B-systems[1]

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Abstract

B-systems are algebras (models) of an essentially algebraic theory that is expected to be constructively equivalent to the essentially algebraic theory of C-systems which is, in turn, constructively equivalent to the theory of contextual categories. The theory of B-systems is closer in its form to the structures directly modeled by contexts and typing judgements of (dependent) type theories and further away from categories than contextual categories and C-systems.

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1   Lft-sets, pre-B-systems and B0-systems

1   Lft-sets

Let us start with the definition of lft-sets. For two natural numbers m, n define

$$m -_{\mathbf{N}} n = max(m - n, 0).$$

Definition 1.1 *[2016.01.27.def1] An lft-set is a collection of data of the following form:*

   1. a set B,

   2. a function $l : B \to \mathbf{N}$,

   3. a function $ft : B \to B$

such that for all $X \in B$ one has $l(ft(X)) = l(X) -_{\mathbf{N}} 1$.

An lft-set is called pointed if the set $\{X \in B, l(X) = 0\}$ is a one element set. In this case the only element of this set is usually denoted by pt.

Lemma 1.2 [2016.02.18.l2] *Let B be an lft-set, $X \in B$ and $n \in \mathbf{N}$. Then*

$$l(ft^n(X)) = l(X) -_{\mathbf{N}} n$$

Proof: Obvious induction on n.

For an lft-set B, define the relation $\geq$ on B by the condition that $Y \geq X$ if and only if $l(Y) \geq l(X)$ and

$$X = ft^{l(Y)-l(X)}(Y).$$

Define the relation $>$ on B by the condition that $Y > X$ if and only if $Y \geq X$ and $l(Y) > l(X)$.

Lemma 1.3 [2016.01.27.l1] *For any lft-set B one has:*

1. *the relation $\geq$ is a partial order relation, i.e., it is reflexive, transitive and antisymmetric,*

2. *the relation $>$ is a strict partial order relation, i.e., it is transitive and asymmetric.*

Proof: Straightforward using the corresponding properties of the relations $\geq$ and $>$ on $\mathbf{N}$ and properties of $-_{\mathbf{N}}$.

Lemma 1.4 [2016.02.22.l2] *Let B be an lft-set. The following mixed transitivities hold:*

1. *if $Z > Y$ and $Y \geq X$ then $Z > X$,*

2. *if $Z \geq Y$ and $Y > X$ then $Z > X$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ and $>$ on $\mathbf{N}$.

Lemma 1.5 [2016.02.22.l3] *Let B be an lft-set, $Y \geq X$ in B and $i \in \mathbf{N}$. Then one has:*

1. *if $l(Y) \geq i + l(X)$ then $ft^i(Y) \geq X$,*

2. *if $l(Y) > i + l(X)$ then $ft^i(Y) > X$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ and $>$ on $\mathbf{N}$.

Lemma 1.6 [2016.01.27.l6] *Let B be an lft-set and $Y > X$ in B. Then $ft(Y) \geq X$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ on $\mathbf{N}$.

Lemma 1.7 [2016.01.29.l3] *Let B be an lft-set, $X \in B$ and $n \in \mathbf{N}$. Then $X \geq ft^n(X)$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ on $\mathbf{N}$.

Lemma 1.8 [2016.01.29.l2] *Let B be an lft-set, $X \in B$, $n > 0$ and $l(ft^n(X)) > 0$. Then $X > ft^n(X)$.*

Proof: From Lemma 1.7 we know that $X \geq ft^n(X)$. It remains to show that $l(X) > l(ft^n(X))$. By Lemma 1.2, $l(ft^n(X)) = max(l(X) - n, 0)$ which implies that under the condition of the lemma $l(ft^n(X)) = l(X) - n$ and since $n > 0$ we have that $l(X) > l(ft^n(X))$.

Definition 1.9 *[2016.01.27.def2] Let B, B' be lft-sets. A morphism of lft-sets $f : B \to B'$ is a function $f : B \to B'$ such that for all $X \in B$ one has $l(f(X)) = l(X)$ and $l(ft(X)) = ft(l(X))$.*

We let $Mor_{lft}(B, B')$ denote the set of morphisms of lft-sets from B to B'.

Lemma 1.10 *[2016.03.15.l5] Let $f : B \to B'$ be a morphism of lft-sets, $X \in B$ and $j \in \mathbf{N}$. Then one has*

$$f(ft^j(X)) = ft^j(f(X))$$

Proof: By induction on j.

Lemma 1.11 *[2017.01.27.l3] Let $f : B \to B'$ be a morphism of lft-sets and $X, Y \in B$. Then one has:*

  1. if $Y \geq X$ then $f(Y) \geq f(X)$,

  2. if $Y > X$ one has $f(Y) > f(X)$,

Proof: Straightforward from Lemma 1.10.

Lemma 1.12 *[2016.01.27.l2] One has:*

  1. for any lft-set B the identity function $Id_B : B \to B$ is a morphism of lft-sets,

  2. for any lft sets B, B', B'' and morphisms $f : B \to B'$, $f' : B' \to B''$ the composition of functions $f \circ f'$ is a morphism of lft-sets.

Proof: Straightforward using the properties of $-_{\mathbf{N}}$.

Let $lft(U)$ be the set of lft-sets in the universe U.

Problem 1.13 *[2016.01.27.prob1] Let U be a universe. To construct a category $LFT(U)$ with the set of objects $lft(U)$.*

Construction 1.14 *[2016.01.27.constr1a]*We define

$$Ob(LFT(U)) = lft(U)$$

$$Mor(LFT(U)) = \amalg_{B,B' \in lft(U)} Mor_{lft}(B, B')$$

with the obvious domain and codomain functions and the identity function and the composition function being defined from the identity and composition of functions between sets using Lemma 1.12.

The proofs of the associativity and the identity axioms of a category are straightforward.

We can not use $\cup$ in this definition instead of $\amalg$ because the sets $Mor(B, B')$ need not be disjoint for different B, B'. For example, if B' has one element of each length then the set $Mor(B, B')$ depends on the set B and the length function l but is independent on the ft function on B.

Therefore there is no category with the set of objects $lft(U)$ and the set of morphisms between any two lft-sets being the set Mor_{lft} of Definition 1.9. Instead in our category the set of morphisms from B to B' is the set of iterated pairs of the form $((B, B'), f)$ where f is a function $B \to B'$ that satisfies the conditions of Definition 1.9. This set is in the obvious bijective correspondence with the set of morphisms from B to B' and we will use both directions of this bijection as coercions - if an element of $Mor_{LFT(U)}(B, B')$ occurs in a position where an element of $Mor_{lft}(B, B')$ should be it is replaced by its image in $Mor_{lft}(B, B')$ under the corresponding function of the bijection and vice versa.

This completes Construction 2.2.

In what follows we fix a universe and write LFT instead of $LFT(U)$ and lft instead of $lft(U)$.

2 Pre-B-systems

Definition 2.1 [2016.01.27.def7] *A B-system carrier is a triple $(B, \widetilde{B}, \partial)$ where B is an lft-set, $\widetilde{B}$ is a set and $\partial : \widetilde{B} \to B$ is a function such that for all $r \in \widetilde{B}$ one has $l(\partial(r)) > 0$.*

Remark 2.2 [2016.03.31.rem2] Elements of a B-system carrier $\mathbf{B} = (B, \widetilde{B}, \partial)$ and connecting them relation $\leq$ can be shown diagrammatically as follows:

$$
\begin{array}{ccc}
Y & & \partial(r) \\
\downarrow & & \downarrow{\uparrow r} \\
X & & ft(\partial(r))
\end{array}
$$

where the first diagram shows a pair $X, Y \in B$ such that $X \leq Y$ and the second one an element $r \in \widetilde{B}$.

Definition 2.3 [2016.01.27.def3] *Let $(B, \widetilde{B}, \partial)$ be a B-system carrier. We set:*

$$T_{dom} = \{X, Y \in B, l(X) \geq 1, ft(X) < Y\} \quad \widetilde{T}_{dom} = \{X \in B, s \in \widetilde{B}, (X, \partial(s)) \in T_{dom}\}$$

$$S_{dom} = \{r \in \widetilde{B}, Y \in B, \partial(r) < Y\} \quad \widetilde{S}_{dom} = \{r, s \in \widetilde{B}, (r, \partial(s)) \in S_{dom}\}$$

$$\delta_{dom} = \{X \in B, l(X) \geq 1\}$$

Definition 2.4 [2014.10.10.def1] *A non-unital pre-B-system is a B-system carrier together with functions $T, \widetilde{T}, S$ and $\widetilde{S}$ of the form:*

$$T : T_{dom} \to B \quad \widetilde{T} : \widetilde{T}_{dom} \to \widetilde{B}$$

$$S : S_{dom} \to B \quad \widetilde{S} : \widetilde{S}_{dom} \to \widetilde{B}$$

Definition 2.5 [2014.10.20.def1] *A pre-B-system is a non-unital pre-B-system together with a function*

$$\delta : \delta_{dom} \to B$$

Definition 2.6 [2016.27.def8] *A morphism of B-system carriers* $\mathbf{f} : (B, \widetilde{B}, \partial) \to (B', \widetilde{B}', \partial')$ *is a pair* $(\widetilde{f}, f)$ *where* $\widetilde{f} : \widetilde{B} \to \widetilde{B}'$ *is a function,* $f : B \to B'$ *is a morphism of lft-sets and for any* $s \in \widetilde{B}$ *one has*

$$\partial'(\widetilde{f}(s)) = f(\partial(s))$$

Problem 2.7 [2016.01.27.prob9] *For a morphism of B-system carriers* $\mathbf{f} : (B, \widetilde{B}, \partial) \to (B', \widetilde{B}', \partial')$ *to construct functions*

$$f_T : T_{dom} \to T'_{dom} \quad f_{\widetilde{T}} : \widetilde{T}_{dom} \to \widetilde{T}'_{dom}$$

$$f_S : S_{dom} \to S'_{dom} \quad f_{\widetilde{S}} : \widetilde{S}_{dom} \to \widetilde{S}'_{dom}$$

$$f_\delta : \delta_{dom} \to \delta'_{dom}$$

Construction 2.8 [2016.01.27.constr8] For $(X, Y) \in T_{dom}$ we set $f_T(X, Y) = (f(X), f(Y))$. The condition that $f_T(X, Y) \in T'_{dom}$ follows immediately from the fact that f is an lft-set morphism and Lemma 1.11.

