Preliminary title: B-presheaves[1]

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Contents

1 Functor Sig and isomorphisms $Sig\mathcal{O}b_i$ and $Sig\widetilde{\mathcal{O}b}_i$

Let $\mathcal{G}$ be a presheaf on a C-system CC. For $\Gamma \in CC$ we set

$$[\mathbf{2016.08.30.eq7}]Sig(\mathcal{G})(\Gamma) = \amalg_{T\in Ob_1(\Gamma)}\mathcal{G}(T) \tag{1.1}$$

and for $f : \Gamma' \to \Gamma$

$$[\mathbf{2016.08.30.eq8}]Sig(\mathcal{G})(f)(T, g) = (f^*(T), \mathcal{G}(q(f, T))(T)) \tag{1.2}$$

Lemma 1.1 *[**2016.08.28.l1**] The data described above defines a presheaf of sets, that is, one has:*

1. *for* $\Gamma \in CC$,
$$Sig(\mathcal{G})(Id_\Gamma) = Id_{Sig(\mathcal{G})(\Gamma)}$$
2. *for* $f' : \Gamma'' \to \Gamma'$, $f : \Gamma' \to \Gamma$ *one has*
$$Sig(\mathcal{G})(f' \circ f) = Sig(\mathcal{G})(f) \circ Sig(\mathcal{G})(f')$$

Proof: For the identity we have

$$Sig(\mathcal{G})(Id_\Gamma)(T, g) = (Id^*_\Gamma(T), \mathcal{G}(q(Id_\Gamma, T))(g)) = (T, g)$$

where the second equality is by axioms of the C-system structure and for the composition we have

$$Sig(\mathcal{G})(f')(Sig(\mathcal{G}(f)(T, g))) = Sig(\mathcal{G})(f')(f^*(T), \mathcal{G}(q(f, T))(g)) =$$

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$$((f')^*(f^*(T)), \mathcal{G}(q(f', f^*(T)))(\mathcal{G}(q(f,T))(g))) = ((f')^*(f^*(T)), \mathcal{G}(q(f', f^*(T)) \circ q(f,T))(g)) =$$
$$((f' \circ f)^*(T), \mathcal{G}(q(f' \circ f, T))(g)) = Sig(\mathcal{G})(f' \circ f)(T, g)$$

where the first two equalities are by definition of $Sig(\mathcal{G})$, the third by the composition property of $\mathcal{G}$, the fourth by the axioms of a C-system and the fifth again by the definition of $Sig(\mathcal{G})$.

This completes the proof of Lemma 1.1.

One defines Sig on morphisms of presheaves $r : \mathcal{G} \to \mathcal{G}'$ by the family of morphisms

$$[\mathbf{2016.08.30.eq9}] Sig(r)_\Gamma(T, g) = (T, r_T(g)) \tag{1.3}$$

For $r : \mathcal{G} \to \mathcal{G}'$ and $f : \Gamma' \to \Gamma$, we have

$$Sig(\mathcal{G})(f) \circ Sig(r)_{\Gamma'} = Sig(r)_\Gamma \circ Sig(\mathcal{G}')(f)$$

that is, the families of functions $Sig(r)_\Gamma$ parametrized by $\Gamma \in CC$ form a morphism of presheaves.

For $\mathcal{G} \in PreShv(CC)$ we have

$$Sig(Id_{\mathcal{G}})_\Gamma(T, g) = (T, (Id_{\mathcal{G}})_T(g)) = (T, g)$$

and for $r : \mathcal{G} \to \mathcal{G}'$, $r' : \mathcal{G}' \to \mathcal{G}''$ we have

$$Sig(r \circ r')_\Gamma(T, g) = (T, (r \circ r')_T(g)) = (T, r'_T(r_T(g))) = Sig(r')(Sig(r)(T, g))$$

These two equalities show that the covariant functor data from $PreShv$ to $PreShv$ given by Sig on presheaves and Sig on morphisms of presheaves is a functor that we also denote by

$$Sig : PreShv(CC) \to PreShv(CC)$$

Problem 1.2 *[**2016.08.30.prob1**] For $i \geq 0$ to construct an isomorphism of presheaves*

$$SigOb_i : Sig(\mathcal{O}b_i) \to \mathcal{O}b_{i+1}$$

Before we provide a construction we prove a lemma that will be also used in Construction 1.4.

Lemma 1.3 *[**2016.09.01.l1**] Let $\Gamma \in CC$. Then one has:*

1. *if $T \in \mathcal{O}b_1(\Gamma)$ and $X \in \mathcal{O}b_i(T)$ then $X \in \mathcal{O}b_{i+1}(\Gamma)$,*
2. *if $X \in \mathcal{O}b_{i+1}(\Gamma)$ then $ft^i(X) \in \mathcal{O}b_1(\Gamma)$ and $X \in \mathcal{O}b_i(ft^i(X))$.*

Proof: The first assertion follows from the equalities $l(X) = l(T) + i = l(\Gamma) + 1 + i$ and $ft^{i+1}(X) = ft(ft^i(X)) = ft(T) = \Gamma$.

