}_2 & \xrightarrow{\widetilde{u}_2} & int^{\circ}(D_p(-, \widetilde{U})) \\ \partial \downarrow & & \downarrow int^{\circ}(D_p(-,p)) \\ \mathcal{O}b_2 & \xrightarrow{u_2} & int^{\circ}(D_p(-, U)) \end{array}$$

commutes.

Proof: For $\Gamma \in CC$ and $o \in \widetilde{\mathcal{O}b_2}(\Gamma)$ we have

$$(int^{\circ}(D_p(-, p)))(\widetilde{u}_2(o)) = (int^{\circ}(D_p(-, p)))(u_{1,\Gamma}(ft(\partial(o))), \widetilde{u}_{1,ft(\partial(o))}(o)) =$$

$$(u_{1,\Gamma}(ft(\partial(o))), \widetilde{u}_{1,ft(\partial(o))}(o) \circ p) = (u_{1,\Gamma}(ft(\partial(o))), u_{1,ft(\partial(o))}(\partial(o)))$$

where the third equality is by the commutativity of (2.1.18) and

$$u_{2,\Gamma}(\partial(o)) = (u_{1,\Gamma}(ft(\partial(o))), u_{1,ft(\partial(o))}(\partial(o)))$$

This completes the proof of the lemma.

Remark 2.1.16 [**2015.07.29.rem2**] Isomorphisms $u_i, \widetilde{u}_i$ for $i = 1, 2$ generalize easily to all $i > 0$ if one defines, inductively,

$$Yo_{n+1}(V)(X) = \amalg_{F:X\to U} Yo_n(V)((X;F))$$

Moreover, if we define $Hom_n(X,Y)$ as $Yo_n(Y)(X)$ then there are composition functions

$$Hom_n(X,Y) \times Hom_m(Y,Z) \to Hom_{n+m}(X,Z)$$

that are likely to satisfy the unity and associativity axioms such that one obtains, from any universe category $(\mathcal{C}, p)$, a new category $(\mathcal{C}, p)_*$ with the same collection of objects and morphisms between two objects given by

$$Hom_{(\mathcal{C},p)_*}(X,Y) = \amalg_{n\geq 1} Hom_m(X,Y)$$

In this paper we will not need Yo_n for $n > 2$ and we defer the study of this structure until the future papers.

When $\mathcal{C}$ is a locally cartesian closed category (see appendix), the functors $D_p(-,V)$ become representable providing us with a way to describe operations such as Π and λ on $CC(\mathcal{C}, p)$ in terms of morphisms between objects in $\mathcal{C}$.

For a morphism $p : \widetilde{U} \to U$ in a locally cartesian closed category and an object V of this category let

$$I_p(V) := \underline{Hom}_U((\widetilde{U}, p), (U \times V, pr_1))$$

and let

$$prI_p(V) = p\triangle pr_1 : I_p(V) \to U$$

be the morphism that defines $I_p(V)$ as an object over U.

Remark 2.1.17 [**2016.04.23.rem1**] In [4] generalized polynomial functors are defined as functors isomorphic to functors of the form I_p.

Note that I_p depends on the choice of a locally cartesian closed structure on $\mathcal{C}$. On the other hand, the construction of the functors $D_p(X,V)$ requires a universe structure on p but does not require a locally cartesian closed structure on $\mathcal{C}$.

The computations below are required in order to establish the connections between the constructions that use the locally cartesian closed structure and the constructions that use universe structures. In particular, we have to deal with the fact that for $F : X \to U$ the fiber product $(X, F) \times_U (\widetilde{U}, p)$ that we have from the structure of a category with fiber products on $\mathcal{C}$ need not be equal to $(X; F)$ that we have from the universe structure on p.

Let $p : \widetilde{U} \to U$ be a universe and V an object of $\mathcal{C}$. We assume that $\mathcal{C}$ is equipped with a locally cartesian closed structure. For $F : X \to U$ there is a unique morphism

$$\iota_F : (X; F) \to (X, F) \times_U (\widetilde{U}, p)$$

such that $\iota_F \circ pr_1 = p_{X,F}$ and $\iota_F \circ pr_2 = Q(F)$ which is a particular case of the morphisms ι, ι' of Lemma 3.4.1.

The evaluation morphism in the case of $I_p(V)$ is of the form

$$evI_p : (I_p(V), prI_p(V)) \times_U (U \times V, pr_1) \to U \times V$$

Define a morphism

$$st_p(V) : (I_p(V); prI_p(V)) \to V$$

as the composition:

$$st_p(V) := \iota_{prI_p(V)} \circ evI_p(V) \circ pr_2$$

We will need to use some properties of these morphisms.

Lemma 2.1.18 *[**2015.04.14.l2a**] Let $f : V \to V'$ be a morphism, then one has*

$$Q(I_p(f), prI_p(V')) \circ st_p(V') = st_p(V) \circ f$$

Proof: Let $pr = prI_p(V)$, $pr' = prI_p(V')$, $\iota = \iota_{pr}$, $\iota' = \iota_{pr'}$, $ev = evI_p(V)$ and $ev' = evI_p(V')$. Then we have to verify that the outer square of the following diagram commutes:

$$\begin{array}{ccccccc}
(I_p(V); pr) & \xrightarrow{\iota} & (I_p(V), pr) \times_U (\widetilde{U}, p) & \xrightarrow{ev} & U \times V & \xrightarrow{pr_2} & V \\
{\scriptstyle Q(I_p(f), pr')}\downarrow & & {\scriptstyle I_p(f) \times Id_{\widetilde{U}}}\downarrow & & {\scriptstyle Id_U \times f}\downarrow & & \downarrow{\scriptstyle f} \\
(I_p(V'); pr') & \xrightarrow{\iota'} & (I_p(V'), pr') \times_U (\widetilde{U}, p) & \xrightarrow{ev'} & U \times V' & \xrightarrow{pr_2} & V'
\end{array}$$

The commutativity of the left square is a particular case of Lemma 3.4.1. The commutativity of the right square is an immediate corollary of the definition of $Id_U \times f$. The commutativity of the middle square is a particular case of the axiom of locally cartesian closed structure that says that morphisms ev_Y^X are natural in Y.

Problem 2.1.19 *[**2015.03.29.prob1**] Let $(\mathcal{C}, p, pt)$ be a locally cartesian closed universe category. To construct, for all $V \in \mathcal{C}$, isomorphisms of presheaves*

$$\eta_V : D_p(-, V) \to Yo(I_p(V))$$

that are natural in V, i.e., such that for all $r : V \to V'$, $X \in \mathcal{C}$ and $d \in D_p(X, V)$ one has

$$[\mathbf{2016.09.11.eq1}]\eta_{V,X}(d) \circ I_p(r) = \eta_{V',X}(D_p(X, r)(d)) \qquad (2.1.19)$$

Construction 2.1.20 [**2015.03.29.constr1**] We need to construct bijections

$$\eta_{V,X} : D_p(X, V) \to Hom(X, I_p(V))$$

such that (2.1.19) holds and for any $f : X' \to X$ and $d \in D_p(X, V)$ one has

$$[\mathbf{2016.09.11.eq2}]f \circ \eta_{V,X}(d) = \eta_{V,X'}(D_p(f, V)(d)) \qquad (2.1.20)$$

We will construct bijections

$$\eta^!_{V,X} : Hom(X, I_p(V)) \to D_p(X, V)$$

such that for all $g : X \to I_p(V)$ one has:

1. for all $r : V \to V'$ one has

$$[\mathbf{2016.09.11.eq3}]D_p(X,r)(\eta^!(g)) = \eta^!(g \circ I_p(r)) \tag{2.1.21}$$

2. for all $f : X' \to X$ one has

$$[\mathbf{2016.09.11.eq4}]D_p(f,V)(\eta^!(g)) = \eta^!(f \circ g) \tag{2.1.22}$$

and then define $\eta_{V,X}$ as the inverse to $\eta^!_{V,X}$. One proves easily that (2.1.19) implies (2.1.21) and (2.1.20) implies (2.1.22).

For $g : X \to I_p(V)$ we set

$$\eta^!_{V,X}(g) := (g \circ prI_p(V), Q(g, prI_p(V)) \circ st_p(V))$$

To see that this is a bijection observe first that it equals to the composition

$$Hom(X, I_p(V)) \to \amalg_{F:X\to U} Hom_U((X,F),(I_p(V), prI_p(V))) \to \amalg_{F:X\to U} Hom((X;F),V)$$

where the first function is of the form $g \mapsto (g \circ prI_p(V), g)$ and the second is the sum over all $F : X \to U$ of functions $g \mapsto Q(g, prI_p(V)) \circ st_p(V)$. The first function is a function of the form $A \to \amalg_{b\in B} h^{-1}(b)$, which is defined and is a bijection for any function of sets $h : A \to B$. It remains to show that the second one is a bijection for every F.