For $(X, s) \in \widetilde{T}_{dom}$ we set $f_{\widetilde{T}}(X, s) = (f(X), \widetilde{f}(s))$. The condition that $f_{\widetilde{T}}(X, s) \in \widetilde{T}'_{dom}$ follows immediately from the fact that $(f, \widetilde{f})$ is a morphism of B-system carriers and and Lemma 1.11.

The proofs for the remaining three subsets are equally easy corollaries of the definitions and Lemma 1.11.

Definition 2.9 [2016.01.27.def4] *Let* $\mathbf{B}, \mathbf{B}'$ *be pre-B-systems. A homomorphism of non-unital pre-B-systems* $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ *is a morphism* $\mathbf{f} = (f, \widetilde{f})$ *of the B-system carriers such that one has*

for $(X, Y) \in T_{dom}, \ f(T(X, Y)) = T'(f_T(X, Y)), \quad$ for $(X, s) \in \widetilde{T}_{dom}, \ \widetilde{f}(\widetilde{T}(X, s)) = \widetilde{T}'_{dom}(f_{\widetilde{T}}(X, s))$

for $(r, Y) \in S_{dom}, \ f(S(r, Y)) = S'(f_S(r, Y)) \quad$ for $(r, s) \in \widetilde{S}_{dom}, \ \widetilde{f}(\widetilde{S}(r, s)) = \widetilde{S}'(f_{\widetilde{S}}(r, s))$

A homomorphism of pre-B-systems is a morphism $\mathbf{f} = (f, \widetilde{f})$ *of B-system carriers that is a homomorphism of non-unital pre-B-systems and such that one has:*

$$\text{for } X \in \delta_{dom}, \ \widetilde{f}(\delta(X)) = \delta'(f(X))$$

Lemma 2.10 [2016.01.27.l5] *One has:*

1. *Let* $\mathbf{B}$ *be a non unital pre-B-system (resp. pre-B-system) and* $(Id_{\widetilde{B}}, Id_B)$ *be the identity morphism of the underlying pre-B-system carries. Then* $(Id_{\widetilde{B}}, Id_B)$ *is a homomorphism of non-unital pre-B-systems (resp. pre-B-systems).*

2. *Let* $\mathbf{B}, \mathbf{B}', \mathbf{B}''$ *be non-unital pre-B-systems (resp. pre-B-systems) and* $\mathbf{f} : \mathbf{B} \to \mathbf{B}', \mathbf{f}' : \mathbf{B}' \to \mathbf{B}''$ *be two homomorphism of non-unital pre-B-systems (resp. pre-B-systems). Then the composition of the underlying homomorphisms of B-system carriers is a homomorphism of non-unital pre-B-systems (resp. pre-B-systems).*

Proof: The proof is straightforward but long since all five conditions of Definition 2.9 have to be verified.

2.1   Construction of the sets $T_{ext,dom}$ and operations T_{ext}

Let
$$T_{ext,dom} = \{X, Y \in B, l(X) \geq 1, Y \geq ft(X)\}$$
Given a function $T : T_{dom} \to B$ let us define the extended version of T as the function
$$T_{ext} : T_{ext,dom} \to B$$
given by the rule:

1. if $Y = ft(X)$ then $T_{ext}(X, Y) = X$,

2. if $Y > ft(X)$ then $T_{ext}(X, Y) = T(X, Y)$.

2.2   Operations T_{ext} and homomorphisms of B-system carriers

Lemma 2.11 [2016.03.15.l1] *Let $f : B \to B'$ be a morphism of lft-sets. Then for $(X, Y) \in T_{ext,dom}$ one has $(f(X), f(Y)) \in T'_{ext,dom}$.*

Proof: Immediate from definitions.

Lemma 2.12 [2016.03.15.l2] *Let $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ be a homomorphism of pre-B-systems. Then for $(X, Y) \in T_{ext,dom}$ one has*
$$f(T_{ext}(X, Y)) = T_{ext}(f(X), f(Y))$$

Proof: Straightforward by case.

2.3   Construction of the sets $S_{ext,dom}$ and operations S_{ext}

Let
$$S_{ext,dom} = \{r \in \widetilde{B}, Y \in B, \partial(r) \leq Y\}$$
Given a function $S : S_{dom} \to B$ let us define the extended version of S as the function
$$S_{ext} : S_{ext,dom} \to B$$
given by the rule:

1. if $\partial(r) = Y$ then $S_{ext}(r, Y) = ft(\partial(r))$,

2. if $\partial(r) < Y$ then $S_{ext}(r, Y) = S(r, Y)$.

Lemma 2.13 [2016.04.14.l1] *For a B0-system $\mathbf{B}$ one has*
$$ft(\partial(r)) \leq S_{ext}(r, Y)$$

Proof: If $\partial(r) = Y$ then we have equality if $\partial(r) < Y$ then we have $ft(\partial(r)) < S_{ext}(r, Y)$ by the B0-system axiom.

2.4 Operations S_{ext} and homomorphisms of B-system carriers

Lemma 2.14 *[2016.03.27.l3] Let $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ be a homomorphism of pre-B-system carriers. Then for $(r, Y) \in S_{ext,dom}$ one has $(\widetilde{f}(r), f(Y)) \in S'_{ext,dom}$.*

Proof: Immediate from definitions.

Lemma 2.15 *[2016.03.27.l4] Let $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ be a homomorphism of pre-B-systems. Then for $(r, Y) \in S_{ext,dom}$ one has*

$$f(S_{ext}(r, Y)) = S_{ext}(\widetilde{f}(r), f(Y))$$

Proof: Straightforward by case.

3 B0-systems

The complex of axioms that define B0-systems among all pre-B-systems is as follows:

Definition 3.1 *[2014.10.16.def1.fromold] [2014.10.16.def1] [2016.01.29.def1] A non-unital pre-B-system is called a non-unital B0-system if the following conditions hold:*

1. *For $(X, Y) \in T_{dom}$ one has:*

   (a) $T(X, Y) > X$,
   (b) if $ft(Y) > ft(X)$ then $ft(T(X, Y)) = T(X, ft(Y))$,
   (c) $l(T(X, Y)) = l(Y) + 1$.

2. *For $(X, s) \in \widetilde{T}_{dom}$ one has:*

   (a) $\partial(\widetilde{T}(X, s)) = T(X, \partial(s))$,
   (b) $l(\partial(\widetilde{T}(X, s))) = l(\partial(s)) + 1$.

3. *For $(r, Y) \in S_{dom}$ one has:*

   (a) $S(r, Y) > ft(\partial(r))$,
   (b) if $ft(Y) > \partial(r)$ then $ft(S(r, Y)) = S(r, ft(Y))$,
   (c) $l(S(r, Y)) = l(Y) - 1$.

4. *For $(r, s) \in \widetilde{S}_{dom}$ one has*

   (a) $\partial(\widetilde{S}(r, s)) = S(r, \partial(s))$,
   (b) $l(\widetilde{S}(r, s)) = l(\partial(s)) - 1$.

Remark 3.2 [2016.03.31.rem1] Elements participating in the operations $T, \widetilde{T}, S$ and $\widetilde{S}$ on a B0-system can be shown by the following diagrams

$$
\begin{array}{ccc}
T(X,Y) & & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & ft(X)
\end{array}
\qquad
\begin{array}{ccc}
T(X,\partial(s)) & & \partial(s) \\
\widetilde{T}(X,s)\uparrow\downarrow & & \downarrow\uparrow s \\
ft(T(X,\partial(s))) & & ft(\partial(s)) \\
\downarrow & & \downarrow \\
X & \longrightarrow & ft(X)
\end{array}
$$

$$
\begin{array}{ccc}
S(r,Y) & & Y \\
\downarrow & & \downarrow \\
ft(\partial(r)) & \xleftarrow{r} & \partial(r)
\end{array}
\qquad
\begin{array}{ccc}
S(r,\partial(s)) & & \partial(s) \\
\widetilde{S}(r,s)\uparrow\downarrow & & \downarrow\uparrow s \\
ft(S(r,\partial(s))) & & ft(\partial(s)) \\
\downarrow & & \downarrow \\
ft(\partial(r)) & \xleftarrow{r} & \partial(r)
\end{array}
$$

Remark 3.3 [2016.01.29.rem1] The axioms of a B0-system given in Definition 3.1 are not independent. If the axioms 1(a) and 2(b) hold then the axiom 2(a) holds and if the axioms 3(a) and 4(b) hold then the axiom 4(a) holds. The axioms are presented there in this form to make it possible to prove facts about various operations in B0-systems independently from each other.