To prove the second assertion let $X \in \mathcal{O}b_{i+1}(\Gamma)$. Since $l(X) \geq i$ we have $l(ft^i(X)) = l(X) - i = l(\Gamma) + (i+1) - i = l(\Gamma) + 1$. The equality $ft^1(ft^i(X)) = ft^{i+1}(X) = \Gamma$ is obvious and we conclude that $ft^i(X) \in \mathcal{O}b_1(\Gamma)$. Next, again because $l(X) \geq i$, we have $l(X) = l(ft^i(X)) + i$ and since $ft^i(X) = ft^i(X)$ we have that $X \in \mathcal{O}b_i(ft^i(X))$.

Construction 1.4 [**2016.08.30.constr1**] Let $\Gamma \in CC$. Then $Sig(\mathcal{O}b_i)(\Gamma)$ is the set of pairs (T, X) where $T \in \mathcal{O}b_1(\Gamma)$ and $X \in \mathcal{O}b_i(T)$. By Lemma 1.3(1), the formula

$$[\mathbf{2016.09.01.eq4}] Sig\mathcal{O}b_{i,\Gamma}(T, X) = X \tag{1.4}$$

defines a function $Sig(\mathcal{O}b)_i(\Gamma) \to \mathcal{O}b_{i+1}(\Gamma)$.

Conversely, by Lemma 1.3(2), the formula

$$[\mathbf{2016.09.01.eq5}] Sig\mathcal{O}b_{i,\Gamma}^{-1}(X) = (ft^i(X), X) \tag{1.5}$$

defines a function $\mathcal{O}b_{i+1}(\Gamma) \to Sig(\mathcal{O}b_i)(\Gamma)$.

If $\Phi = Sig\mathcal{O}b_{i,\Gamma}$ and $\Psi = Sig\mathcal{O}b_{i,\Gamma}^{-1}$ then

$$\Phi(\Psi(X)) = \Phi((ft^i(X), X)) = X$$

and

$$\Psi(\Phi(T, X)) = \Psi(X) = (ft^i(X), X) = (T, X)$$

where the last equality follows from the equality $T = ft^i(X)$. We conclude that $Sig\mathcal{O}b_{i,\Gamma}$ and $Sig\mathcal{O}b_{i,\Gamma}^{-1}$ are mutually inverse bijections.

It remains to verify that the family of bijections $SigOb_{i,\Gamma}$ parametrized by $\Gamma \in CC$ is a morphism of presheaves, that is, that for any $f : \Gamma' \to \Gamma$ and $(T, X) \in Sig(\mathcal{O}b_i)(\Gamma)$ we have

$$[\mathbf{2016.08.30.eq10}] \mathcal{O}b_{i+1}(f)(SigOb_{i,\Gamma}((T, X))) = SigOb_{i,\Gamma'}(Sig(\mathcal{O}b_i)(f)((T, X))) \tag{1.6}$$

We compute

$$\mathcal{O}b_{i+1}(f)(SigOb_{i,\Gamma}((T, X))) = f^*(X)$$

$$SigOb_{i,\Gamma'}(Sig(\mathcal{O}b_i)(f)((T, X))) = SigOb_{i,\Gamma'}(f^*(T), q(f, T)^*(X)) = q(f, T)^*(X)$$

and (1.6) follows from [**?**, Lemma 2.7]. This completes Construction 1.4.

As an immediate corollary of Construction 1.4 we obtain the fact that the family of functions (1.5) parametrized by $\Gamma \in CC$ is an isomorphism of presheaves that is the inverse isomorphism to $SigOb_i$.

Problem 1.5 *[***2016.08.30.prob2***] For $i \geq 1$ to construct an isomorphism of presheaves*

$$Sig\widetilde{Ob}_i : Sig(\widetilde{\mathcal{O}b}_i) \to \widetilde{\mathcal{O}b}_{i+1}$$

Construction 1.6 [**2016.09.01.constr2**] For $\Gamma \in CC$ we have

$$Sig(\widetilde{\mathcal{O}b}_i)(\Gamma) = \{(T, o) \,|\, T \in \mathcal{O}b_1(\Gamma),\ o \in \widetilde{\mathcal{O}b}_i(T)\}$$

For $(T, o) \in Sig(\widetilde{\mathcal{O}b}_i)(\Gamma)$ we have $o \in \widetilde{\mathcal{O}b}_{i+1}(\Gamma)$. Indeed, $\partial(o) \in \widetilde{Ob}_i(T)$ and, by Lemma 1.3(1), $\partial(o) \in \widetilde{Ob}_{i+1}(\Gamma)$. Since $i + 1 > 0$ we have $\partial(o) > \Gamma$ and therefore $o \in \widetilde{\mathcal{O}b}_{i+1}(\Gamma)$ and the formula

$$[\mathbf{2016.09.01.eq6}] Sig\widetilde{Ob}_{i,\Gamma}(T, o) = o \tag{1.7}$$

defines a function $Sig(\widetilde{\mathcal{O}b}_i)(\Gamma) \to \widetilde{\mathcal{O}b}_i(\Gamma)$.