By definition of the $\underline{Hom}$ structure we know that for each F the function

$$Hom_U((X,F),(I_p(V), prI_p(V))) \to Hom_U(((X,F) \times_U (\widetilde{U}, p), -), (U \times V, pr_1))$$

given by $g \mapsto (g \times Id_{\widetilde{U}}) \circ evI_p(V)$ is a bijection. We also know that the function

$$Hom_U(((X,F) \times_U (\widetilde{U}, p), F \diamond p), (U \times V, pr_1)) \to Hom((X,F) \times_U (\widetilde{U}, p), V)$$

is a bijection. Since ι_F is an isomorphism the composition with it is a bijection. Now we have two functions

$$Hom_U((X,F),(I_p(V), prI_p(V))) \to Hom((X;F),V)$$

given by $g \mapsto \iota_F \circ (g \times Id_{\widetilde{U}}) \circ evI_p(V) \circ p_V$ and $g \mapsto Q(g, prI_p(V)) \circ st_p(V)$ of which the first one is the bijection. It remains to show that these functions are equal. For this it is sufficient to show that

$$Q(g, prI_p(V)) \circ \iota_{prI_p(V)} = \iota_F \circ (g \times Id_{\widetilde{U}})$$

which follows easily from computing compositions with the projections pr_1 to $I_p(V)$ and pr_2 to $\widetilde{U}$.

We now have to check the behavior of $\eta^!$ with respect to morphisms in X and V.

Let $pr = prI_p(V)$ and $pr' = prI_p(V')$. For $f : V' \to V$ and $f : X \to I_p(V)$ we have

$$D_p(X,f)(\eta^!(g)) = D_p(X,f)(g \circ pr, Q(g,pr) \circ st_p(V)) = (g \circ pr, Q(g,pr) \circ st_p(V) \circ f)$$

and

$$\eta^!(g\circ I_p(f)) = (g\circ I_p(f)\circ pr', Q(g\circ I_p(f), pr')\circ st_p(V'))$$

We have $pr = I_p(f)\circ pr'$ because $I_p(f)$ is a morphism over U. It remains to check that

$$Q(g, pr)\circ st_p(V)\circ f = Q(g\circ I_p(f), pr')\circ st_p(V')$$

By [9, Lemma 2.5] we have

$$Q(g\circ I_p(f), pr') = Q(g, pr)\circ Q(I_p(f), pr')$$

and the remaining equality

$$Q(g, pr)\circ st_p(V)\circ f = Q(g, pr)\circ Q(I_p(f), pr')\circ st_p(V')$$

follows from Lemma 2.1.18.

Consider now $f : X'\to X$. Then

$$D_p(f, V)(\eta^!(g)) = D_p(f, V)(g\circ pr, Q(g, pr)\circ st_p(V)) = (f\circ g\circ pr, Q(f, g\circ pr)\circ Q(g, pr)\circ st_p(V))$$

$$\eta^!(f\circ g) = (f\circ g\circ pr, Q(f\circ g, pr)\circ st_p(V))$$

and the required equality follows from [9, Lemma 2.5].

Problem 2.1.21 *[***2015.03.17.prob3***] For a locally cartesian closed closed $\mathcal{C}$ and a universe $p : \widetilde{U}\to U$ in $\mathcal{C}$ to construct isomorphisms of presheaves*

$$\mu_2 : \mathcal{O}b_2 \to int^{\circ}(Yo(I_p(U)))$$

and

$$\widetilde{\mu}_2 : \widetilde{\mathcal{O}b}_2 \to int^{\circ}(Yo(I_p(\widetilde{U})))$$

such that the square

$$\begin{array}{ccc} \widetilde{\mathcal{O}b}_2 & \xrightarrow{\widetilde{\mu}_2} & int^{\circ}(Yo(I_p(\widetilde{U}))) \\ \partial\downarrow & & \downarrow int^{\circ}(Yo(I_p(p))) \\ \mathcal{O}b_2 & \xrightarrow{\mu_2} & int^{\circ}(Yo(I_p(U))) \end{array}$$

commutes.

Construction 2.1.22 *[***2015.03.17.constr2***]*Compose isomorphism u_2 (resp. $\widetilde{u}_2$) with the isomorphism $int^{\circ}(\eta_U)$ (resp. $int^{\circ}(\eta_{\widetilde{U}})$). The explicit formulas for μ_2 and $\widetilde{\mu}_2$ are

$$\mu_2(T) = \eta(u_2(T))$$

$$\widetilde{\mu}_2(o) = \eta(\widetilde{u}(o))$$

Remark 2.1.23 [**2015.03.29.rem2**] The previous constructions related to Ob_2 and $\widetilde{\mathcal{O}b}_2$ can be generalized to Ob_n and $\widetilde{\mathcal{O}b}_n$ for all $n > 0$. For example, there are isomorphisms

$$\mu_{n+1} : Ob_{n+1} \to int^{\circ}(I_p^n(U))$$

$$\widetilde{\mu}_{n+1} : \widetilde{\mathcal{O}b}_{n+1} \to int^{\circ}(I_p^n(\widetilde{U}))$$

where I_p^n is the n-th iteration of the functor I_p and $\mu_1 = u_1$ and $\widetilde{\mu}_1 = \widetilde{u}_1$. The functors $Yo_n(V)$ of Remark 2.1.16 in the case of a locally cartesian closed universe category $(\mathcal{C}, p)$ are representable by objects $I_p^n(V)$.

2.2 (Π, λ)-structures on the C-systems $CC(\mathcal{C}, p)$

We will show now how to construct (Π, λ)-structures on C-systems of the form $CC(\mathcal{C}, p)$ for a locally cartesian closed universe category $(\mathcal{C}, p)$.

Definition 2.2.1 *[**2015.03.29.def1**] Let $\mathcal{C}$ be a locally cartesian closed universe category, pt be the final object of $\mathcal{C}$ and $p : \widetilde{U} \to U$ the universe. A P-structure on p is a pair of morphisms*

$$\widetilde{P} : I_p(\widetilde{U}) \to \widetilde{U}$$
$$P : I_p(U) \to U$$

such that the square

$$[\mathbf{2009.prod.square}] \quad \begin{array}{ccc} I_p(\widetilde{U}) & \xrightarrow{\widetilde{P}} & \widetilde{U} \\ \downarrow{\scriptstyle I_p(p)} & & \downarrow{\scriptstyle p} \\ I_p(U) & \xrightarrow{P} & U \end{array} \tag{2.2.1}$$

is pullback square.

Problem 2.2.2 *[**2015.03.17.prob0**] Let $\mathcal{C}$ be a locally cartesian closed category, pt be a final object in $\mathcal{C}$ and $p : \widetilde{U} \to U$ a universe. Let $(\widetilde{P}, P)$ be a P-structure on p. To construct a (Π, λ)-structure on $CC(\mathcal{C}, p)$.*

Construction 2.2.3 [**2015.03.17.constr3**]The diagram

$$\begin{array}{ccccccc} \widetilde{\mathcal{O}b}_2 & \xrightarrow{\widetilde{\mu}_2} & int^\circ(Yo(I_p(\widetilde{U}))) & \xrightarrow{int^\circ(Yo(\widetilde{P}))} & int^\circ(Yo(\widetilde{U})) & \xrightarrow{\mu_1^{-1}} & \widetilde{\mathcal{O}b}_1 \\ \partial\downarrow & & \downarrow{\scriptstyle int^\circ(Yo(I_p(p)))} & & \downarrow{\scriptstyle int^\circ(Yo(p))} & & \downarrow\partial \\ \mathcal{O}b_2 & \xrightarrow{\mu_2} & int^\circ(Yo(I_p(U))) & \xrightarrow{int^\circ(Yo(P))} & int^\circ(Yo(U)) & \xrightarrow{\mu_1^{-1}} & \mathcal{O}b_1 \end{array}$$

shows that for a P-structure $(\widetilde{P}, P)$ the pair of morphisms

$$\Pi = \widetilde{\mu}_2 \circ int^\circ(Yo(\widetilde{P})) \circ \widetilde{\mu}_1^{-1}$$
$$\lambda = \mu_2^{-1} \circ int^\circ(Yo(P)) \circ \mu_1^{-1}$$

is a (Π, λ)-structure on $CC(\mathcal{C}, p)$.