Definition 3.4 [2014.10.20.def2] *A pre-B-system is called a B0-system if the underlying non-unital pre-B-system is a non-unital B0-system and for all $X \in \delta_{dom}$ one has*

$$[\textbf{2009.12.27.eq1}]\partial(\delta(X)) = T(X,X) \tag{1}$$

Lemma 3.5 [2016.03.11.l1] *Let B be an lft-set, T_{dom} be the corresponding set of pairs (X,Y) in B and $T : T_{dom} \to B$ be a function satisfying the conditions of Definition 3.1(1). Let $(X,Y) \in T_{dom}$ be such that $ft(Y) = ft(X)$. Then*

$$ft(T(X,Y)) = X$$

Proof: We know that $T(X,Y) > X$. In particular, $ft^{l(T(X,Y))-l(X)}(T(X,Y)) = X$. On the other hand $l(T(X,Y)) = l(Y)+1$. Since $Y > ft(X)$ we know that $l(Y) \geq 1$. Therefore $l(ft(Y)) = l(Y)-1$ and similarly from $l(X) \geq 1$ we have $l(ft(X)) = l(X) - 1$. Therefore $ft(Y) = ft(X)$ implies that $l(Y) = l(X)$ and

$$l(T(X,Y)) - l(X) = l(Y) + 1 - l(X) = l(X) + 1 - l(X) = 1$$

We conclude that $ft(T(X,Y)) = X$.

Lemma 3.6 [2016.03.19.l2] *Let B be an lft-set, T_{dom} be the corresponding set of pairs (X,Y) in B and $T : T_{dom} \to B$ be a function satisfying the conditions of Definition 3.1(1). Then if $(X,Y),(X,Y') \in T_{dom}$ and $Y \leq Y'$ (resp. $Y < Y'$) then $T(X,Y) \leq T(X,Y')$ (resp. $T(X,Y) < T(X,Y')$).*

Proof: Note first that by Definition 3.1(1a) we have $l(T(X,Y)) \leq l(T(X,Y'))$ if $l(Y) \leq l(Y')$ and $l(T(X,Y)) < l(T(X,Y'))$ if $l(Y) < l(Y')$. In particular, it is sufficient to consider the $Y \leq Y'$ case. Then we need to prove that

$$[\textbf{2016.03.19.eq1}] ft^{l(T(X,Y'))-l(T(X,Y))}(T(X,Y')) = T(X,Y) \tag{2}$$

We have $l(T(X,Y')) - l(T(X,Y)) = (l(Y')+1) - (l(Y)+1) = l(Y') - l(Y)$. By an easy induction one proves that $ft^i(T(X,Y')) = T(X, ft^i(Y'))$ if $(X, ft^i(Y')) \in T_{dom}$. In our case

$$ft^{l(T(X,Y'))-l(T(X,Y))}(Y') = ft^{(l(Y')+1)-(l(Y)+1)}(Y') = ft^{l(Y')-l(Y)}(Y') = Y$$

and since $(X,Y) \in T_{dom}$ by the assumption, (2) holds.

Lemma 3.7 *[2016.03.21.l3] Let* **B** *be a B-system carrier and* $S : S_{dom} \to B$ *an operation satisfying the conditions of Definition 3.1(3). Let* $(r,Y) \in S_{dom}$ *be such that* $\partial(r) = ft(Y)$. *Then*

$$ft(S(r,Y)) = ft(\partial(r)).$$

Proof: We know that $ft(\partial(r)) < S(r,Y)$. Therefore $ft^{l(S(r,Y))-l(ft(\partial(r)))}(S(r,Y)) = ft(\partial(r))$. From the definition of a B-system carrier we know that $l(\partial(r)) > 0$. Therefore $l(ft(\partial(r))) = l(\partial(r)) - 1$. We also have $l(S(r,Y)) = l(Y) - 1$. We also have that $l(ft(Y)) = l(\partial(Y)) > 0$ and therefore $l(\partial(r)) = l(ft(Y)) = l(Y) - 1 > 0$. Together this implies that

$$l(S(r,Y)) - l(ft(\partial(r))) = (l(Y)-1) - (l(\partial(r))-1) = (l(Y)-1) - ((l(Y)-1)-1) = 1$$

We conclude that $ft(S(r,Y)) = ft(\partial(r))$.

Lemma 3.8 *[2016.03.19.l5] Let* **B** *be a B-system carrier and* $S : S_{dom} \to B$ *an operation satisfying the conditions of Definition 3.1(3). Let* $(r,Y),(r,Y') \in S_{dom}$ *and suppose that* $Y \leq Y'$ *(resp.* $Y < Y'$*). Then* $S(r,Y) \leq S(r,Y')$ *(resp.* $S(r,Y) < S(r,Y')$*).*

Proof: Note first that $l(S(r,Y')) - l(S(r,Y)) = (l(Y')-1) - (l(Y)-1) = l(Y') - l(Y)$. In particular, if $S(r,Y) \leq S(r,Y')$ and $l(Y) < l(Y')$ then $S(r,Y) < S(r,Y')$ and so it is sufficient to prove the case of $\leq$. By an easy induction one proves that if $\partial(r) < ft^i(Y')$ then $ft^i(S(r,Y')) = S(r, ft^i(Y'))$. In our case

$$ft^{l(S(r,Y'))-l(S(r,Y))}(Y') = ft^{l(Y')-l(Y)}(Y') = Y$$

and since $\partial(r) < Y$ by the assumption we conclude that

$$ft^{l(S(r,Y'))-l(S(r,Y))}(S(r,Y')) = S(r,Y)$$

that is $S(r,Y) \leq S(r,Y')$.

3.1 Properties of T_{ext} when T satisfies the B0-system conditions

Lemma 3.9 *[2016.03.09.l1] Let* **B** *be a B-system carrier and let* $T : T_{dom} \to B$ *be an operation satisfying the conditions of Definition 3.1(1). Let* $(X,Y) \in T_{ext,dom}$. *Then one has:*

1. *if* $ft(X) = Y$ *then* $T_{ext}(X,Y) = X$,

2. *if $ft(X) < Y$ then $T_{ext}(X,Y) = T(X,Y)$,*

3. *if $l(Y) > 0$ and $ft(X) \leq ft(Y)$ then $ft(T_{ext}(X,Y)) = T_{ext}(X, ft(Y))$,*

4. *if $Y \leq Y'$ then $(X,Y') \in T_{ext,dom}$ and $T_{ext}(X,Y) \leq T_{ext}(X,Y')$,*

5. $T_{ext}(X,Y) \geq X$,

6. $l(T_{ext}(X,Y)) = l(Y) + 1$.

Proof: Elements involved in the operation T_{ext} under the assumptions of the lemma can be seen on the following diagram:

$$
\begin{array}{ccc}
T_{ext}(X,Y) & & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & ft(X)
\end{array}
$$

Consider now the assertions of the lemma:

1. This follows from the construction of T_{ext}.

2. This follows from the construction of T_{ext}.

3. If $ft(X) < ft(Y)$ then by Lemma 1.4 we have $ft(X) < Y$ and $T_{ext}(X,Y) = T(X,Y)$ and $T_{ext}(X, ft(Y)) = T(X, ft(Y))$ and the assertion follows from Definition 3.1(1). If $ft(X) = ft(Y)$ then, since $l(Y) > 0$ we have $l(ft(Y)) = l(Y) - 1$, therefore $ft(Y) < Y$ and $ft(X) < Y$. Using the first and second assertions of the lemma we see that in this case we need to show that $ft(T(X,Y)) = X$. By Definition 3.1(1) we have $T(X,Y) \geq X$, i.e.,

$$ft^{l(T(X,Y))-l(X)}(T(X,Y)) = X$$

Next we have $l(T(X,Y)) = l(Y) + 1$. Since $l(X) > 0$ and $l(Y) > 0$ from $ft(X) = ft(Y)$ we have $l(X) = l(Y)$. We conclude that $ft^1(T(X,Y)) = X$.

4. That $(X,Y') \in T_{ext,dom}$ follows from Lemma 1.3. To prove that $T_{ext}(X,Y) \leq T_{ext}(X,Y')$ we proceed by induction on $j = l(Y') - l(Y)$.

   If $j = 0$ then since $Y \leq Y'$ we have $Y = Y'$ and the assertion is obvious.

   For the successor of j we have $l(Y') - l(Y) = j + 1$. In particular, $l(Y') > 0$ and $l(ft(Y')) = l(Y') - 1$. By Lemma 1.5 we have $Y \leq ft(Y')$ and therefore, by Lemma 1.3, $ft(X) \leq ft(Y')$ and the third assertion of the present lemma applies giving us $ft(T_{ext}(X,Y')) = T_{ext}(X, ft(Y'))$. We now have

$$T_{ext}(X,Y) \leq T_{ext}(X, ft(Y')) = ft(T_{ext}(X,Y')) \leq T_{ext}(X,Y')$$

   where the first inequality is by the inductive assumption since $l(ft(Y')) - l(Y) = j$. We conclude by Lemma 1.3.

5. It follows by case on whether $ft(X) < Y$ or $ft(X) = Y$ from Definition 3.1(1) and the first assertion of the lemma.

6. It follows by case on whether $ft(X) < Y$ or $ft(X) = Y$ from Definition 3.1(1) and the first assertion of the lemma.

3.2 Properties of S_{ext} when S satisfies the B0-system conditions

Lemma 3.10 *[2016.03.27.l5] Let* **B** *be a B-system carrier and let* $S : S_{dom} \to B$ *be an operation satisfying the conditions of Definition 3.1(3). Let* $(r, Y) \in S_{ext,dom}$. *Then one has:*

1. *if* $\partial(r) = Y$ *then* $S_{ext}(r, Y) = ft(\partial(r))$,

2. *if* $\partial(r) < Y$ *then* $S_{ext}(r, Y) = S(r, Y)$,

3. *if* $\partial(r) \leq ft(Y)$ *then* $ft(S_{ext}(r, Y)) = S_{ext}(r, ft(Y))$,

4. *if* $Y \leq Y'$ *then* $(r, Y') \in S_{ext,dom}$ *and* $S_{ext}(r, Y) \leq S_{ext}(r, Y')$,

5. $S_{ext}(r, Y) \geq ft(\partial(r))$,

6. $l(S_{ext}(r, Y)) = l(Y) - 1$.