If $o \in \widetilde{\mathcal{O}b}_{i+1}(\Gamma)$ then $\partial(o) \in \mathcal{O}b_{i+1}(\Gamma)$ and, by Lemma 1.3(2), $ft^i(\partial(o)) \in \widetilde{Ob}_1(\Gamma)$ and $\partial(o) \in \widetilde{Ob}_i(ft^i(\partial(o)))$. If $i > 0$ we also have $\partial(o) > ft^i(\partial(o))$ and therefore the formula

$$[\mathbf{2016.09.01.eq7}] Sig\widetilde{Ob}_{i,\Gamma}^{-1}(o) = (ft^i(\partial(o)), o) \tag{1.8}$$

defines a function $\widetilde{\mathcal{O}b}_i(\Gamma) \to Sig(\widetilde{\mathcal{O}b}_i)(\Gamma)$.

One verifies in the same way as in Construction 1.4 that $Sig\widetilde{Ob}_{i,\Gamma}$ and $Sig\widetilde{Ob}_{i,\Gamma}^{-1}$ are mutually inverse bijections.

It remains to verify that the family of functions $Sig\widetilde{Ob}_{i,\Gamma}$ parametrized by $\Gamma \in CC$ is a morphism of functors, that is, that for $f : \Gamma' \to \Gamma$ and $(T, o) \in Sig\mathcal{O}b_{i,\Gamma}$ one has

$$[\mathbf{2016.09.01.eq2b}] \widetilde{\mathcal{O}b}_{i+1}(f)(Sig\widetilde{Ob}_{i,\Gamma}(T, o)) = Sig\widetilde{Ob}_{i,\Gamma'}(Sig(\widetilde{\mathcal{O}b}_i)(f)(T, o)) \tag{1.9}$$

Computing we get

$$\widetilde{\mathcal{O}b}_{i+1}(f)(Sig\widetilde{Ob}_{i,\Gamma}(T, o)) = \widetilde{\mathcal{O}b}_{i+1}(f)(o) = f^*(o)$$

$$Sig\widetilde{Ob}_{i,\Gamma'}(Sig(\widetilde{\mathcal{O}b}_i)(f)(T, o)) = Sig\widetilde{Ob}_{i,\Gamma'}(f^*(T), q(f, T)^*(o)) = q(f, T)^*(o)$$

and we conclude that (1.9) holds by (**??**).

This completes Construction 1.6.

As an immediate corollary of Construction 1.6 we obtain the fact that the family of functions (1.8) parametrized by $\Gamma \in CC$ is an isomorphism of presheaves that is the inverse isomorphism to $Sig\widetilde{Ob}_i$.

Define Sig^i by induction setting $Sig^0 = Id_{PreShv(CC)}$ and $Sig^{i+1} = Sig^i \circ Sig$. Then, by induction, we obtain an isomorphism

$$Sig\mathcal{O}b^i : Sig^i(\mathcal{O}b_1) \to \mathcal{O}b_{i+1}$$

where $Sig\mathcal{O}b^0 = Id_{\mathcal{O}b_1}$ and $Sig\mathcal{O}b^{i+1}$ is the composition

$$Sig^{i+1}(\mathcal{O}b_1) = Sig^1(Sig^i(\mathcal{O}b_1)) \xrightarrow{Sig^1(Sig\mathcal{O}b^i)} Sig^1(\mathcal{O}b_{i+1}) = Sig(\mathcal{O}b_{i+1}) \xrightarrow{Sig\mathcal{O}b_{i+1}} \mathcal{O}b_{i+2}$$

In exactly the same way we construct isomorphisms

$$Sig\widetilde{\mathcal{O}b}^i : Sig^i(\widetilde{\mathcal{O}b_1}) \to \widetilde{\mathcal{O}b_{i+1}}$$

2 Functors D_p

Let $\mathcal{C}$ be a category with a universe p. For any $\mathcal{G} \in PreShv(\mathcal{C})$ we define contravariant functor data $D_p(\mathcal{G})$ from $\mathcal{C}$ to $Sets$ on objects by

$$[\mathbf{2016.08.30.eq4}] D_p(\mathcal{G})(X) := \amalg_{F:X\to U}\mathcal{G}((X;F)) \tag{2.1}$$

and on morphisms, for $f : X' \to X$, by

$$[\mathbf{2016.08.30.eq5}] D_p(\mathcal{G})(f) : (F, g) \mapsto (f \circ F, \mathcal{G}(Q(f, F))(g)) \tag{2.2}$$

Lemma 2.1 *[**2016.08.26.l2**] The data described above defines a presheaf of sets, that is, one has:*

1. *for $X \in \mathcal{C}$,*
$$D_p(\mathcal{G})(Id_X) = Id_{D_p(\mathcal{G})(X)}$$
2. *for $f' : X'' \to X'$, $f : X' \to X$ one has*
$$D_p(\mathcal{G})(f' \circ f) = D_p(\mathcal{G})(f) \circ D_p(\mathcal{G})(f')$$