There is an important class of cases when the function from P-structures on p to (Π, λ)-structures on $CC(\mathcal{C}, p)$ is a bijection.

Lemma 2.2.4 *[**2016.09.09.l1**] Let $(\mathcal{C}, p)$ be a universe category such that the functor*

$$Yo \circ int^\circ : \mathcal{C} \to PreShv(CC(\mathcal{C}, p))$$

is fully faithful. Then the function from the P-structures on p to the (Π, λ)-structures on $CC(\mathcal{C}, p)$ defined by Construction 2.2.3 is a bijection.

Proof: Let $\widetilde{\phi}$ be the inverse to $(Yo \circ int^{\circ})_{I_p(\widetilde{U}),\widetilde{U}}$ and ϕ be the inverse to $(Yo \circ int^{\circ})_{I_p(U),U}$. Given a (Π, λ)-structure (Π, λ) let

$$[\mathbf{2016.09.09.eq1}] \begin{aligned} \widetilde{P} &= \widetilde{\phi}(\widetilde{\mu}_2^{-1} \circ \Pi \circ \widetilde{\mu}_1) \\ P &= \phi(\mu_2^{-1} \circ \lambda \circ \mu_1) \end{aligned} \tag{2.2.2}$$

Then $\widetilde{P} : I_p(\widetilde{U}) \to \widetilde{U}$ and $P : I_p(U) \to U$. Let S be the square that $\widetilde{P}$ and P form with $I_p(p)$ and p. One verifies that $(Yo \circ int^{\circ})(S)$ is isomorphic to the square formed by Π and λ and as a square isomorphic to a pullback square is a pullback square.

The functor $Yo \circ int^{\circ}$ is assumed to be fully faithful and if the image of a square under a fully faithful functor is a pullback then the square itself is a pullback. We conclude that formulas (2.2.2) define a function from (Π, λ)-structures to P-structures.

It remains to verify that this function is inverse on both sides to the function of Construction 2.2.3 which is straightforward from its definition. The lemma is proved.

3 Functoriality of P-structures and (Π, λ)-structures

3.1 Universe category functors and the D_p and I_p constructions

Let $(\mathcal{C}, p, pt)$ and $(\mathcal{C}', p', pt')$ be two universe categories. Recall from [9] that a functor of universe categories from $(\mathcal{C}, p, pt)$ to $(\mathcal{C}', p', pt')$ is a triple $\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi})$ where Φ is a functor $\mathcal{C} \to \mathcal{C}'$ and $\phi : \Phi(U) \to U'$, $\widetilde{\phi} : \Phi(\widetilde{U}) \to \widetilde{U}'$ are two morphisms such that Φ takes the final object to a final object, pullback squares based on p to pullback squares and such that the square

$$[\mathbf{2015.03.21.sq1}] \quad \begin{array}{ccc} \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(U) & \xrightarrow{\phi} & U' \end{array} \tag{3.1.1}$$

is a pullback square.

For X, V in $\mathcal{C}$ we have the functoriality function

$$\Phi : Hom(X, V) \to Hom(\Phi(X), \Phi(V))$$

Problem 3.1.1 *[**2015.04.12.prob1**] For a universe category functor $\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi})$, to define, for all $X, V \in \mathcal{C}$, functions*

$$\mathbf{\Phi}^2_{V,X} : D_p(X, V) \to D_{p'}(\Phi(X), \Phi(V))$$

Construction 3.1.2 [**2015.04.12.constr1**] Let $(F_1 : X \to U, F_2 : (X; F_1) \to V)$ be an element in $D_p(X, V)$. Consider $(\Phi(X); \Phi(F_1) \circ \phi)$. Since the square (3.1.1) is a pullback square there is a unique morphism q such that $q \circ \widetilde{\phi} = Q(\Phi(F_1) \circ \phi)$ and $q \circ \Phi(p) = p_{\Phi(X),\Phi(F_1)\circ\phi} \circ \Phi(F_1)$ and then the left hand side square in the diagram

$$\begin{array}{ccccc} (\Phi(X); \Phi(F_1) \circ \phi) & \xrightarrow{q} & \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ \downarrow{\scriptstyle p_{\Phi(X),\Phi(F_1)\circ\phi}} & & {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(X) & \xrightarrow{\Phi(F_1)} & \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

is a pullback square. Together with the fact that Φ takes pullback squares based on p to pullback squares we obtain a unique morphism, which is an isomorphism,

$$\iota : (\Phi(X); \Phi(F_1) \circ \phi) \to \Phi(X; F_1)$$

such that

$$[\mathbf{2015.04.08.eq1}]\iota \circ \Phi(p_{X,F_1}) = p_{\Phi(X),\Phi(F_1)\circ\phi} \tag{3.1.2}$$

$$[\mathbf{2015.04.08.eq2}]\iota \circ \Phi(Q(F_1)) \circ \widetilde{\phi} = Q(\Phi(F_1) \circ \phi) \tag{3.1.3}$$

and we define:

$$\mathbf{\Phi}^2_{V,X}(F_1, F_2) := (\Phi(F_1) \circ \phi, \iota \circ \Phi(F_2))$$

The following lemma proves that the family of functions $\mathbf{\Phi}^2_{V,X}$ parametrized by $X \in \mathcal{C}$ is a morphism of presheaves of the form

$$\mathbf{\Phi}^2_V : D_p(-, V) \to \Phi^\circ(D_{p'}(-, \Phi(V)))$$

Lemma 3.1.3 *[***2015.03.23.l1***] Let Φ be as above, $f : X' \to X$ be a morphism and V be an object of $\mathcal{C}$. Then the square*

$$\begin{array}{ccc} D_p(X, V) & \xrightarrow{D_p(f,V)} & D_p(X', V) \\ {\scriptstyle \mathbf{\Phi}^2_{V,X}}\downarrow & & {\scriptstyle \mathbf{\Phi}^2_{V,X'}}\downarrow \\ D_{p'}(\Phi(X), \Phi(V)) & \xrightarrow{D_{p'}(\Phi(f),\Phi(V))} & D_{p'}(\Phi(X'), \Phi(V)) \end{array}$$

commutes.

Proof: We will omit the indexes at $\mathbf{\Phi}^2$. We have to show that for any $d \in D_p(X, V)$ one has

$$D_{p'}(\Phi(f), \Phi(V))(\mathbf{\Phi}^2(d)) = \mathbf{\Phi}^2(D_p(f, V)(d))$$

Let $d = (F_1, F_2)$. Then

$$D_{p'}(\Phi(f), \Phi(V))(\mathbf{\Phi}^2(d)) = D_{p'}(\Phi(f), \Phi(V))(\Phi(F_1) \circ \phi, \iota \circ \Phi(F_2)) =$$

$$(\Phi(f)\circ\Phi(F_1)\circ\phi, q'\circ\iota\circ\Phi(F_2))$$

and

$$\mathbf{\Phi}^2(D_p(f,V)(F_1,F_2)) = \mathbf{\Phi}^2(f\circ F_1, q\circ F_2) =$$
$$(\Phi(f\circ F_1)\circ\phi, \iota'\circ\Phi(q\circ F_2))$$

where

$$\iota : (\Phi(X);\Phi(F_1)\circ\phi)\rightarrow\Phi(X;F_1) \qquad \iota' : (\Phi(X');\Phi(f\circ F_1)\circ\phi)\rightarrow\Phi(X';f\circ F_1)$$

$$q : (X';f\circ F_1)\rightarrow(X;F_1) \qquad q' : (\Phi(X');\Phi(f)\circ\Phi(F_1)\circ\phi)\rightarrow(\Phi(X);\Phi(F_1)\circ\phi)$$

are the morphisms defined in Construction 3.1.2. We have

$$\Phi(f)\circ\Phi(F_1)\circ\phi = \Phi(f\circ F_1)\circ\phi$$

and it remains to check that

$$q'\circ\iota\circ\Phi(F_2) = \iota'\circ\Phi(q\circ F_2)$$

or that $q'\circ\iota = \iota'\circ\Phi(q)$. The codomain of both morphisms is $\Phi(X;F_1)$ that by our assumption on Φ is a pullback of p' and $\Phi(F_1)\circ\phi$. Therefore it is sufficient to verify that the compositions of these two morphisms with the projections to $\widetilde{U}'$ and $\Phi(X)$ coincide.

This is done by a direct computation from definitions.