Proof: Elements involved in the operation S_{ext} under the assumptions of the lemma can be seen on the following diagram:

$$
\begin{array}{ccc}
S_{ext}(r, Y) & & Y \\
\downarrow & & \downarrow \\
ft(\partial(r)) & \xleftarrow{\;\;r\;\;} & \partial(r)
\end{array}
$$

Consider now the assertions of the lemma:

1. This follows from the construction of S_{ext}.

2. This follows from the construction of S_{ext}.

3. If $\partial(r) < ft(Y)$ then by Lemma 1.4 we have $\partial(r) < Y$ and $S_{ext}(r, Y) = S(r, Y)$ and $S_{ext}(r, ft(Y)) = S(r, ft(Y))$ and the assertion follows from Definition 3.1(3). If $\partial(r) = ft(Y)$ then, since $l(\partial(r)) > 0$ we have $l(ft(Y)) = l(Y) - 1$, therefore $ft(Y) < Y$ and $\partial(r) < Y$. Using the first and second assertions of the lemma we see that in this case we need to show that $ft(S(r, Y)) = ft(\partial(r))$. By Definition 3.1(3) we have $S(r, Y) \geq ft(\partial(r))$, i.e.,
$$ft^{l(S(r,Y))-l(ft(\partial(r)))}(S(r, Y)) = ft(\partial(r))$$
Next we have $l(S(r, Y)) = l(Y) - 1$ and $l(ft(\partial(r))) = l(ft^2(Y))$. Since $Y = ft(\partial(r))$ and $l(\partial(r)) > 0$ we have $l(ft^2(Y)) = l(Y) - 2$. We conclude that $ft^1(S(r, Y)) = ft(\partial(r))$.

4. That $(r, Y') \in S_{ext,dom}$ follows from Lemma 1.3. To prove that $S_{ext}(r, Y) \leq S_{ext}(r, Y')$ we proceed by induction on $j = l(Y') - l(Y)$.

   If $j = 0$ then since $Y \leq Y'$ we have $Y = Y'$ and the assertion is obvious.

   For the successor of j we have $l(Y') - l(Y) = j + 1$. In particular, $l(Y') > 0$ and $l(ft(Y')) = l(Y') - 1$. By Lemma 1.5 we have $Y \leq ft(Y')$ and therefore, by Lemma 1.3, $\partial(r) \leq ft(Y')$ and the third assertion of the present lemma applies giving us $ft(S_{ext}(r, Y')) = S_{ext}(r, ft(Y'))$. We now have
   $$S_{ext}(r, Y) \leq S_{ext}(r, ft(Y')) = ft(S_{ext}(r, Y')) \leq S_{ext}(r, Y')$$
   where the first inequality is by the inductive assumption since $l(ft(Y')) - l(Y) = j$. We conclude by Lemma 1.3.

5. It follows by case on whether $\partial(r) < Y$ or $\partial(r) = Y$ from Definition 3.1(3) and the first assertion of the lemma.

6. It follows by case on whether $\partial(r) < Y$ or $\partial(r) = Y$ from Definition 3.1(3) and the first assertion of the lemma.

3.3   Construction of sets T^*_{dom} and operations T^*

Define the set T^*_{dom} by the formula:

$$T^*_{dom} = \{X, Y, Z \in B, X \geq Y, Z \geq Y\}.$$

Problem 3.11 *[2016.02.18.prob3] Let B be an lft-set and $T : T_{dom} \to B$ a function satisfying the conditions of Definition 3.1(1). To define a function*

$$T^* : T^*_{dom} \to B$$

such that:

1. *$T^*(X, Y, Z) \geq X$,*

2. *if $l(Z) \geq 1$ and $ft(Z) \geq Y$ then $ft(T^*(X, Y, Z)) = T^*(X, Y, ft(Z))$,*

3. *$l(T^*(X, Y, Z)) - l(X) = l(Z) - l(Y)$.*

Elements involved in operation T^* can be shown on a diagram as follows:

$$
\begin{array}{ccc}
T^*(X, Y, Z) & & Z \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

Construction 3.12 [2016.02.18.constr3] We proceed by induction on $j = l(X) - l(Y)$.

For $j = 0$ we set $T^*(X, Y, Z) = Z$. The proofs of the conditions are obvious.

For $j = 1$ we set $T^*(X, Y, Z) = T_{ext}(X, Z)$. The fact that $(X, Z) \in T_{ext,dom}$ is easy to prove. The conditions are the conditions proved in Lemma 3.9.

For the successor of $j > 0$ we set

$$[\mathbf{2016.02.20.eq2}] T^*(X, Y, Z) = T_{ext}(X, T^*(ft(X), Y, Z)) \tag{3}$$

For the formula (3) to be well defined we need to show that for $(X, Y, Z) \in T^*_{dom}$ we have

$$[\mathbf{2016.02.20.eq4}](ft(X), Y, Z) \in T^*_{dom} \tag{4}$$

and

$$[\mathbf{2016.02.20.eq5}](X, T^*(ft(X), Y, Z)) \in T_{ext,dom} \tag{5}$$

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The condition (4) follows from Lemma 1.5(1).

The condition (5) is equivalent to $l(X) \geq 1$ and

$$T^*(ft(X), Y, Z) \geq ft(X)$$

That $l(X) \geq 1$ follows from $l(X) \geq j + 1$.

That $T^*(ft(X), Y, Z) \geq ft(X)$ follows from the inductive assumption.

Let us prove the conditions for $j + 1$. The first is immediate from Lemma 3.13

$$T^*(X, Y, Z) = T_{ext}(X, T^*(ft(X), Y, Z)) \geq X$$

To prove the second condition let $ft(Z) \geq Y$ and $l(Z) \geq 1$. Then, by the inductive assumption we have

$$ft(T^*(ft(X), Y, Z)) = T^*(ft(X), Y, ft(Z))$$

Therefore,

$$ft(T^*(X, Y, Z)) = ft(T_{ext}(X, T^*(ft(X), Y, Z))) = T_{ext}(X, ft(T^*(ft(X), Y, Z))) =$$

$$T_{ext}(X, T^*(ft(X), Y, ft(Z))) = T^*(X, Y, ft(Z))$$

where the second equality follows from Lemma 3.9(3) whose condition is satisfied because

$$ft(T^*(ft(X), Y, Z)) = T^*(ft(X), Y, ft(Z)) \geq ft(X).$$

For the last condition we have

$$l(T^*(X, Y, Z)) - l(X) = l(T_{ext}(X, T^*(ft(X), Y, Z))) - l(X) = (l(T^*(ft(X), Y, Z)) + 1) - l(X) =$$

$$(l(T^*(ft(X), Y, Z)) + 1) - (l(ft(X)) + 1) = l(T^*(ft(X), Y, Z)) - l(ft(X)) = l(Z) - l(Y)$$

This completes Construction 3.12.

Lemma 3.13 *[2016.03.09.l2]/[2016.04.08.l1] In the context of Problem 3.11 let $X, Y, Z, W \in B$ be such that $X \geq Y$, $Y \leq Z \leq W$. Then one has*

1. *$T^*(X, Y, Z) \leq T^*(X, Y, W)$,*

2. *$l(T^*(X, Y, W)) - l(T^*(X, Y, Z)) = l(W) - l(Z)$,*

3. *if $Z < W$ then $T^*(X, Y, Z) < T^*(X, Y, W)$*

4. *if $Y = Z$ then $T^*(X, Y, Z) = X$.*

Proof: To prove the first assertion we proceed by induction on $j = l(W) - l(Z)$.

For $j = 0$ we have $W = Z$ and $T^*(X, Y, Z) = T^*(X, Y, W)$.

For the successor of $j \geq 0$ we have $l(W) - l(Z) > 0$ and therefore $Z \leq ft(W)$ and $Y \leq Z \leq ft(W)$. Therefore $T^*(X, Y, ft(W))$ is defined and by property (2) of Problem 3.11 we have $ft(T^*(X, Y, W)) = T^*(X, Y, ft(W))$.

By the inductive assumption we have $T^*(X, Y, Z) \le T^*(X, Y, ft(W))$ and since $ft(T^*(X, Y, W)) \le T^*(X, Y, W)$ we conclude that $T^*(X, Y, Z) \le T^*(X, Y, W)$.

For the second assertion consider that we have, from Problem 3.11(3), that

$$l(T^*(X, Y, W)) = l(W) + (l(X) - l(Y))$$

$$l(T^*(X, Y, Z)) = l(Z) + (l(X) - l(Y))$$

subtracting we obtain the second assertion.

The third assertion follows easily from the first and the second ones.

The fourth assertion follows easily from the first and the second ones.