Proof: For the identity we have

$$D_p(\mathcal{G})(Id_X)(F, g) = (Id_X \circ F, \mathcal{G}(Q(Id_X, F))(g)) = (F, g)$$

where the second equality is by (**??**) and for the composition we have

$$D_p(\mathcal{G})(f')(D_p(\mathcal{G}(f)(F, g))) = D_p(\mathcal{G})(f')(f \circ F, \mathcal{G}(Q(f, F))(g)) =$$

$$(f' \circ f \circ F, \mathcal{G}(Q(f', f \circ F))(\mathcal{G}(Q(f, F))(g))) = (f' \circ f \circ F, \mathcal{G}(Q(f', f \circ F) \circ Q(f, F))(g)) =$$
$$(f' \circ f \circ F, \mathcal{G}(Q(f' \circ f, F))(g)) = D_p(\mathcal{G})(f' \circ f)(F, g)$$

where the first two equalities are by definition of $D_p(\mathcal{G})$, the third by the composition property of $\mathcal{G}$, the fourth by (**??**) and the fifth again by the definition of $D_p(\mathcal{G})$.

This completes the proof of Lemma 2.1.

The sets $D_p(-)(X)$ are also covariantly functorial in $r : \mathcal{G} \to \mathcal{G}'$ according to the formula

$$[\mathbf{2016.08.30.eq6}] D_p(r)_X(F, g) = (F, r_{(X;F)}(g)) \tag{2.3}$$

For $r : \mathcal{G} \to \mathcal{G}'$ and $f : X \to X'$, we have

$$D_p(\mathcal{G})(f) \circ D_p(r)_X = D_p(r)_{X'} \circ D_p(\mathcal{G}')(f)$$

that is, the families of functions $D_p(r)_X$ parametrized by $X \in \mathcal{C}$ form a morphism of presheaves.

For $\mathcal{G} \in PreShv(\mathcal{C})$ we have

$$D_p(Id_{\mathcal{G}})_X(F,g) = (F,(Id_{\mathcal{G}})_{(X;F)}(g)) = (F,g)$$

and for $r : \mathcal{G} \to \mathcal{G}'$, $r' : \mathcal{G}' \to \mathcal{G}''$ we have

$$D_p(r \circ r')_X(F,g) = (F,(r \circ r')_{(X;F)}(g)) = (F, r'_{(X;F)}(r_{(X;F)}(g))) = D_p(r')(D_p(r)(F,g))$$

These two equalities show that the covariant functor data from $PreShv$ to $PreShv$ given by D_p on presheaves and D_p on morphisms of presheaves is a functor that we also denote by

$$D_p : PreShv(\mathcal{C}) \to PreShv(\mathcal{C})$$

This functor is defined on any category $\mathcal{C}$ with a universe morphism p.

3 Isomorphisms SD_p

Problem 3.1 [**2016.08.28.prob1**] *For a universe category $(\mathcal{C}, p)$ to construct an isomorphism of functors $PreShv(\mathcal{C}) \to PreShv(CC)$ of the form*

$$SD_p : int^{\circ} \circ Sig \to D_p \circ int^{\circ}$$

Construction 3.2 [**2016.08.28.constr1**] We have to construct, for any $\mathcal{G} \in PreShv(\mathcal{C})$, an isomorphism of presheaves on CC of the form

$$SD_{p,\mathcal{G}} : Sig(int \circ \mathcal{G}) \to int \circ D_p(\mathcal{G})$$

and to show that these isomorphisms are natural in $\mathcal{G}$, that is, that for a morphism of presheaves $r : \mathcal{G} \to \mathcal{G}'$ one has

$$SD_{p,\mathcal{G}} \circ int^{\circ}(D_p(r)) = Sig(int^{\circ}(r)) \circ SD_{p,\mathcal{G}'}$$

That means that for any $\mathcal{G}$ and any $\Gamma \in CC$ we need to construct a bijection $SD_{p,\mathcal{G},\Gamma}$, which we will denote $\phi_{\mathcal{G},\Gamma}$ for the duration of the proof, of the form

$$\phi_{\mathcal{G},\Gamma} : Sig(int \circ \mathcal{G})(\Gamma) \to (int \circ D_p(\mathcal{G}))(\Gamma) = D_p(\mathcal{G})(int(\Gamma))$$

and to show that two conditions hold:

1. for any $f : \Gamma' \to \Gamma$ we have

$$[\mathbf{2016.08.30.eq1}]\phi_{\mathcal{G},\Gamma} \circ D_p(\mathcal{G})(int(f)) = Sig(int \circ \mathcal{G})(f) \circ \phi_{\mathcal{G},\Gamma'} \tag{3.1}$$