If we consider D_p as a functor $\mathcal{C}\rightarrow PreShv(\mathcal{C})$ then the following lemma shows that the family of functor morphisms $\mathbf{\Phi}^2_V$ parametrized by $V\in\mathcal{C}$ form a functor morphism of the form

$$\mathbf{\Phi}^2 : D_p\rightarrow\Phi\circ D_{p'}\circ\Phi^{\circ}$$

Lemma 3.1.4 *[***2015.04.10.l3***] Let* $\mathbf{\Phi}$ *be as above,* X *an object of* $\mathcal{C}$ *and* $f : V\rightarrow V'$ *a morphism. Then the square*

$$\begin{array}{ccc} D_p(X,V) & \xrightarrow{D_p(X,f)} & D_p(X,V') \\ \downarrow{\scriptstyle \mathbf{\Phi}^2_{V,X}} & & \downarrow{\scriptstyle \mathbf{\Phi}^2_{V',X}} \\ D_{p'}(\Phi(X),\Phi(V)) & \xrightarrow{D_p(\Phi(X),\Phi(f))} & D_{p'}(\Phi(X),\Phi(V')) \end{array}$$

commutes.

Proof: We will omit the indexes at $\mathbf{\Phi}^2$. Let $d=(F_1,F_2)\in D_p(X,V)$. We have to show that

$$\mathbf{\Phi}^2(D_p(X,f)(F_1,F_2)) = D_p(\Phi(X),\Phi(f))(\mathbf{\Phi}^2(F_1,F_2))$$

We have:

$$\mathbf{\Phi}^2(D_p(X,f)(F_1,F_2)) = \mathbf{\Phi}^2((F_1,F_2\circ f)) = (\Phi(F_1)\circ\phi, \iota\circ\Phi(F_2\circ f)) =$$

$$(\Phi(F_1)\circ\phi, \iota\circ\Phi(F_2)\circ\Phi(f)) = D_p(\Phi(X),\Phi(f))(\mathbf{\Phi}^2(F_1,F_2))$$

Note that in the problem below no assumption is made about the compatibility of Φ with the locally cartesian closed structures on $\mathcal{C}$ and $\mathcal{C}'$.

Problem 3.1.5 *[**2015.03.21.prob1**] Assume that $\mathcal{C}$ and $\mathcal{C}'$ are locally cartesian closed universe categories. For $\mathbf{\Phi}$ as above and $V\in\mathcal{C}$ to construct a morphism*

$$\chi_{\mathbf{\Phi}}(V):\Phi(I_p(V))\to I_{p'}(\Phi(V))$$

Construction 3.1.6 [**2015.03.21.constr1**] Consider the sequence of functions

$$\begin{array}{ccc} D_p(I_p(V),V) & \xrightarrow{\mathbf{\Phi}^2_{I_p(V),V}} & D_{p'}(\Phi(I_p(V)),\Phi(V)) \\ \eta^!_{V,I_p(V)}\uparrow & & \downarrow\eta_{\Phi(I_p(V)),\Phi(V)} \\ Hom(I_p(V),I_p(V)) & & Hom(\Phi(I_p(V)),I_{p'}(\Phi(V))) \end{array}$$

where η and its inverse $\eta^!$ are the bijections of Construction 2.1.20. Applying it to $Id_{I_p(V)}$ we obtain

$$\chi_{\mathbf{\Phi}}(V)=\eta_{\Phi(I_p(V)),\Phi(V)}(\mathbf{\Phi}^2_{I_p(V),V}(\eta^!_{V,I_p(V)}(Id_{I_p(V)})))$$

Let us show that $\chi_{\mathbf{\Phi}}$ are natural in V.

Lemma 3.1.7 *[**2015.04.10.l4**] For $\mathbf{\Phi}$ as above let $f:V_1\to V_2$ be a morphism. Then the square*

$$\begin{array}{ccc} \Phi(I_p(V_1)) & \xrightarrow{\chi_{\Phi}(V_1)} & I_{p'}(\Phi(V_1)) \\ \Phi(I_p(f))\downarrow & & \downarrow I_{p'}(\Phi(f)) \\ \Phi(I_p(V_2)) & \xrightarrow{\chi_{\Phi}(V_2)} & I_{p'}(\Phi(V_2)) \end{array}$$

commutes.

Proof: Let $X_i=I_p(V_i)$ for $i=1,2$. In the following computations we often omit the indexes that can be recovered from the context. We have:

$$\chi(V_1)\circ I_{p'}(\Phi(V_1))=\eta(\mathbf{\Phi}^2(\eta^!(Id_{X_1})))\circ I_{p'}(\Phi(f))=\eta'(D_{p'}(\Phi(X_1),\Phi(f))(\mathbf{\Phi}^2(\eta^!(Id_{X_1}))))$$

where the second equality is by (2.1.20) with respect to $\Phi(f)$. Then

$$\eta(D_p(X_1,\Phi(f))(\mathbf{\Phi}^2(\eta^!(Id_{X_1}))))=\eta(\mathbf{\Phi}^2(D_p(X_1,f)(\eta^!(Id_{X_1}))))=$$

$$\eta(\mathbf{\Phi}^2(\eta^!(Id_{X_1}\circ I_p(f)))=\eta(\mathbf{\Phi}^2(\eta^!(I_p(f))))$$

where the first equality holds by Lemma 3.1.4 and the second by (2.1.21) with respect to f.

On the other hand:

$$\Phi(I_p(f))\circ\chi(V_2)=\Phi(I_p(f))\circ\eta(\mathbf{\Phi}^2(\eta^!(Id_{X_2})))=$$

$$\eta'(D_{p'}(\Phi(I_p(f)),\Phi(X_2))(\mathbf{\Phi}^2(\eta^!(Id_{X_2}))))$$

where the second equality is by (2.1.20) with respect to $\Phi(I_p(f))$. Then

$$\eta(D_{p'}(\Phi(I_p(f)),\Phi(X_2))(\mathbf{\Phi}^2(\eta^!(Id_{X_2}))))=\eta(\mathbf{\Phi}^2(D_p(I_p(f),X_2)(\eta^!(Id_{X_2}))))=$$

$$\eta(\mathbf{\Phi}^2(\eta^!(I_p(f)\circ Id_{X_2})))=\eta(\mathbf{\Phi}^2(\eta^!(I_p(f))))$$

where the first equality holds by Lemma 3.1.4 and the second by (2.1.22) with respect to the $I_p(f)$. This completes the proof of Lemma 3.1.7.

Lemma 3.1.8 *[**2015.05.06.l1**] For all $X,V\in\mathcal{C}$ the pentagon*

$$\begin{array}{ccc}
D_p(X,V) & \xrightarrow{\eta_{V,X}} & Hom(X,I_p(V))\\
{\scriptstyle \mathbf{\Phi}^2_{V,X}}\downarrow & & \downarrow{\scriptstyle \Phi}\\
D_{p'}(\Phi(X),\Phi(V)) & & Hom(\Phi(X),\Phi(I_p(V)))\\
{\scriptstyle \eta_{\Phi(V),\Phi(X)}}\downarrow & & \downarrow{\scriptstyle -\circ\chi_{\mathbf{\Phi}}(V)}\\
Hom(\Phi(X),I_{p'}(\Phi(V))) & = & Hom(\Phi(X),I_{p'}(\Phi(V)))
\end{array}$$

commutes, that is, for all $a\in D_p(X,V)$ one has

$$\Phi(\eta(a))\circ\chi_{\mathbf{\Phi}}(V)=\eta(\mathbf{\Phi}^2(a))$$

Proof: By definition of $\chi_{\mathbf{\Phi}}$ and (2.1.20) with respect to $\Phi(\eta(a))$ we have

$$\Phi(\eta(a))\circ\chi_{\mathbf{\Phi}}(V)=\Phi(\eta(a))\circ\eta(\mathbf{\Phi}^2(\eta^!(Id)))=\eta(D_{p'}(\Phi(\eta(a)),\Phi(V))(\mathbf{\Phi}^2(\eta^!(Id_{I_p(V)}))))$$

By Lemma 3.1.3 we further have:

$$\eta(D_{p'}(\Phi(\eta(a)),\Phi(V))(\mathbf{\Phi}^2(\eta^!(Id))))=\eta(\mathbf{\Phi}^2(D_p(\eta(a),V)(\eta^!(Id))))$$

It remains to show that $D_p(\eta(a),V)(\eta^!(Id))=a$. Since η is a bijection we may apply it on both sides and by (2.1.22) with respect to $\eta(a)$ we get

$$\eta(D_p(\eta(a),V)(\eta^!(Id)))=\eta(\eta^!(\eta(a)\circ Id))=\eta(a)\circ Id=\eta(a)$$