3.4   Construction of sets $\widetilde{T}^*_{dom}$ and operations $\widetilde{T}^*$

We will also require a similar construction for $\widetilde{T}$. Let

$$\widetilde{T}^*_{dom} = \{X, Y \in B, s \in \widetilde{B}, (X, Y, ft(\partial(s))) \in T^*_{dom}\}$$

Problem 3.14 *[2016.02.20.prob1] Let $(B, \widetilde{B}, \partial)$ be a B-system carrier and let*

$$T : T_{dom} \to B \quad \widetilde{T} : \widetilde{T}_{dom} \to \widetilde{B}$$

be functions satisfying the conditions of Definition 3.1(1,2). For $j \in \mathbf{N}$, define a function

$$\widetilde{T}^* : \widetilde{T}^*_{dom} \to \widetilde{B}$$

such that:
$$[\mathbf{2016}.\mathbf{02}.\mathbf{22}.\mathbf{eq1}]\partial(\widetilde{T}^*(X, Y, s)) = T^*(X, Y, \partial(s)) \tag{6}$$

Elements involved in operation $\widetilde{T}^*$ can be shown on a diagram as follows:

$$
\begin{array}{ccc}
T^*(X, Y, \partial(s)) & & \partial(s) \\
\widetilde{T}^*(X,Y,s)\uparrow\Big\downarrow & & \Big\downarrow\uparrow s \\
ft(T^*(X, Y, \partial(s))) & & ft(\partial(s)) \\
\Big\downarrow & & \Big\downarrow \\
X & \longrightarrow & Y
\end{array}
$$

Construction 3.15 [2016.02.20.constr1] We proceed by induction on $j = l(X) - l(Y)$.

Observe first that the condition $(X, Y, ft(\partial(s))) \in T^*_{dom}$ is equivalent to the condition that $X \ge Y$ and $ft(\partial(s)) \ge Y$ and that since $l(\partial(s)) > 0$ the latter condition is equivalent to $\partial(s) > Y$.

For $j = 0$ we set $\widetilde{T}^*(X, Y, s) = s$. The proof of the conditions is obvious.

For $j = 1$ we set $\widetilde{T}^*(X, Y, s) = \widetilde{T}(X, s)$. For the right hand side to be defined we need $\partial(s) > ft(X)$ which is satisfied by the observation made above since $ft(X) = Y$. The condition (6) is the condition of Definition 3.1(2a).

For the successor of $j > 0$ we set

$$[\mathbf{2016.02.20.eq3}]\widetilde{T}^*(X, Y, s) = \widetilde{T}(X, \widetilde{T}^*(ft(X), Y, s)) \tag{7}$$

For the formula (7) to be well defined we need to show that assuming $(X, Y, ft(\partial(s))) \in T^*_{dom}$ we have:

$$[\mathbf{2016.02.20.eq7}](ft(X), Y, s) \in \widetilde{T}^*_{dom} \tag{8}$$

and

$$[\mathbf{2016.02.20.eq8}](X, \widetilde{T}^*(ft(X), Y, s)) \in \widetilde{T}_{dom} \tag{9}$$

The condition (8) is equivalent to $(ft(X), Y, ft(\partial(s))) \in T^*_{dom}$ and its proof is identical to the proof of (4) for $Z = ft(\partial(s))$.

The condition (9) is equivalent to $(X, \partial(\widetilde{T}^*(ft(X), Y, s))) \in T_{dom}$. By the inductive assumption we have $\partial(\widetilde{T}^*(ft(X), Y, s)) = T^*(ft(X), Y, \partial(s))$. Therefore we need to show that

$$(X, T^*(ft(X), Y, \partial(s))) \in T_{dom}$$

that is, that $l(X) \geq 1$ and $T^*(ft(X), Y, \partial(s)) > ft(X)$. The first condition follows from the fact that $l(X) = l(Y) + j + 1 \geq 1$. For the second condition we know that $T^*(ft(X), Y, \partial(s)) \geq ft(X)$ from the first condition of Problem 3.11. On the other hand

$$l(T^*(ft(X), Y, \partial(s))) - l(ft(X)) = l(\partial(s)) - l(Y)$$

Since $(ft(X), Y, ft(\partial(s))) \in T^*_{dom}$ we have $ft(\partial(s)) \geq Y$ and therefore $l(ft(\partial(s))) = l(\partial(s)) - 1 \geq l(Y)$. This implies that $l(\partial(s)) - l(Y) > 0$ and $T^*(ft(X), Y, \partial(s)) > ft(X)$.

Let us prove condition (6). We have

$$\partial(\widetilde{T}^*(X, Y, s)) = \partial(\widetilde{T}(X, \widetilde{T}^*(ft(X), Y, s))) = T(X, \partial(\widetilde{T}^*(ft(X), Y, s))) =$$

$$T(X, T^*(ft(X), Y, \partial(s))) = T^*(X, Y, \partial(s))$$

This completes Construction 3.15.

3.5 Operations T^* and $\widetilde{T}^*$ and homomorphisms of pre-B-systems

Lemma 3.16 *[2016.03.15.l3] Let $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ be a homomorphism of B0-systems. Then one has:*

1. *if $(X, Y, Z) \in T^*_{dom}$ then $(f(X), f(Y), f(Z)) \in (T')^*_{dom}$ and*

$$f(T^*(X, Y, Z)) = (T')^*(f(X), f(Y), f(Z))$$

2. *if $(X, Y, s) \in \widetilde{T}^*_{dom}$ then $(f(X), f(Y), \widetilde{f}(s)) \in (\widetilde{T}')^*_{dom}$ and*

$$\widetilde{f}(\widetilde{T}^*_{dom}(X, Y, s)) = (\widetilde{T}')^*(f(X), f(Y), \widetilde{f}(s))$$

Proof: The first parts of both assertions follow immediately from Lemma 1.11.

The second parts require proofs by induction using in the first case Lemma 2.12.

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2 Sets $BMor(X,Y)$, composition operations and the identity elements

1 Construction of the set $BMor(X,Y)$ and operations $fstar$

Problem 1.1 *[2016.02.28.prob1] For a pointed B0-system* **B** *and* $X, Y \in B$ *to define:*

1. *a set that will be denoted* $BMor(X,Y)$,

2. *for all* $f \in BMor(X,Y)$ *and* $Y' \in B$ *such that* $Y \leq Y'$ *an element* $fstar(f, Y') \in B$,

such that

1. $fstar(f, Y) = X$,

2. *if* $Y \leq Y' \leq Y''$ *then* $fstar(f, Y') \leq fstar(f, Y'')$,

3. $l(fstar(f, Y')) - l(X) = l(Y') - l(Y)$,

4. *if* $l(Y') > 0$ *and* $Y \leq ft(Y')$ *then* $ft(fstar(f, Y')) = fstar(f, ft(Y'))$.

Construction 1.2 [**2016.02.28.constr1**][**2016.02.20.def1**] Observe first that condition (4) for given X, Y follows from the conditions (2) and (3). Indeed, from (2) we have $fstar(f, ft(Y')) \leq fstar(f, Y')$, that is,

$$ft^{l(fstar(f,Y'))-l(fstar(f,ft(Y')))}(fstar(f, Y')) = fstar(f, ft(Y'))$$

From (3) we have $l(fstar(f, Y')) - l(fstar(f, ft(Y'))) = l(Y') - l(ft(Y'))$ and since $l(Y') > 0$ we have $l(Y') - l(ft(Y')) = 1$.

We fix X and proceed by induction on $n = l(Y)$ to define pairs $(BMor(X,Y), fstar)$ satisfying conditions (1)-(4).

For $n = 0$ we define $BMor(X,Y) = unit$. Since **B** is pointed we have $Y = pt$ and $X \geq Y$ and therefore $T^*(X, Y, Y')$ is defined. We set

$$fstar(f, Y') = T^*(X, Y, Y')$$

Let us prove the conditions (1)-(3). We have:

1. It follows from Lemma 3.13(4).

2. It follows from Lemma 3.13(1).

3. It follows from Problem 3.11(2).

For the successor of n we have $l(Y) = n + 1$ and $l(ft(Y)) = n$. Using the inductive assumption we set
$$BMor(X, Y) = \{(f, r) \mid f \in BMor(X, ft(Y)), r \in \widetilde{B}(fstar(f, Y))\}$$
For $(f, r) \in BMor(X, Y)$ and $Y' \geq Y$ we set

$$fstar((f, r), Y') = S_{ext}(r, fstar(f, Y'))$$

To see that the right hand side is defined we need to verify that $\partial(r) \leq fstar(f, Y')$. Since $\partial(r) = fstar(f, Y)$ and $Y \leq Y'$ we have that $\partial(r) \leq fstar(f, Y')$ from the inductive assumption of condition (2).

The diagram that shows the elements involved in the construction of $(BMor(X,Y), fstar)$ for $l(Y) = n + 1$ is as follows:

$$
\begin{array}{ccccc}
S_{ext}(r, fstar(f, Y')) & & fstar(f, Y') & & Y' \\
\downarrow & & \downarrow & & \downarrow \\
X & \xleftarrow{\;\;r\;\;} & fstar(f, Y) & & Y \\
& & r\uparrow\downarrow & & \downarrow \\
& & X \xdashrightarrow{\;f\;} & & ft(Y)
\end{array}
$$

Let us prove the properties. We have:

1. By construction $fstar(f, Y) = \partial(r)$ and therefore
$$fstar((f, r), Y) = S_{ext}(r, fstar(f, Y)) = ft(\partial(r)) =$$
$$ft(fstar(f, Y)) = fstar(f, ft(Y)) = X$$
where the second equality is from the definition of S_{ext}, the fourth equality from the inductive assumption of condition (4) which is applicable since $l(Y) = n + 1 > 0$ and the last equality is from the inductive assumption of condition (1).

2. If $Y'' \geq Y'$ then
$$fstar((f, r), Y'') = S_{ext}(r, fstar(f, Y'')) \geq S_{ext}(r, fstar(f, Y')) = fstar((f, r), Y')$$
where the inequality is by the inductive assumption of condition (3) and Lemma 3.10(4).

3. We have
$$l(fstar((f, r), Y')) - l(X) = l(S_{ext}(r, fstar(f, Y'))) - l(X) = l(fstar(f, Y')) - 1 - l(X) =$$
$$l(Y') - l(ft(Y)) - 1 = l(Y') - l(Y)$$
where the second equality is by Lemma 3.10(6), the third by the inductive assumption on condition (3) and the fourth follows from the fact that $l(ft(Y)) = l(Y) - 1$ since $l(Y) = n + 1 > 0$.