2. for any $r : \mathcal{G} \to \mathcal{G}'$ and $\Gamma \in CC$ we have

$$[\mathbf{2016.08.30.eq2}]\phi_{\mathcal{G},\Gamma} \circ D_p(r)_{int(\Gamma)} = Sig(int^{\circ}(r))_{\Gamma} \circ \phi_{\mathcal{G}',\Gamma} \tag{3.2}$$

Both equations are easier to see as the commutativity of the corresponding squares. To construct $\phi_{\mathcal{G},\Gamma}$ we first compute using (2.1) and (1.1)

$$D_p(\mathcal{G})(int(\Gamma)) = \amalg_{F:int(\Gamma)\to U}\mathcal{G}((int(\Gamma);F))$$

and

$$Sig(int\circ\mathcal{G})(\Gamma) = \amalg_{T\in\mathcal{O}b_1(\Gamma)}\mathcal{G}(int(T))$$

We define the function $\phi_{\mathcal{G},\Gamma}$ by the formula

$$[\mathbf{2016.09.01.eq3}]\phi_{\mathcal{G},\Gamma}((T,g)) = (u_{1,\Gamma}(T),g) \tag{3.3}$$

the right hand side is defined because of (**??**) and the function $\phi_{\mathcal{G},\Gamma}$ is a bijection as the total function of the bijection $u_{1,\Gamma}$ and the family of bijections, namely the identity functions.

To prove equality (3.1) we again first compute using (2.2) and (1.2)

$$D_p(\mathcal{G})(int(f))(F,g) = (int(f)\circ F, \mathcal{G}(Q(int(f),F))(g))$$

$$Sig(int\circ\mathcal{G})(f)(T,g) = (f^*(T), \mathcal{G}(int(q(f,T)))(int(T)))$$

Equality (3.1) follows now from (**??**) and (**??**).

To prove equality (3.2) we compute using (2.3) and (1.3)

$$D_p(r)_{int(\Gamma)}(F,g) = (F, r_{(int(\Gamma);F)}(g))$$

$$Sig(int^{\circ}(r))_{\Gamma}(T,g) = (T, r_{int(T)}(g))$$

and (3.2) follows from (**??**).

This completes Construction 3.2.

4 Appendix: presheaves of relative C-systems

Let $\mathcal{O}b(\Gamma)$ denote the set of $X\in CC$ such that $X\geq\Gamma$.

We equip the set $\mathcal{O}b(\Gamma)$ with the length function given by $l_\Gamma(X) = l(X)-l(\Gamma)$ and let

$$\mathcal{O}b_n(\Gamma) = \{X\in\mathcal{O}b(\Gamma)\,|\,l_\Gamma\}$$

We let $\mathcal{M}or(\Gamma)$ denote the set of morphisms over Γ in the sense of Definition **??**.

We equip the set $\mathcal{O}b(\Gamma)$ with the ft-function given by

$$ft_\Gamma(X) = \begin{cases} ft(X) & \text{if } l(X)>l(\Gamma)\\ X & \text{if } l(X)=l(\Gamma)\end{cases}$$

One defines domain, codomain, identity and composition functions on the pair of sets $\mathcal{O}b(\Gamma)$, $\mathcal{M}or(\Gamma)$ as the restrictions of the corresponding functions for CC. That these restrictions are defined for the identity and composition follows from Lemma **??**.

For $X \in \mathcal{O}b(\Gamma)$ define $p^{\Gamma}_X : X \to ft_{\Gamma}(X)$ as follows

$$p^{\Gamma}_X = \begin{cases} p_X & \text{if } l(X) > l(\Gamma) \\ Id_X & \text{if } l(X) = l(\Gamma) \end{cases}$$

As was mentioned above, p_X is a morphism over Γ if $X > \Gamma$. Therefore, $p^{\Gamma}_X \in \mathcal{M}or(\Gamma)$.

For $f : X \to ft(Y)$ over Γ and Y such that $l(Y) > 0$, $q(f, Y)$ is a morphism over Γ by Lemma **??**(3) and one defines $q_{\Gamma}(f, Y)$ as $q(f, Y)$. Finally, for $a : X \to Y$ over Γ such that $Y > \Gamma$, s_a is a morphism over Γ by Lemma **??** and one defines s^{Γ}_a as s_a.

Some of the previous results can be combined into the following theorem that was stated without a proof in [**?**, pp. 240-241] and with a proof in [**?**, Theorem 1.4, p.52].

Theorem 4.1 *(cf. [**?**, Theorem 1.4, p.52])[**2016.04.07.th1**] The structures specified above define a C-system structure on the pair of sets $\mathcal{O}b(\Gamma)$, $\mathcal{M}or(\Gamma)$.*

Proof: Since $\mathcal{O}b(\Gamma)$ and $\mathcal{M}or(\Gamma)$ are subsets in $Ob(CC)$ and $Mor(CC)$ respectively the associativity and the identity axioms for the identities and compositions over Γ follow from the corresponding axioms for CC.

Verification of the conditions of [**?**, Definitions 2.1, 2.3] is completely straightforward.