3.2 More on universe category functors

By [9, Construction 4.7] any universe category functor $\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi})$ from $(\mathcal{C}, p)$ to $(\mathcal{C}', p)$ defines a homomorphism of C-systems

$$H : CC(\mathcal{C}, p) \to CC(\mathcal{C}', p')$$

Let $\psi_0 : pt' \to \Phi(pt)$ be the unique morphism. To define H on objects, one uses the fact that

$$Ob(CC(\mathcal{C}, p)) = \amalg_{n \geq 0} Ob_n(\mathcal{C}, p)$$

and defines $H(n, A)$ as $(n, H_n(A))$ where

$$H_n : Ob_n(\mathcal{C}, p) \to Ob_n(\mathcal{C}', p')$$

To obtain H_n one defines by induction on n, pairs (H_n, ψ_n) where H_n is as above and ψ_n is a family of isomorphisms

$$\psi_n(A) : int_n(H_n(A)) \to \Phi(int_n(A))$$

as follows:

1. for $n = 0$, H_0 is the unique function from one point set to one point set and $\psi_0(A) = \psi_0$,
2. for the successor of n one has

   $$H_{n+1}(A, F) = (H_n(A), \psi_n(A) \circ \Phi(F) \circ \phi)$$

   and $\psi_{n+1}(A, F)$ is the unique morphism $int(H(A, F)) \to \Phi(int(A, F))$ such that

   $$\psi_{n+1}(A, F) \circ \Phi(Q(F)) \circ \widetilde{\phi} = Q(\psi_n(A) \circ \Phi(F) \circ \phi)$$

   and

   $$\psi_{n+1}(A, F) \circ \Phi(p_{int_n(A),F}) = p_{H_n(A),\psi_n(A) \circ \Phi(F) \circ \phi} \circ \psi_n(A)$$

The function $H : Ob(CC(\mathcal{C}, p)) \to Ob(CC(\mathcal{C}', p'))$ is the sum of functions H_n. For $\Gamma = (m, A)$ in $Ob(CC(\mathcal{C}, p))$ we let $\psi(\Gamma) = \psi_m(A)$ such that ψ is the sum of families ψ_n.

The action of H on morphisms is given, for $f : \Gamma' \to \Gamma$, by

$$H(f) = \psi(\Gamma') \circ \Phi(int(f)) \circ \psi(\Gamma)^{-1}$$

We will often write H also for the functions H_n and ψ for the functions ψ_n.

Lemma 3.2.1 *[**2015.03.21.l4**] Let $(\Phi, \phi, \widetilde{\phi})$ be universe category functor. Then:*

1. *for $T \in Ob_1(\Gamma)$ one has*

   $$u_{1,H(\Gamma)}(H(T)) = \psi(\Gamma) \circ \Phi(u_{1,\Gamma}(T)) \circ \phi$$

2. for $o \in \widetilde{Ob}_1(\Gamma)$ *one has*

$$\widetilde{u}_{1,H(\Gamma)}(H(o)) = \psi(\Gamma) \circ \Phi(\widetilde{u}_{1,\Gamma}(o)) \circ \widetilde{\phi}$$

3. for $T \in Ob_2(\Gamma)$ *one has*

$$u_{2,H(\Gamma)}(H(T)) = D_{p'}(\psi(\Gamma), U')(D_{p'}(int(H(\Gamma)), \phi)(\mathbf{\Phi}^2(u_{2,\Gamma}(T))))$$

4. for $o \in \widetilde{Ob}_2(\Gamma)$ *one has*

$$\widetilde{u}_{2,H(\Gamma)}(H(o)) = D_{p'}(\psi(\Gamma), \widetilde{U}')(D_{p'}(int(H(\Gamma)), \widetilde{\phi})(\mathbf{\Phi}^2(\widetilde{u}_{2,\Gamma}(o))))$$

Proof: Let $\Gamma = (n, A)$.
In the case of $T \in Ob_1(\Gamma)$, if $T = (n+1, (A, F))$ then

$$u_1(H(T)) = u_1(n+1, H(A,F)) = u_1(n+1, (H(A), \psi(\Gamma) \circ \Phi(F) \circ \phi)) = \psi(\Gamma) \circ \Phi(F) \circ \phi$$

In the case of $s \in \widetilde{Ob}_1(\Gamma)$, if $F = u_1(\partial(s))$ then

$$\widetilde{u}_1(H(s)) = H(s) \circ Q(u_1(n+1, H(A,F))) = \psi(A) \circ \Phi(s) \circ \psi(A,F)^{-1} \circ Q(\psi(A) \circ \Phi(F) \circ \phi) =$$
$$\psi(A) \circ \Phi(s) \circ \Phi(Q(F)) \circ \widetilde{\phi} = \psi(A) \circ \Phi(s \circ Q(F)) \circ \widetilde{\phi} = \psi(A) \circ \Phi(\widetilde{u}_1(s)) \circ \widetilde{\phi}$$

Where the second equality is by definition of $\psi(A, F)$.
In the case $T \in Ob_2(\Gamma)$, if $T = (n+2, ((A, F_1), F_2))$ then

$$u_2(H(T)) = u_2(n+2, H(((A,F_1),F_2))) = u_2(n+2, (H(A,F_1), \psi(A,F_1) \circ \Phi(F_2) \circ \phi)) =$$
$$u_2(n+2, ((H(A), \psi(A) \circ \Phi(F_1) \circ \phi), \psi(A,F_1) \circ \Phi(F_2) \circ \phi)) =$$
$$(\psi(A) \circ \Phi(F_1) \circ \phi, \psi(A,F_1) \circ \Phi(F_2) \circ \phi)$$

On the other hand

$$D_{p'}(\psi(A), -)D_{p'}(-, \phi)(\mathbf{\Phi}^2(u_2(T))) = D_{p'}(\psi(A), -)D_{p'}(-, \phi)(\mathbf{\Phi}^2(u_2(n+2, ((A,F_1),F_2)))) =$$
$$D_{p'}(\psi(A), -)D_{p'}(-, \phi)(\mathbf{\Phi}^2(F_1, F_2)) = D_{p'}(\psi(A), -)D_{p'}(-, \phi)(\Phi(F_1) \circ \phi, \iota \circ \Phi(F_2)) =$$
$$D_{p'}(\psi(A), -)(\Phi(F_1) \circ \phi, \iota \circ \Phi(F_2) \circ \phi) = (\psi(A) \circ \Phi(F_1) \circ \phi, Q(\psi(A), \Phi(F_1) \circ \phi) \circ \iota \circ \Phi(F_2) \circ \phi)$$

therefore we need to show that

$$[\mathbf{2015.04.12.eq1}]\psi(A, F_1) \circ \Phi(F_2) \circ \phi = Q(\psi(A), \Phi(F_1) \circ \phi) \circ \iota \circ \Phi(F_2) \circ \phi \qquad (3.2.1)$$

which we reduce to $\psi(A, F_1) = Q(\psi(A), \Phi(F_1) \circ \phi) \circ \iota$. The codomain of both sides is $\Phi(int(A, F_1))$. Using the fact that the external square of the diagram

$$\begin{array}{ccccc} \Phi(int(A,F_1)) & \xrightarrow{\Phi(Q(F_1))} & \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ \downarrow{\scriptstyle \Phi(p_{(A,F_1)})} & & \downarrow{\scriptstyle \Phi(p)} & & \downarrow{\scriptstyle p'} \\ \Phi(int(A)) & \xrightarrow{\Phi(F_1)} & \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

is a pullback square we see that equality (3.2.1) would follow from the following two equalities:

$$\psi(A,F_1)\circ\Phi(Q(F_1))\circ\widetilde{\phi}=Q(\psi(A),\Phi(F_1)\circ\phi)\circ\iota\circ\Phi(Q(F_1))\circ\widetilde{\phi}$$

and

$$\psi(A,F_1)\circ\Phi(p_{(A,F_1)})=Q(\psi(A),\Phi(F_1)\circ\phi)\circ\iota\circ\Phi(p_{(A,F_1)})$$

For the first equality we have

$$\psi(A,F_1)\circ\Phi(Q(F_1))\circ\widetilde{\phi}=Q(\psi(A)\circ\Phi(F_1)\circ\phi)$$

by definition of $\psi(\Gamma,F_1)$ and

$$Q(\psi(A),\Phi(F_1)\circ\phi)\circ\iota\circ\Phi(Q(F_1))\circ\widetilde{\phi}=Q(\psi(A),\Phi(F_1)\circ\phi)\circ Q(\Phi(F_1)\circ\phi)=Q(\psi(A)\circ\Phi(F_1)\circ\phi)$$

where the first equality holds by definition of ι and second by the definition of $Q(-,-)$.