Lemma 1.3 *[2016.04.16.l3] For $f \in BMor(X, Y)$ and $Y' \geq Y$ one has $fstar(f, Y') \geq X$.*

Proof: It follows from assertions (1) and (2) of Problem 1.1.

Remark 1.4 [2016.02.28.rem1] To define $BMor(X, Y)$ we need less than the full set of B0-system structures and axioms. All we need is a B-system carrier with operations T and S such that T satisfies conditions of Definition 3.1(1) and S the conditions of Definition 3.1(3).

In the next section we will construct for any C-system CC a B0-system $CB(CC)$ and for any $X, Y \in CC$ a bijection between $BMor_{CB(CC)}(X, Y)$ and $Mor_{CC}(X, Y)$.

2 Operations $\widetilde{fstar}$

Problem 2.1 *[**2016.04.18.prob1**] Let* **B** *be a pointed B0-system. Let* $X, Y \in B$, $f \in BMor(X, Y)$ *and* $s \in \widetilde{B}$ *be such that* $Y \le ft(\partial(s))$ *to construct an element* $\widetilde{fstar}(f, s)$ *such that*

$$[\mathbf{2016.04.18.eq1}]\partial(\widetilde{fstar}(f, s)) = fstar(f, \partial(s)) \tag{10}$$

Construction 2.2 *[**2016.04.18.constr1**] As in the construction of fstar we proceed by induction on* $n = l(Y)$.

For $n = 0$ *we set*

$$\widetilde{fstar}(f, s) = \widetilde{T}^*(X, Y, s)$$

The right hand side is defined since $Y = pt$ *and* $X \ge pt \le ft(\partial(s))$. *The condition (10) holds because of (6).*

For the successor of n *we have* $l(Y) = n + 1$ *and* $f = (f_1, r_1)$ *where* $f_1 \in BMor(X, ft(Y))$ *and* $r_1 \in \widetilde{B}(fstar(f, Y))$. *We set*

$$\widetilde{fstar}((f_1, r_1), s) = \widetilde{S}(r_1, \widetilde{fstar}(f_1, s))$$

The subexpression $\widetilde{fstar}(f_1, s)$ *is defined by the inductive assumption because* $l(ft(Y)) = n$ *and because* $ft(Y) \le Y \le ft(\partial(s))$ *and Lemma 1.3.*

The right hand side is defined because

$$\partial(r_1) = fstar(f_1, Y) \le fstar(f_1, ft(\partial(s))) = ft(fstar(f_1, \partial(s))) = ft(\partial(\widetilde{fstar}(f_1, s)))$$

where the inequality is by condition (2) of Problem 1.1, the second equality is by condition (4) of Problem 1.1 and the third equality is by the inductive assumption on (10).

To prove (10) for the successor of n *we have*

$$\partial(\widetilde{fstar}((f_1, r_1), s)) = \partial(\widetilde{S}(r_1, \widetilde{fstar}(f_1, s))) = S(r_1, \partial(\widetilde{fstar}(f_1, s))) = S(r_1, fstar(f_1, \partial(s))) = fstar((r_1, f_1), \partial(s$$

where the equalities follow from Definition 3.1(2) and the inductive assumption on (10).

3 Sets $BMor(X, Y)$ and homomorphisms of pointed B0-systems

Problem 3.1 *[**2016.03.15.prob2**] Let* $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ *be a homomorphism of pointed B0-systems. For* $X, Y \in B$ *to define a function*

$$\mathbf{f}_{BMor,X,Y} : BMor(X, Y) \to BMor'(f(X), f(Y))$$

Construction 3.2 [**2016.03.15.constr3**] By Construction 1.2 we have

$$BMor(X, Y) = \widetilde{B}^*(X, T^*(X, pt, Y))$$

Applying Construction **??** we obtain the function

$$\mathbf{f}_{\widetilde{B}^*(X,T^*(X,pt,Y))} : \widetilde{B}^*(X, T^*(X, pt, Y)) \to \widetilde{B}'(f(X), f(T^*(X, pt, Y)))$$

Applying Lemma 3.16(1) we get that $f(T^*(X, pt, Y)) = (T')^*(f(X), f(pt), f(Y))$. Since $\mathbf{B}'$ is pointed we have $f(pt) = pt'$ and

$$\widetilde{B}'(f(X), (T')^*(f(X), f(pt'), f(Y))) = BMor'(f(X), f(Y))$$

This completes the construction.

3.1 Composition operation

To construct the operations on the $BMor$ sets that will be related to the composition of morphisms and the identity morphisms in the case of the B0-systems of the form $CB(CC)$ we need the underlying B0-system to satisfy some of the axioms of a B-system.

Everywhere below when we say that an equality involving partially defined operations holds for certain values of the arguments we mean that the left and the right hand side expressions are defined and equal.

3 B-systems

1 Preliminary lemmas

Lemma 1.1 [2016.03.19.l1] *Let* **B** *be a B0-system. Let* $X, Y, Z \in B$ *be such that* $l(X) \geq 1$, $ft(X) < Y$ *and* $ft(Y) < Z$ *then one has:*

1. $(X, Y), (X, Z), (Y, Z) \in T_{dom}$,

2. $(X, T(Y, Z)) \in T_{dom}$,

3. $(T(X, Y), T(X, Z)) \in T_{dom}$.

Proof: We have:

1. The inclusion $(X, Y) \in T_{dom}$ follows from the definitions.

   To show that $(X, Z) \in T_{dom}$ we need to prove that $ft(X) < Z$. We have $ft(X) < Y$ and therefore $ft(X) \leq ft(Y)$ by Lemma 1.6. Then $ft(X) \leq ft(Y) < Z$ implies $ft(X) < Z$ by Lemma 1.4(1).

   To show that $(Y, Z) \in T_{dom}$ we need only to prove $l(Y) \geq 1$ which follows from $Y > ft(X)$.

2. We have that $l(X) \geq 1$. We need to prove that $ft(X) < T(Y, Z)$. We have that $ft(X) < Y$ and $Y < T(Y, Z)$ which implies $ft(X) < T(Y, Z)$ by Lemma 1.3(2).

3. We have $T(X, Y) > X$ and therefore $l(T(X, Y)) \geq 1$. Next we need to prove that $ft(T(X, Y)) < T(X, Z)$. Since $ft(X) < Y$ by Lemma 1.6 we have $ft(X) \leq ft(Y)$. Therefore one of the two cases occurs. In the first case $ft(X) < ft(Y)$ in which case $ft(T(X, Y)) = T(X, ft(Y))$ and since $ft(Y) < Z$ the inequality $ft(T(X, Y)) < T(X, Z)$ follows from Lemma 3.6. In the second case $ft(X) = ft(Y)$. Then $ft(T(X, Y)) = X$ by Lemma 3.5 and $X < T(X, Z)$ by Definition 3.1(1b).

Lemma 1.2 [2016.03.19.l3] *Let* **B** *be a B0-system. Let* $X, Y \in B$, $s \in \widetilde{B}$ *be such that* $l(X) \geq 1$, $ft(X) < Y$ *and* $ft(Y) < \partial(s)$. *Then one has:*

1. $(X, Y) \in T_{dom}$, $(X, s), (Y, s) \in \widetilde{T}_{dom}$,

2. $(X, \widetilde{T}(Y, s)) \in \widetilde{T}_{dom}$,

3. $(T(X,Y), \widetilde{T}(X,s)) \in \widetilde{T}_{dom}$.

Proof: We have:

1. These assertions follow by applying Lemma 1.1(1) to $X, Y, \partial(s)$.

2. Since we know that $l(X) \geq 1$ we need to show that $ft(X) < \partial(\widetilde{T}(Y,s))$. We have $\partial(\widetilde{T}(Y,s)) = T(Y, \partial(s))$. From $ft(X) < Y$ and $Y < T(Y, \partial(s))$ we get that $ft(X) < T(Y, \partial(s))$ applying Lemma 1.3(2).

3. We have $l(T(X,Y)) \geq 1$ because $T(X,Y) > X$. It remains to show that $ft(T(X,Y)) < \partial(\widetilde{T}(X,s))$. We have $\partial(\widetilde{T}(X,s)) = T(X, \partial(s))$ and the required inequality follows from Lemma 1.1(3) applied to $X, Y, \partial(s)$.

Lemma 1.3 [2016.03.19.l4] *Let* **B** *be a B0-system. Let* $r, s \in \widetilde{B}$, $Y \in B$ *be such that* $\partial(r) < \partial(s)$ *and* $\partial(s) < Y$. *Then one has*

1. $(r,s) \in \widetilde{S}_{dom}$, $(r,Y), (s,Y) \in S_{dom}$,

2. $(r, S(s,Y)) \in S_{dom}$,

3. $(\widetilde{S}(r,s), S(r,Y)) \in S_{dom}$.

Proof: We have:

1. $(r,s) \in \widetilde{S}_{dom}$ and $(s,Y) \in S_{dom}$ is immediate from the assumptions. To show that $(r,Y) \in S_{dom}$ we need to prove that $\partial(r) < Y$. From $\partial(r) < \partial(s)$ and $\partial(s) < Y$ we obtain $\partial(r) < Y$ by Lemma 1.3(2).

2. We need to show that $\partial(r) < S(s,Y)$. We have $ft(\partial(s)) < S(s,Y)$ and from $\partial(r) < \partial(s)$ we have that $\partial(r) \leq ft(\partial(s))$ by Lemma 1.6. Using Lemma 1.4(1) we get $\partial(r) < S(s,Y)$.