We denote the C-system of Theorem 4.1 by $CC(\Gamma)$ such that we have

$$Ob(CC(\Gamma)) = \mathcal{O}b(\Gamma)$$

$$Mor(CC(\Gamma)) = \mathcal{M}or(\Gamma)$$

A detailed definition of a homomorphism of C-systems is given in [**?**, Definition 3.1].

Theorem 4.2 *(cf. [**?**, Theorem 1.6, p.55]) [**2015.06.15.th1**] The functions*

$$f^* : \mathcal{O}b(\Gamma) \to \mathcal{O}b(\Gamma')$$

$$f^* : \mathcal{M}or(\Gamma) \to \mathcal{M}or(\Gamma')$$

*corresponding to a morphism $f : \Gamma' \to \Gamma$ and defined above Lemma **??** and in Lemma **??** respectively, form a homomorphism of C-systems.*

Proof: The commutation with the length function is (**??**). The commutation with the ft follows from (**??**) and the fact that $f^*(\Gamma) = \Gamma$.

The commutation with the domain and codomain functions are automatic. The commutation with the identities is (**??**) and the commutation with compositions is (**??**). The commutation with p-morphisms follows from (**??**). The the commutation with q-morphisms is shown in Lemma **??**. The commutation with s-morphisms is (**??**).

To emphasize the difference between f^* on objects and on morphisms and to keep in sync with the notations that we use for the object and morphism components of a functor we may sometimes write f^*_{Ob} and f^*_{Mor}.

The content of the following theorem is very close to the content of [**?**, Theorem 1.7, p. 60]. For a universe U that is an element of our ambient universe UU let $Csys(U)$ be the category of C-systems in U, that is, the category whose set of objects is the set of C-systems CC such that $Ob(CC)$ and $Mor(CC)$ are elements of U and whose morphisms are homomorphisms of C-systems.

Theorem 4.3 *[**2016.05.06.th1**] Let U be a universe such that for $X \in U$ and $Y \subset X$ one has $Y \in U$ and let CC be a C-system in U. Then the family of C-systems $CC(\Gamma)$ and homomorphisms of C-systems f^* defined by Theorems 4.1 and 4.2 form a presheaf (a contravariant functor) on the category underlying CC with values in $Csys(U)$.*

Proof: Since two homomorphisms of C-systems are equal when the underlying functions on the sets of objects and morphisms are equal we need to show that:

1. for $\Gamma \in Ob(CC)$ and $X \in \mathcal{O}b(\Gamma)$ one has $Id^*_{\Gamma}(X) = X$ and for $a \in \mathcal{M}or(\Gamma)$ one has $Id^*_{\Gamma}(a) = a$,
2. for $g : \Gamma'' \to \Gamma'$, $f : \Gamma' \to \Gamma$ and $X \in \mathcal{O}b(\Gamma)$ one has $g^*(f^*(X)) = (g \circ f)^*(X)$ and for $a \in \mathcal{M}or(\Gamma)$ one has $g^*(f^*(a)) = (g \circ f)^*(a)$.

It follows from Lemmas **??** and **??**.

We do not introduce a special notation for the presheaf defined by Theorem 4.3. We will be mostly concerned with its objects part that we denote by

$$\mathcal{O}b_{CC} : CC \to Sets(U)$$

and often write as $\mathcal{O}b$ when CC is clear from the context.

Note that in the context of a given C-system CC the notation $\mathcal{O}b$ refers to a presheaf of sets on CC while the notation Ob refers to the set of objects of CC.

Let CC be a C-system. Recall that $\widetilde{\mathcal{O}b}(CC)$ is the subset in $Mor(CC)$ of elements o such that

$$[\mathbf{2016.05.16.eq3}] l(codom(o)) > 0 \tag{4.1}$$

$$[\mathbf{2016.05.16.eq4}] ft(codom(o)) = dom(o) \tag{4.2}$$

$$[\mathbf{2016.05.16.eq5}] o \circ p_{codom(o)} = Id_{dom(o)} \tag{4.3}$$

and that for $o \in \widetilde{\mathcal{O}b}(CC)$ we let $\partial(o)$ denote $codom(o)$.

Lemma 4.4 *[**2016.05.12.l1**] Let $F : CC_1 \to CC_2$ be a homomorphism of C-systems. Then for $o \in \widetilde{\mathcal{O}b}(CC_1)$ one has $F(o) \in \widetilde{\mathcal{O}b}(CC_2)$.*

Proof: We need to verify the conditions $l(codom(F(o))) > 0$, $dom(F(o)) = ft(codom(F(o)))$ and $F(o) \circ p_{codom(F(o))} = Id_{dom(F(o))}$. All three follow immediately from the definition of a homomorphism of C-systems and the conditions (4.1)-(4.3) satisfied by o.

Lemma 4.4 shows that the restriction of the morphism component of a homomorphism of C-systems F defines a function $F_{\widetilde{Ob}} : \widetilde{Ob}(CC_1) \to \widetilde{Ob}(CC_2)$ that we will often write simply as F.