For the second equality we have

$$\psi(A,F_1)\circ\Phi(p_{(A,F_1)})=p_{H(A,F_1)}\circ\psi(A)$$

by definition of $\psi(A,F_1)$ and

$$Q(\psi(A),\Phi(F_1)\circ\phi)\circ\iota\circ\Phi(p_{(A,F_1)})=Q(\psi(A),\Phi(F_1)\circ\phi)\circ p_{\Phi(int(A)),\Phi(F_1)\circ\phi}=p_{H(A,F_1)}\circ\psi_\Gamma$$

by definitions of Q and ι.

The case of $o\in\widetilde{Ob}_2(\Gamma)$ is strictly parallel to the case of $T\in Ob_2(\Gamma)$ with $\Phi(F_2)\circ\phi$ at the end of the formulas replaced by $\Phi(\widetilde{F}_2)\circ\widetilde{\phi}$ where instead of $F_2: int(A,F_1)\rightarrow U$ one has $\widetilde{F}_2: int(A,F_1)\rightarrow\widetilde{U}$ with $\widetilde{F}_2=\widetilde{u}_{1,ft(\partial(o))}(o)$.

For $(\Phi,\phi,\widetilde{\phi})$ as above let us denote by

$$\xi_{\mathbf{\Phi}}:\Phi(I_p(U))\rightarrow I_{p'}(U')$$

the composition $\chi_{\mathbf{\Phi}}(U)\circ I_{p'}(\phi)$ and by

$$\widetilde{\xi}_{\mathbf{\Phi}}:\Phi(I_p(\widetilde{U}))\rightarrow I_{p'}(\widetilde{U}')$$

the composition $\chi_{\mathbf{\Phi}}(\widetilde{U})\circ I_p(\widetilde{\phi})$.

Lemma 3.2.2 *[***2015.05.06.l2***] Let $(\Phi,\phi,\widetilde{\phi})$ be a universe category functor and $\Gamma\in Ob(CC(\mathcal{C},p))$. Then one has:*

1. *for $T\in Ob_2(\Gamma)$*
$$\mu_2(H(T))=\psi(\Gamma)\circ\Phi(\mu_2(T))\circ\xi_{\mathbf{\Phi}}$$
2. *for $o\in\widetilde{Ob}_2(\Gamma)$*
$$\widetilde{\mu}_2(H(o))=\psi(\Gamma)\circ\Phi(\widetilde{\mu}_2(o))\circ\widetilde{\xi}_{\mathbf{\Phi}}$$

Proof: We have

$$\mu_2(H(T)) = \eta(u_2(H(T))) = \eta(D_{p'}(\psi(\Gamma), \_)(D_{p'}(\_, \phi)(\mathbf{\Phi}^2(u_2(T))))) =$$

$$\psi(\Gamma) \circ \eta(\mathbf{\Phi}^2(u_2(T))) \circ I_{p'}(\phi)$$

where the first equality holds by the definition of μ_2 (cf. Construction 2.1.22), the second equality holds by Lemma 3.2.1(3) and the third by the naturality of η. Next

$$\eta(\mathbf{\Phi}^2(u_2(T))) \circ I_{p'}(\phi) = \Phi(\eta(u_2(T))) \circ \chi_\Phi(U) \circ I_{p'}(\phi) = \Phi(\eta(u_2(T))) \circ \xi_{\mathbf{\Phi}} = \Phi(\mu_2(T)) \circ \xi_{\mathbf{\Phi}}$$

where the first equality holds by Lemma 3.1.8, the second one by the definition of $\xi_{\mathbf{\Phi}}$ and the third one by the definition of μ_2.

The proof of the second part of the lemma is strictly parallel to the proof of the first part.

3.3 Functoriality properties of the (Π, λ)-structures constructed from P-structures

Let us show the functoriality properties of the (Π, λ) structures of Construction 2.2.3.

Definition 3.3.1 *[**2016.09.13.def1**] Let $H : CC \to CC'$ be a homomorphism of C-systems. Let (Π, λ) and (Π', λ') be pre-(Π, λ)-structures on CC and CC' respectively.*

The H is called a homomorphism of C-systems with pre-(Π, λ) and (Π', λ') if the following two squares commute

$$\begin{array}{ccc} \mathcal{O}b_2 & \xrightarrow{\Pi} & \mathcal{O}b_1 \\ {\scriptstyle H\mathcal{O}b_2}\downarrow & & \downarrow{\scriptstyle H\mathcal{O}b_1} \\ H^\circ(\mathcal{O}b_2) & \xrightarrow{H^\circ(\Pi')} & H^\circ(\mathcal{O}b_1) \end{array} \qquad \begin{array}{ccc} \widetilde{\mathcal{O}b}_2 & \xrightarrow{\lambda} & \widetilde{\mathcal{O}b}_1 \\ {\scriptstyle H\widetilde{\mathcal{O}b}_2}\downarrow & & \downarrow{\scriptstyle H\widetilde{\mathcal{O}b}_1} \\ H^\circ(\widetilde{\mathcal{O}b}_2) & \xrightarrow{H^\circ(\lambda')} & H^\circ(\widetilde{\mathcal{O}b}_1) \end{array}$$

If (Π, λ) and (Π', λ') are (Π, λ)-structures then H is called a homomorphism of C-systems with (Π, λ)-structures if it is a homomorphism of C-systems with the corresponding pre-(Π, λ)-structures.

Unfolding the definition of $H\mathcal{O}b_i$ and $H\widetilde{\mathcal{O}b}_i$ in Definition 3.3.1 we see that H is a homomorphism of C-systems with pre-(Π, λ)-structures if and only if for all $\Gamma \in CC$ one has

1. for all $T \in \mathcal{O}b_2(\Gamma)$ one has

$$[\mathbf{2016.09.13.eq1}]H(\Pi_\Gamma(T)) = \Pi'_{H(\Gamma)}(H(T)) \qquad (3.3.1)$$

2. for all $o \in \widetilde{\mathcal{O}b}_2(\Gamma)$ one has

$$[\mathbf{2016.09.13.eq2}]H(\lambda_\Gamma(o)) = \lambda'_{H'(\Gamma)}(H(o)) \qquad (3.3.2)$$

Theorem 3.3.2 *[**2015.03.21.th1**] Let $(\Phi, \phi, \widetilde{\phi})$ be as above and let $(P, \widetilde{P})$, $(P', \widetilde{P}')$ be as in Problem 2.2.2 for $\mathcal{C}$ and $\mathcal{C}'$ respectively.*

Assume that the squares

$$[\mathbf{2015.03.23.sq1}]\quad \begin{array}{ccc} \Phi(I_p(U)) & \xrightarrow{\xi_\Phi} & I_{p'}(U') \\ {\scriptstyle \Phi(P)}\downarrow & & \downarrow{\scriptstyle P'} \\ \Phi(U) & \xrightarrow{\phi} & U \end{array} \qquad \begin{array}{ccc} \Phi(I_p(\widetilde{U})) & \xrightarrow{\widetilde{\xi}_\Phi} & I_{p'}(\widetilde{U}') \\ {\scriptstyle \Phi(\widetilde{P})}\downarrow & & \downarrow{\scriptstyle \widetilde{P}'} \\ \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U} \end{array} \tag{3.3.3}$$

commute. Then the homomorphism

$$H(\Phi, \phi, \widetilde{\phi}) : CC(\mathcal{C}, p) \to CC(\mathcal{C}', p')$$

is a homomorphism of C-systems with (Π, λ)-structures.

Proof: We have to show that for all $\Gamma \in Ob(CC(\mathcal{C}, p))$, $T \in Ob_2(\Gamma)$ and $o \in \widetilde{Ob}_2(\Gamma)$ the equalities (3.3.1) and (3.3.2) hold. We will prove the first equality. The proof of the second is strictly parallel to the proof of the first.