3. We need to show that $\partial(\widetilde{S}(r,s)) < S(r,Y)$. We have $\partial(\widetilde{S}(r,s)) = S(r, \partial(s))$. It remains to show that $S(r, \partial(s)) < S(r,Y)$. It follows from our assumption $\partial(s) < Y$ and Lemma 3.8.

Lemma 1.4 [2016.03.21.l1] *Let* **B** *be a B0-system. Let* $r, s, t \in \widetilde{B}$ *be such that* $\partial(r) < \partial(s)$ *and* $\partial(s) < \partial(t)$. *Then one has*

1. $(r,s), (r,t), (s,t) \in \widetilde{S}_{dom}$,

2. $(r, \widetilde{S}(s,t)) \in \widetilde{S}_{dom}$,

3. $(\widetilde{S}(r,s), \widetilde{S}(r,t)) \in \widetilde{S}_{dom}$.

Proof: We have:

1. This follows by applying Lemma 1.3(1) to $r, s, \partial(t)$.

2. We need to show that $\partial(r) < \partial(\widetilde{S}(s,t))$. We have $\partial(\widetilde{S}(s,t)) = S(s,\partial(t))$. We can now apply the proof of Lemma 1.3(2) to $r, s, \partial(t)$.

3. We need to show that $\partial(\widetilde{S}(r,s)) < \partial(\widetilde{S}(r,t))$. We have $\partial(\widetilde{S}(r,t)) = \widetilde{S}(r,\partial(t))$ and can now apply the proof of Lemma 1.3(3) to $r, s, \partial(t)$.

Lemma 1.5 *[2016.03.21.l2] For any $r \in \widetilde{B}$, $Y, Z \in B$ such that $\partial(r) < Y$ and $ft(Y) < Z$ one has:*

1. $(r, Y), (r, Z) \in S_{dom}$ *and* $(Y, Z) \in T_{dom}$,

2. $(r, T(Y, Z)) \in S_{dom}$,

3. $(S(r, Y), S(r, Z)) \in T_{dom}$.

Proof: We have:

1. The inclusions $(r, Y) \in S_{dom}$ and $(Y, Z) \in T_{dom}$ are immediate from the definitions. It remains to show that $(r, Z) \in S_{dom}$, that is, $\partial(r) < Z$. We have $\partial(r) < Y$ and therefore by Lemma 1.6 we have $\partial(r) \le ft(Y)$. Together with $ft(Y) < Z$, Lemma 1.4(1) gives us $\partial(r) < Z$.

2. We need to show that $\partial(r) < T(Y, Z)$. We have that $Y < T(Z, Y)$ and $\partial(r) < Y$ and applying Lemma 1.3(2) we obtain that $\partial(r) < T(Z, Y)$.

3. We need to show that $l(S(r, Y)) \ge 1$ and $ft(S(r, Y)) < S(r, Z)$. We have $l(S(r, Y)) = l(Y) - 1$. We also have that $l(\partial(r)) > 0$ and $l(Y) > l(\partial(r))$. Therefore $l(Y) \ge 2$ and $l(Y) - 1 \ge 1$. Next we have, $\partial(r) < Y$ and therefore $\partial(r) \le ft(Y)$ by Lemma 1.6. We have two cases. If $\partial(r) < ft(Y)$ then $ft(S(r, Y)) = S(r, ft(Y))$ and since $ft(Y) < Z$ we have that $S(r, ft(Y)) < S(r, Z)$ by Lemma 3.8. If $\partial(r) = ft(Y)$ then $ft(S(r, Y)) = ft(\partial(r))$ by Lemma 3.7 and we know that $ft(\partial(r)) < S(r, Z)$.

Lemma 1.6 *[2016.03.21.l4] For any $r, s \in \widetilde{B}$, $Y \in B$ such that $\partial(r) < Y$ and $ft(Y) < \partial(s)$ one has:*

1. $(r, Y) \in S_{dom}$, $(r, s) \in \widetilde{S}_{dom}$ *and* $(Y, s) \in \widetilde{T}_{dom}$,

2. $(r, \widetilde{T}(Y, s)) \in \widetilde{S}_{dom}$,

3. $(S(r, Y), \widetilde{S}(r, s)) \in \widetilde{T}_{dom}$.

Proof: We have:

1. This follows by applying Lemma 3.7(1) to $r, Y, \partial(s)$.

2. We need to show that $\partial(r) < \partial(\widetilde{T}(Y, s))$. We have that $\partial(\widetilde{T}(Y, s)) = T(Y, \partial(s))$ and the rest of the proof is the same as the proof of Lemma 3.7(2) for $r, Y, \partial(s)$.

3. We need to show that $l(S(r, Y)) \ge 1$ and $ft(S(r, Y)) < \partial(\widetilde{S}(r, s))$. We have $\partial(\widetilde{S}(r, s)) = S(r, \partial(s))$ and the rest of the proof is the same as the proof of Lemma 3.7(3) for $r, Y, \partial(s)$.

Lemma 1.7 [2016.03.21.l5a] *Let* **B** *be a B0-system. Then for any* $X, Y \in B$, $r \in \widetilde{B}$ *such that* $l(X) \geq 1$, $ft(X) < \partial(r)$, $\partial(r) < Y$ *one has:*

1. $(X, r) \in \widetilde{T}_{dom}$, $(X, Y) \in T_{dom}$, $(r, Y) \in S_{dom}$,

2. $(X, S(r, Y)) \in T_{dom}$,

3. $(\widetilde{T}(X, r), T(X, Y)) \in S_{dom}$.

Proof: We have:

1. The inclusions $(X, r) \in \widetilde{T}$ and $(r, Y) \in S_{dom}$ are immediate from the definitions. It remains to show that $(X, Y) \in T_{dom}$. We know that $l(X) \geq 1$ and need to prove that $ft(X) < Y$. We have $ft(X) < \partial(r)$ and $\partial(r) < Y$ and applying Lemma 1.3(2) we get $ft(X) < Y$.

2. We know that $l(X) \geq 1$. We need to show that $ft(X) < S(r, Y)$. We have $ft(X) < \partial(r)$. By Lemma 1.6 we get that $ft(X) \leq ft(\partial(r))$. We also have that $ft(\partial(r)) < S(r, Y)$. Combining these two inequalities and applying Lemma 1.4(1) we get that $ft(X) < S(r, Y)$.

3. We need to show that $\partial(\widetilde{T}(X, r)) < T(X, Y)$. We have $\partial(\widetilde{T}(X, r)) = T(X, \partial(r))$ and $\partial(r) < Y$. By Lemma 3.6 we conclude that $T(X, \partial(r)) < T(X, Y)$.

Lemma 1.8 [2016.03.21.l5] *Let* **B** *be a B0-system. Then for any* $X \in B$, $r, s \in \widetilde{B}$ *such that* $l(X) \geq 1$, $ft(X) < \partial(r)$, $\partial(r) < \partial(s)$ *one has:*

1. $(X, r), (X, s) \in \widetilde{T}_{dom}$, $(r, s) \in \widetilde{S}_{dom}$,

2. $(X, \widetilde{S}(r, s)) \in \widetilde{T}_{dom}$,

3. $(\widetilde{T}(X, r), \widetilde{T}(X, s)) \in \widetilde{S}_{dom}$.

Proof: We have:

1. The inclusions $(X, r) \in \widetilde{T}$ and $(r, s) \in \widetilde{S}_{dom}$ are immediate from the definitions. It remains to show that $(X, s) \in \widetilde{T}_{dom}$. This follows by applying Lemma 1.6(1) to $X, \partial(s), r$.

2. We need to show that $(X, \partial(\widetilde{S}(r, s))) \in T_{dom}$. We have $\partial(\widetilde{S}(r, s)) = S(s, \partial(s))$ and the required inclusion follows from Lemma 1.6(2) applied to $X, \partial(s), r$.

3. We need to show that $\partial(\widetilde{T}(X, r)) < \partial(\widetilde{T}(X, s))$. We have $\partial(\widetilde{T}(X, s)) = T(X, \partial(s))$ and the required inclusion follows from Lemma 1.6(3) applied to $X, \partial(s), r$.

Lemma 1.9 [2016.03.21.l6] *Let* **B** *be a B0-system. Then for any* $r \in \widetilde{B}$ *and* $Y \in B$ *such that* $ft(\partial(r)) < Y$ *one has*

1. $(\partial(r), Y) \in T_{dom}$,

2. $(r, T(\partial(r), Y)) \in S_{dom}$.

Proof: The first inclusion follows immediately from the definitions since $l(\partial(r)) \geq 1$ by the definition of a B-system carrier.

To prove the second inclusion we need to show that $\partial(r) < T(\partial(r), Y)$ which is immediate from the definition of a B0-system.

Lemma 1.10 [2016.03.21.l7] *Let* **B** *be a B0-system. Then for any* $r, s \in \widetilde{B}$ *such that* $ft(\partial(r)) < \partial(s)$ *one has*

1. $(\partial(r), s) \in \widetilde{T}_{dom}$,

2. $(r, \widetilde{T}(\partial(r), s)) \in \widetilde{S}_{dom}$.

Proof: The first inclusion follows immediately from the definitions since $l(\partial(r)) \geq 1$ by the definition of a B-system carrier.

To prove the second inclusion we need to show that $\partial(r) < \partial(\widetilde{T}(\partial(r), s))$. We have $\partial(\widetilde{T}(\partial(r), s)) = T(\partial(r), \partial(s))$ and $\partial(r) < T(\partial(r), \partial(s))$ by the definition of a B0-system.