Theorem 4.5 *[***2016.05.12.th1***] Let U be a universe as above. Then the family*[3] *of sets $\widetilde{Ob}(CC)$ parametrized by $CC \in Csys(U)$ together with functions $F_{\widetilde{Ob}}$ defined by Lemma 4.4 form a covariant functor from $Csys(U)$ to $Sets(U)$.*

Proof: It follows from the fact that $\widetilde{Ob}$ is a subset of Mor and the definition of the identity homomorphisms and the composition of homomorphisms of C-systems.

For Γ in CC we will abbreviate the notation $\widetilde{Ob}(CC(\Gamma))$ to $\widetilde{Ob}(\Gamma)$.

Lemma 4.6 *[***2016.05.16.l1***] Let $\Gamma \in CC$. Then one has*

$$[\mathbf{2016.05.16.eq1}]\widetilde{Ob}(\Gamma) = \mathcal{M}or(\Gamma) \cap \widetilde{Ob}(CC) \tag{4.4}$$

Proof: Note first that all three sets appearing in (4.4) are subsets of $Mor(CC)$.

We need to prove three inclusions

$$[\mathbf{2016.05.16.eq6}]\widetilde{Ob}(\Gamma) \subset \mathcal{M}or(\Gamma) \tag{4.5}$$

$$[\mathbf{2016.05.16.eq7}]\widetilde{Ob}(\Gamma) \subset \widetilde{Ob}(CC) \tag{4.6}$$

$$[\mathbf{2016.05.16.eq8}]\mathcal{M}or(\Gamma) \cap \widetilde{Ob}(CC) \subset \widetilde{Ob}(\Gamma) \tag{4.7}$$

The inclusion (4.5) is by definition.

For $o \in Mor(CC)$ to be an element of $\widetilde{Ob}(CC)$ we need it to satisfy the conditions (4.1)-(4.3). Since $o \in \widetilde{Ob}(\Gamma)$ we have

$$[\mathbf{2016.05.16.eq9}]l_\Gamma(codom(o)) > 0 \qquad ft_\Gamma(codom(o)) = dom(o) \qquad o \circ p^\Gamma_{codom(o)} = Id_{dom(o)} \tag{4.8}$$

Since $l(X) \geq l_\Gamma(X)$ for any X over Γ, the inequality implies (4.1). The two equalities imply (4.2) and (4.3) since for X over Γ such that $l_\Gamma(X) > 0$ we have $ft_\Gamma(X) = ft(X)$ and $p^\Gamma_X = p_X$. It proves (4.6).

[3]By a family of sets parametrized by X we mean a function from X to the universe. Given a family of sets $F : X \to U$, a family of elements of sets $F(x)$ parametrized by X is a function G from X to $\cup_{x \in X} F(x)$ such that for all $x \in X$ one has $G(x) \in F(x)$.

Let $o \in \mathcal{M}or(\Gamma) \cap \widetilde{\mathcal{O}b}(CC)$. Now we need to prove (4.8). Since $l(codom(o)) > 0$ we have $l(dom(o)) = l(ft(codom(o))) = l(codom(o)) - 1$. Since o is over Γ, $dom(o) \geq \Gamma$ and in particular $l(dom(o)) \geq l(\Gamma)$. Therefore,

$$l_\Gamma(codom(o)) = l(codom(o)) - l(\Gamma) = 1 + (l(dom(o)) - l(\Gamma)) \geq 1 > 0$$

The two equalities in (4.8) follow from (4.2) and (4.3) since for X over Γ such that $l_\Gamma(X) > 0$ we have $ft_\Gamma(X) = ft(X)$ and $p^\Gamma_X = p_X$. It proves (4.7) and completes the proof of the lemma.

Lemma 4.7 *[**2016.05.16.l2**] An element o of $\widetilde{\mathcal{O}b}(CC)$ is a morphism over Γ if and only if $dom(o) \geq \Gamma$.*

The set $\widetilde{\mathcal{O}b}(\Gamma)$ is the subset of elements $o \in \widetilde{\mathcal{O}b}(CC)$ such that $dom(o) \geq \Gamma$.

Proof: The "only" if part of the first assertion follows from the fact that the domain of a morphism over Γ is an object over Γ.

If $dom(o) \geq \Gamma$ then

$$codom(o) \geq ft(codom(o)) = dom(o) \geq \Gamma$$

that is, $codom(o)$ is an object over Γ. Next we have

$$o \circ p(codom(o), \Gamma) = o \circ p_{codom(o)} \circ p(ft(codom(o), \Gamma) = o \circ p_{codom(o)} \circ p(dom(o), \Gamma) =$$

$$Id_{dom(o)} \circ p(dom(o), \Gamma) = p(dom(o), \Gamma)$$

that is, o is a morphism over Γ. It proves the "if" part of the first assertion.

The second assertion follows from the first and Lemma 4.6.