By definition we have:

$$H(\Pi(T)) = H(u_1^{-1}(\eta(u_2(T)) \circ P)) = (u_1)^{-1}(\psi(\Gamma) \circ \Phi(\eta(u_2(T)) \circ P) \circ \phi) =$$

$$(u_1)^{-1}(\psi(\Gamma) \circ \Phi(\eta(u_2(T))) \circ \Phi(P) \circ \phi)$$

where the second equality holds by Lemma 3.2.1(1) and

$$\Pi'(H(T)) = (u_1)^{-1}(u_2(H(T)) \circ P') = (u_1)^{-1}(\eta'(u_2(H(T))) \circ P')$$

Let us show that

$$\eta'(u_2(H(T))) \circ P' = \psi(\Gamma) \circ \Phi(\eta(u_2(T))) \circ \Phi(P) \circ \phi$$

By Lemma 3.2.2(1) we have

$$\eta'(u_2(H(T))) \circ P' = \psi(\Gamma) \circ \Phi(\eta(u_2(T))) \circ \xi_\Phi \circ P'$$

It remains to show that

$$\xi_\Phi \circ P' = \Phi(P) \circ \phi$$

which is our assumption about the commutativity of the square first square in (3.3.3).

3.4 Appendix. Categories with pullbacks and locally cartesian closed categories

Lemma 3.4.1 *[**2015.04.16.l1**] Let $\mathcal{C}$ be a category. Consider four pullback squares*

$$
\begin{array}{ccc}
pb_i & \xrightarrow{pr_{Y,i}} & Y \\
{\scriptstyle pr_{X,i}}\downarrow & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{f} & Z
\end{array}
\qquad
\begin{array}{ccc}
pb'_i & \xrightarrow{pr_{Y',i}} & Y' \\
{\scriptstyle pr_{X,i}}\downarrow & & \downarrow{\scriptstyle g'} \\
X' & \xrightarrow{f'} & Z
\end{array}
$$

where $i = 1, 2$. Let $a : X' \to X$ and $b : Y' \to Y$ be such that $a \circ f = f'$ and $b \circ g = g'$. Let $\iota : pb_1 \to pb_2$ be the unique morphism such that $\iota \circ pr_{X_2} = pr_{X,1}$ and $\iota \circ pr_{Y,1} = pr_{Y,2}$ and similarly for $\iota' : pb'_1 \to pb'_2$. Let $c_i(a,b) : pb'_i \to pb_i$ be the unique morphisms such that $c_i(a,b) \circ pr_{X,i} = pr_{X',i} \circ a$ and $c_i(a,b) \circ pr_{Y,i} = b \circ pr_{Y',i}$. Then the square

$$
\begin{array}{ccc}
pb'_1 & \xrightarrow{c_1(a,b)} & pb_1 \\
{\scriptstyle \iota'}\downarrow & & \downarrow{\scriptstyle \iota} \\
pb'_2 & \xrightarrow{c_2(a,b)} & pb_2
\end{array}
$$

commutes, i.e., $c_1(a,b) \circ \iota = \iota' \circ c_2(a,b)$.

Proof: Since pb_2 is a pullback it is sufficient to prove that

$$c_1(a,b) \circ \iota \circ pr_{X,2} = \iota' \circ c_2(a,b) \circ pr_{X,2}$$

and

$$c_1(a,b) \circ \iota \circ pr_{Y,2} = \iota' \circ c_2(a,b) \circ pr_{Y,2}$$

For the first one we have:

$$c_1(a,b) \circ \iota \circ pr_{X,2} = c_1(a,b) \circ pr_{X,1} = pr_{X',1} \circ a$$

and

$$\iota' \circ c_2(a,b) \circ pr_{X,2} = \iota' \circ pr_{X',2} \circ a = pr_{X',1} \circ a$$

The verification of the second equality is similar.

Definition 3.4.2 *[**2015.04.22.def1**] A category with pullbacks or a category with fiber products is a category together with, for all pairs of morphisms of the form $f : X \to Z$, $g : Y \to Z$, pullback squares*

$$
\begin{array}{ccc}
(X,f) \times_Z (Y,g) & \xrightarrow{pr_2^{(X,f),(Y,g)}} & Y \\
{\scriptstyle pr_1^{(X,f),(Y,g)}}\downarrow & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{f} & Z
\end{array}
$$

We will often abbreviate these main notations in various ways. The morphism $pr_2 \circ g = pr_1 \circ f$ from $(X,f) \times (Y,g)$ to Z is denoted by $f \diamond g$.

Given a category with pullbacks, morphisms $g_i : Y_i \to Z$, $i = 1, 2$ and morphisms $a : X_1 \to Y_1$, $b : X_2 \to Y_2$ denote by

$$(a \times b)^{g_1,g_2} : ((X_1, a \circ g_1) \times_Z (X_2, b \circ g_2), (a \circ g_1) \diamond (b \circ g_2)) \to ((Y_1, g_1) \times_Z (Y_2, g_2), g_1 \diamond g_2)$$

the unique morphism over Z such that

$$(a \times b)^{g_1,g_2} \circ pr_1 = pr_1 \circ a$$

and

$$(a \times b)^{g_1,g_2} \circ pr_2 = pr_2 \circ b$$

To show that $(a \times b)^{g_1,g_2}$ exists we need to check that

$$pr_1 \circ a \circ g_1 = pr_2 \circ b \circ g_2$$

which is immediate from the definition of the pullback.

Lemma 3.4.3 [**2015.05.14.l1**] *In the setting introduced above suppose that we have in addition $a' : X_1' \to X_1$ and $b' : X_2' \to X_2$. Then one has*

$$((a' \circ a) \times (b' \circ b))^{g_1,g_2} = (a' \times b')^{a \circ g_1, b \circ g_2} \circ (a \times b)^{g_1,g_2}$$

Proof: Straightforward rewriting to compute the compositions of both sides with $pr_1^{g_1,g_2}$ and $pr_2^{g_1,g_2}$.

Definition 3.4.4 [**2015.03.27.def1**] *A locally cartesian closed structure on a (pre-)category $\mathcal{C}$ is a collection of data of the form:*

1. *A structure of a category with pullbacks on $\mathcal{C}$.*
2. *For all f, g of the form $f : X \to Z$, $g : Y \to Z$, an object $\underline{Hom}_Z((X, f), (Y, g))$ and a morphism*
$$f \triangle g : \underline{Hom}_Z((X, f), (Y, g)) \to Z$$
together with morphisms of the form
$$\underline{Hom}((X, f), a) : \underline{Hom}((X, f), (Y, g)) \to \underline{Hom}((X, f), (Y', g'))$$
for all $a : (Y, g) \to (Y', g')$ over Z, that make $\underline{Hom}((X, f), -)$ into a functor from $\mathcal{C}/Z$ to $\mathcal{C}$.
3. *For all f, g as above a morphism*
$$ev_{(Y,g)}^{(X,f)} : (\underline{Hom}_Z((X, f), (Y, g)), f \triangle g) \times (X, f) \to (Y, g)$$
over Z such that for all $h : W \to Z$ the function
$$adj_{(Y,g)}^{(W,h),(X,f)} : Hom_Z((W, h), (\underline{Hom}_Z((X, f), (Y, g)), f \triangle g)) \to$$
$$Hom_Z(((W, h) \times (X, f), h \diamond f), (Y, g))$$
given by
$$u \mapsto (u \times Id_X)^{f \triangle g, f} \circ ev_{(Y,g)}^{(X,f)}$$
is a bijection and such that the morphisms $ev_{(Y,g)}^{(X,f)}$ are natural in Y.

A locally cartesian closed (pre-)category is a (pre-)category together with a locally cartesian closed structure on it.

If a locally cartesian closed category is given with a final object pt we will write $X \times Y$ for $(X, \pi_X) \times_{pt} (Y, \pi_Y)$ where π_X and π_Y are the unique morphisms from X and Y respectively to pt.