Lemma 4.8 *[**2016.05.16.l3**] Let $o \in \widetilde{\mathcal{O}b}(\Gamma)$. Then o is a morphism over Γ and for $f : \Gamma' \to \Gamma$ one has $f^*(o) \in \widetilde{\mathcal{O}b}(\Gamma')$.*

Proof: That o is a morphism over Γ follows from the definition $\widetilde{\mathcal{O}b}(\Gamma) = \widetilde{\mathcal{O}b}(CC(\Gamma))$. By Theorem 4.2, (f^*_{Ob}, f^*_{Mor}) form a homomorphism of C-systems. By Lemma 4.4 a homomorphism of C-systems takes $\widetilde{\mathcal{O}b}$ to $\widetilde{\mathcal{O}b}$.

To keep in sync with the notations f^*_{Ob} and f^*_{Mor} for the functions f^* on objects and morphisms we will sometimes use the notation $f^*_{\widetilde{\mathcal{O}b}}$ for the function $\widetilde{\mathcal{O}b}(\Gamma) \to \widetilde{\mathcal{O}b}(\Gamma')$ that is defined in view of Lemma 4.8.

Theorem 4.9 *[**2016.05.06.l1**] Let U be universe as above and CC be a C-system in U. Then the family of sets $\widetilde{\mathcal{O}b}(\Gamma)$ parametrized by $\Gamma \in CC$ together with the family of functions $f^*_{\widetilde{\mathcal{O}b}}$ is a presheaf (contravariant functor) on the category underlying CC with values in $Sets(U)$.*

Proof: Since $\widetilde{\mathcal{O}b}(CC)$ is a subset in $Mor(CC)$ it follows from Lemma **??**.

We denote the presheaf defined by Theorem 4.9 by $\widetilde{\mathcal{O}b}_{CC}$ or, when CC is clear from the context, by $\widetilde{\mathcal{O}b}$.

As in the case of $\mathcal{O}b$ and Ob, in the context of a given C-system CC, we have a presheaf of sets $\widetilde{\mathcal{O}b}$ and a set $\widetilde{\mathcal{O}b}$.

The function $\partial : \widetilde{\mathcal{O}b}(CC) \to Ob(CC)$ takes elements of $\widetilde{\mathcal{O}b}(\Gamma)$ to elements of $\mathcal{O}b(\Gamma)$. We will sometimes denote the resulting functions $\widetilde{\mathcal{O}b}(\Gamma) \to \mathcal{O}b(\Gamma)$ by ∂_Γ. Since ∂ is simply the restriction of *codom* to $\widetilde{\mathcal{O}b}$ it commutes with the functions $F_{\widetilde{\mathcal{O}b}}, F_{Ob}$ for homomorphisms of C-systems. In particular, the functions ∂_Γ commute with the functions $f^*_{\widetilde{\mathcal{O}b}}, f^*_{\mathcal{O}b}$ and form a homomorphism of presheaves $\partial : \widetilde{\mathcal{O}b}_{CC} \to \mathcal{O}b_{CC}$.

We will also consider the sets $Ob_n(CC)$ of objects of length n in $Ob(CC)$, the sets $\mathcal{O}b_n(\Gamma)$ of objects of length n in $\mathcal{O}b(\Gamma)$, the sets $\widetilde{\mathcal{O}b}_n(CC)$ of elements o in $\widetilde{\mathcal{O}b}(CC)$ such that $l(\partial(o)) = n$ and the sets $\widetilde{\mathcal{O}b}_n(\Gamma)$ of elements o in $\widetilde{\mathcal{O}b}(\Gamma)$ such that $l_\Gamma(\partial(o)) = n$.

The sets $Ob_n(CC)$ and $\widetilde{\mathcal{O}b}_n(CC)$ are mapped to the corresponding sets for CC' by homomorphisms $CC \to CC'$ of C-systems forming covariant functors from $Csys(U)$ to $Sets(U)$ for appropriate universes U.

In particular, the sets $\mathcal{O}b_n(\Gamma)$ and $\widetilde{\mathcal{O}b}_n(\Gamma)$ are mapped to the sets $\mathcal{O}b_n(\Gamma')$ and $\widetilde{\mathcal{O}b}_n(\Gamma')$ by the functions $f^*_{\mathcal{O}b}$ and $f^*_{\widetilde{\mathcal{O}b}}$ forming presheaves (contravariant functors) on CC with values in $Sets(U)$ for appropriate universes U.

The function ∂ maps elements of $\widetilde{\mathcal{O}b}_n$ to elements of $\widetilde{\mathcal{O}b}_n$ defining functions $\widetilde{\mathcal{O}b}_n \to Ob_n$ that we will sometimes denote by ∂_n. Similarly, the functions ∂_Γ map elements of $\widetilde{\mathcal{O}b}_n(\Gamma)$ to elements of $\mathcal{O}b_n(\Gamma)$ defining functions $\widetilde{\mathcal{O}b}_n(\Gamma) \to \mathcal{O}b_n(\Gamma)$ that we will sometimes denote $\partial_{\Gamma,n}$. More often we will write ∂_n and $\partial_{\Gamma,n}$ simply as ∂.

The proofs of all these facts are straightforward. Finally, we want to make a comment that will be important for the proper interpretation of some of our constructions.