By definition the objects $(\underline{Hom}((X,f),(Y,g)), f\triangle g)$ of $\mathcal{C}/Z$ are functorial only in (Y,g). Their functoriality in (X,f) is a consequence of a lemma. For $f : X \to Z$, $f' : X' \to Z$, $g : Y \to Z$ and $h : X' \to X$ such that $h \circ f = f'$ let

$$\underline{Hom}_Z(h,(Y,g)) : \underline{Hom}_Z((X,f),(Y,g)) \to \underline{Hom}_Z((X',f'),(Y,g))$$

be the unique function whose adjoint

$$adj(\underline{Hom}_Z(h,(Y,g))) : (\underline{Hom}_Z((X,f),(Y,g)), f\triangle g) \times_Z (X',f') \to (Y,g)$$

equals $(Id_{\underline{Hom}_Z((X,f),(Y,g))} \times h)^{f\triangle g, f} \circ ev_Y^X$. Then one has:

Lemma 3.4.5 *[***2015.04.10.l1***] The morphisms $\underline{Hom}_Z(h,(Y,g))$ satisfy the equations*

$$\underline{Hom}_Z(h,(Y,g)) \circ (f'\triangle g) = f\triangle g$$

and the equations

$$\underline{Hom}_Z(h_1 \circ h_2,(Y,g)) = \underline{Hom}(h_2,(Y,g)) \circ \underline{Hom}(h_1,(Y,g))$$
$$\underline{Hom}_Z(Id,(Y,g)) = Id$$

making $\underline{Hom}_Z(-,(Y,g))$ into a contravariant functor from $\mathcal{C}/Z$ to itself. In addition, for each $h' : (Y,g) \to (Y,g')$ the square

$$\begin{array}{ccc}
\underline{Hom}_Z((X',f'),(Y,g)) & \xrightarrow{\underline{Hom}_Z((X',f'),h')} & \underline{Hom}_Z((X',f'),(Y',g')) \\
{\scriptstyle \underline{Hom}_Z(h,(Y,g))}\downarrow & & \downarrow{\scriptstyle \underline{Hom}_Z(h,(Y',g'))} \\
\underline{Hom}_Z((X,f),(Y,g)) & \xrightarrow{\underline{Hom}_Z((X,f),h')} & \underline{Hom}_Z((X,f),(Y',g'))
\end{array}$$

commutes.

Proof: It is a particular case of [5, Theorem 3, p.100]. The commutativity of the square is a part of the "bifunctor" claim of the theorem.

Lemma 3.4.6 *[***2015.04.20.l2***] In a locally cartesian closed category let $f : X \to Z$, $f' : X' \to Z$, $g : Y \to Z$ be objects over Z and let $a : X' \to X$ be a morphism over Z. Then the square*

$$\begin{array}{ccc}
(\underline{Hom}((X,f),(Y,g)), f\triangle g) \times_Z (X',f') & \xrightarrow{1} & (\underline{Hom}((X,f),(Y,g)), f\triangle g) \times_Z (X,f) \\
{\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle ev} \\
(\underline{Hom}_Z((X',f'),(Y,g)), f'\triangle g) \times_Z (X',f') & \xrightarrow{ev'} & Y
\end{array}$$

where 1 is $(Id_{\underline{Hom}((X,f),(Y,g))} \times a)^{f\triangle g, f}$ and 2 is $(\underline{Hom}(a,(Y,g)) \times Id_{X'})^{f'\triangle g, f'}$, commutes.

Proof: Let us show that both paths in the square are adjoints to $\underline{Hom}(a,(Y,g))$. For the path that goes through the upper right corner it follows from the definition of $\underline{Hom}(a,(Y,g))$ as the morphism whose adjoint is $(Id \times a) \circ ev$. For the path that goes through the lower left corner it follows from the definition of adjoint applied to $\underline{Hom}(a,(Y,g))$. Indeed, the adjoint to this morphism is

$$adj(\underline{Hom}(a,(Y,g))) = (\underline{Hom}(a,(Y,g)) \times Id_{X'}) \circ ev'$$

Lemma 3.4.7 *[**2015.05.12.l2**] Let $\mathcal{C}$ be a locally cartesian closed category. Let Z, $(X,f),(Y,g),(W,h)$ be as above.*

1. *Let (Y',g') be an object over Z and $a:(Y,g) \to (Y',g')$ a morphism over Z. Then for any $b \in Hom_Z((W,h), \underline{Hom}_U((X,f),(Y,g)))$ one has*
$$adj(b \circ \underline{Hom}_Z((X,f),a)) = adj(b) \circ a$$
2. *Let (X',f') be an object over Z and $a:(X',f') \to (X,f)$ a morphism over Z. Then for any $b \in Hom_Z((W,h), \underline{Hom}_U((X,f),(Y,g)))$ one has*
$$adj(b \circ \underline{Hom}_Z(a,(Y,g))) = (Id_W \times a)^{h,f} \circ adj(b)$$
3. *Let (W',h') be an object over Z and $a:(W',h') \to (W,h)$ a morphism over Z. Then for any $b \in Hom_Z((W,h), \underline{Hom}_U((X,f),(Y,g)))$ one has*
$$adj(a \circ b) = (a \times Id_X)^{h,f} \circ adj(b)$$

Proof: The proof of the first case is given by

$$adj(b \circ \underline{Hom}_Z((X,f),a)) = ((b \circ \underline{Hom}_Z((X,f),a)) \times Id_X)^{f\triangle g',f} \circ ev^{(X,f)}_{(Y',g')} =$$

$$(b \times Id_X)^{f\triangle g,f} \circ (\underline{Hom}_Z((X,f),a)) \times Id_X)^{f\triangle g',f} \circ ev^{(X,f)}_{(Y',g')} =$$

$$(b \times Id_X)^{f\triangle g,f} \circ ev^{(X,f)}_{(Y,g)} \circ a = adj(b) \circ a$$

where the second equality holds by Lemma 3.4.3 and the third equality by the naturality axiom for morphisms $ev^{(X,f)}_{(Y,g)}$ in (Y,g).

The proof of the second case is given by the following sequence of equalities where we use the notation Hm for $\underline{Hom}_Z(a,(Y,g))$ as well as a number of other abbreviations:

$$adj(b \circ Hm) = ((b \circ Hm) \times Id) \circ ev = (b \times Id) \circ (Hm \times Id) \circ ev = (b \times Id) \circ adj(Hm) =$$

$$(b \times Id) \circ (Id \times a) \circ ev = (b \times a) \circ ev = (Id \times a) \circ (b \times Id) \circ ev = (Id \times a) \circ adj(b)$$

The proof of the third case is given by

$$adj(a \circ b) = ((a \circ b) \times Id_X) \circ ev^{(X,f)}_{(Y,g)} = (a \times Id_X) \circ (b \times Id_X) \circ ev^{(X,f)}_{(Y,g)} =$$

$$(a \times Id_X) \circ adj(b)$$

where the second equality holds by Lemma 3.4.3.

Lemma is proved.

Example 3.4.8 [**2015.05.20.ex1**] The following example shows that there can be many different structures of a category with pullbacks on a (pre-)category and also many locally cartesian closed structures.

Let us take as our (pre-)category the (pre-)category *preStn* whose objects are natural numbers and $Hom(n, m) = Hom(\{0, \dots, n-1\}, \{0, \dots, m-1\})$.

Since every isomorphism class contains exactly one object every auto-equivalence of this category is an automorphism. Let Φ be such an automorphism. It is easy to see that it must be identity on the set of objects. Let $X = \{0, 1\}$. Consider Φ on $End(X)$. Since Φ must respect identities and compositions, Φ must take $Aut(X)$ to itself and must act on it by identity. If 1 and σ are the two elements of $Aut(X)$ we conclude that $\Phi(1) = 1$ and $\Phi(\sigma) = \sigma$.

Let us choose now any structure str_0 of a category with pullbacks on *preStn* and let us consider two structures str_1 and str_σ that are obtained by choosing all the pullbacks as in str_0 except for the square for the pair (Id_X, Id_X) which we choose to be, correspondingly, as follows:

[**2015.05.20.sq1**]
$$\begin{array}{ccc} X & \xrightarrow{Id_X} & X \\ {\scriptstyle Id_X}\downarrow & & \downarrow{\scriptstyle Id_X} \\ X & \xrightarrow{Id_X} & X \end{array} \quad \text{for } str_1 \text{ and } \quad \begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle Id_X} \\ X & \xrightarrow{Id_X} & X \end{array} \quad \text{for } str_\sigma. \qquad (3.4.1)$$

The preceding discussion of the auto-equivalences of *preStn* shows that there is no auto-equivalence which would transform str_1 into str_σ.

The (pre-)category *preStn* also has a locally cartesian closed structure that can be modified so that its underlying pullback structures are str_1 and str_σ. This shows that *preStn* has at least two locally cartesian closed structures that are not interchanged by auto-equivalences of *preStn*.

Remark 3.4.9 [**2015.05.20.rem1**] The previous example has a continuation in the univalent foundations where there is a notion of a category and pre-category. There one expects it to be true that the type of pullback structures and the type of locally cartesian closed structures on a category (as opposed to those on a general pre-category) are of h-level 1, i.e., classically speaking are either empty or contain only one element.

In addition any such structure on a pre-category should define a structure of the same kind on the Rezk completion of this pre-category with all the different structures on the pre-category becoming equal on the Rezk completion. In the case of the previous example the Rezk completion of *preStn* is the category $FSets$ of finite sets and in view of the univalence axiom for finite sets the two pullback squares of 3.4.1 will become equal in $FSets$.

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