Lemma 1.11 [2016.03.21.l8] *Let* **B** *be a (unital) B0-system. Then for any* $X, Y \in B$ *such that* $l(X) \geq 1$ *and* $ft(X) < Y$ *one has:*

1. $Y \in \delta_{dom}$,

2. $(X, \delta(Y)) \in \widetilde{T}_{dom}$,

3. $(X, Y) \in T_{dom}$,

4. $T(X, Y) \in \delta_{dom}$.

Proof: We have:

1. Since $ft(X) < Y$ we have $l(Y) \geq 1$, i.e., $Y \in \delta_{dom}$.

2. $l(X) \geq 1$ as one of our conditions. It remains to show that $ft(X) < \partial(\delta(Y))$. We have $\partial(\delta(Y)) = T(Y, Y)$. Next, we have $ft(X) < Y$ and $Y < T(Y, Y)$ and by Lemma 1.3(2) we conclude that $ft(X) < T(Y, Y)$.

3. This inclusion follows immediately from our conditions $l(X) \geq 1$ and $ft(X) < Y$.

4. We have $l(T(X, Y)) = l(Y) + 1 \geq 1$.

Lemma 1.12 [2016.03.21.l9] *Let* **B** *be a (unital) B0-system. Then for any* $r \in \widetilde{B}$, $Y \in B$ *such that* $\partial(r) < Y$ *one has:*

1. $Y \in \delta_{dom}$,

2. $(r, \delta(Y)) \in \widetilde{S}_{dom}$,

3. $(r, Y) \in S_{dom}$,

4. $S(r, Y) \in \delta_{dom}$.

Proof: We have:

1. Since $Y > \partial(s)$ we have $l(Y) \geq 1$.

2. We need to check that $\partial(r) < \partial(\delta(Y))$. We have $\partial(\delta(Y)) = T(Y, Y)$ and $Y < T(Y, Y)$. We have $\partial(r) < Y$ and $Y < T(Y, Y)$ and by Lemma 1.3(2) we get $\partial(r) < T(Y, Y)$.

3. We need to check that $\partial(r) < Y$ which is one of our conditions.

4. We need to check that $l(S(r, Y)) \geq 1$. We have $l(S(r, Y)) = l(Y) - 1$. Since $Y > \partial(r)$ and $l(\partial(r)) \geq 1$ we have $l(Y) \geq 2$ and $l(Y) - 1 \geq 1$.

Lemma 1.13 [2016.03.23.l1] *Let* **B** *be a B0-system. Then for any* $r \in \widetilde{B}$ *one has:*

1. $\partial(r) \in \delta_{dom}$,

2. $(r, \delta(\partial(r))) \in \widetilde{S}_{dom}$.

Proof: We have:

1. $l(\partial(r)) \geq 1$ by the definition of a B-system carrier and therefore $\partial(r) \in \delta_{dom}$.

2. We have to show that $\partial(r) < \partial(\delta(\partial(r)))$. By the definition of a B0-system we have $\partial(\delta(\partial(r))) = T(\partial(r), \partial(r))$ and $\partial(r) < T(\partial(r), \partial(r))$ by another part of the same definition.

Lemma 1.14 [2016.03.23.l2] *Let* **B** *be a B0-system. Then for any* $X, Y \in B$ *such that* $l(X) \geq 1$ *and* $X < Y$ *one has:*

1. $X \in \delta_{dom}$,

2. $(X, Y) \in T_{dom}$,

3. $(\delta(X), T(X, Y)) \in S_{dom}$.

Proof: We have:

1. The inclusion $X \in \delta_{dom}$ follows directly from the assumption.

2. Tor $(X, Y) \in T_{dom}$ we need to show that $l(X) \geq 1$ and $ft(X) < Y$. The first inequality is an assumption. For the second inequality we have $ft(X) \leq X$ by Lemma 1.7 and together with $X < Y$, Lemma 1.4(1) gives us $ft(X) < Y$.

3. We need to show that $\partial(\delta(X)) < T(X, Y)$. We have $\partial(\delta(X)) = T(X, X)$ and $T(X, X) < T(X, Y)$ by Lemma 3.6.

Lemma 1.15 [2016.03.23.l3] *Let* **B** *be a B0-system. Then for any* $X \in B$, $s \in \widetilde{B}$ *such that* $l(X) \geq 1$ *and* $X < \partial(s)$ *one has:*

1. $X \in \delta_{dom}$,

2. $(X, s) \in \widetilde{T}_{dom}$,

3. $(\delta(X), \widetilde{T}(X, s)) \in \widetilde{S}_{dom}$.

Proof: Using the fact that $\partial(\widetilde{T}(X, s)) = T(X, \partial(s))$ this lemma follows from Lemma 1.14 applied to X and $\partial(s)$.

2 Definition of B-systems

Definition 2.1 [2014.10.16.def2] [was.2014.06.18.eq2.to.eq11] *Let* **B** *be a non-unital B0-system. Define the following conditions on* **B***:*

1. *The TT-condition. For all* $X, Y \in B$ *such that* $l(X) \geq 1$ *and* $ft(X) < Y$ *one has*

   (a) *for all* $Z \in B$ *such that* $ft(Y) < Z$ *one has*

   $$T(T(X, Y), T(X, Z)) = T(X, T(Y, Z))$$

   where the left and the right hand sides are defined in view of Lemma 1.1.

   (b) *for all* $s \in \widetilde{B}$ *such that* $ft(Y) < \partial(s)$ *one has*

   $$\widetilde{T}(T(X, Y), \widetilde{T}(X, s)) = \widetilde{T}(X, \widetilde{T}(Y, s))$$

   where the left and the right hand sides are defined in view of Lemma 1.2.

2. *The SS-condition. For all* $r, s \in \widetilde{B}$ *such that* $\partial(r) < \partial(s)$ *one has*

   (a) *for all* $Y \in B$ *such that* $\partial(s) < Y$

   $$S(\widetilde{S}(r, s), S(r, Y)) = S(r, S(s, Y))$$

   where the left and the right hand sides are defined in view of Lemma 1.3.

   (b) *for all* $t \in \widetilde{B}$ *such that* $\partial(s) < \partial(t)$ *one has*

   $$\widetilde{S}(\widetilde{S}(r, s), \widetilde{S}(r, t)) = \widetilde{S}(r, \widetilde{S}(s, t))$$

   where the left and the right hand sides are defined in view of Lemma 1.4.

3. *The TS-condition. For any* $r \in \widetilde{B}$ *and* $Y \in B$ *such that* $\partial(r) < Y$ *one has*

   (a) *for all* $Z \in B$ *such that* $ft(Y) < Z$

   $$T(S(r, Y), S(r, Z)) = S(r, T(Y, Z))$$

   where the left and the right hand sides are defined in view of Lemma 1.5.

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(b) for all $r \in \widetilde{B}$ such that $ft(Y) < \partial(r)$

$$\widetilde{T}(S(r,Y), \widetilde{S}(r,s)) = \widetilde{S}(r, \widetilde{T}(Y,s))$$

where the left and the right hand sides are defined in view of Lemma 1.6.

4. *The ST-condition.* For any $X \in B$ and $r \in \widetilde{B}$ such that $l(X) \geq 1$ and $ft(X) < \partial(r)$ one has

(a) for all $Y \in B$ such that $\partial(r) < Y$ one has

$$S(\widetilde{T}(X,r), T(X,Y)) = T(X, S(r,Y))$$

where the left and the right hand sides are defined in view of Lemma 1.7.
(b) for all $s \in \widetilde{B}$ such that $\partial(r) < \partial(s)$ one has

$$\widetilde{S}(\widetilde{T}(X,r), \widetilde{T}(X,s)) = \widetilde{T}(X, \widetilde{S}(r,s))$$

where the left and the right hand sides are defined in view of Lemma 1.8.

5. *The STid-condition.* For any $r \in \widetilde{B}$ one has

(a) for all $Y \in B$ such that $ft(\partial(r)) < Y$ one has

$$S(r, T(\partial(r), Y)) = Y$$

where the left and the right hand sides are defined in view of Lemma 1.9.
(b) for all $s \in \widetilde{B}$ such that $ft(\partial(r)) < \partial(s)$ one has

$$\widetilde{S}(r, \widetilde{T}(\partial(r), s)) = s$$

where the left and the right hand sides are defined in view of Lemma 1.10.

Definition 2.2 *[2014.10.20.def3] Let* **B** *be a unital B0-system. Define the following conditions on* **B***:*

1. *The δT-condition.* For any $X, Y \in B$ such that $l(X) \geq 1$ and $ft(X) < Y$ one has

$$\widetilde{T}(X, \delta(Y)) = \delta(T(X,Y))$$

where the left and the right hand sides are defined in view of Lemma 1.11.

2. *The δS-condition.* For any $r \in \widetilde{B}$ and $Y \in B$ such that $\partial(r) < Y$ one has

$$\widetilde{S}(r, \delta(Y)) = \delta(S(r,Y))$$

where the left and the right hand sides are defined in view of Lemma 1.12.

3. *The δSid-condition.* For any $r \in \widetilde{B}$ one has

$$\widetilde{S}(r, \delta(\partial(r))) = r$$

where the left hand sides is defined in view of Lemma 1.13.

4. *The $S\delta T$-condition.* For any $X \in B$ such that $l(X) \geq 1$ one has

(a) for $Y \in B$ such that $X < Y$ one has:

$$S(\delta(X), T(X,Y)) = R$$

where the left hand sides is defined in view of Lemma 1.14.

(b) for $s \in \widetilde{B}$ such that $X < \partial(s)$ one has

$$\widetilde{S}(\delta(X), \widetilde{T}(X,r)) = r$$

where the left hand sides is defined in view of Lemma 1.15.

Remark 2.3 [2014.06.14.rem2] *In the case of a syntactic B0-system, the conditions defined above can be shown as follows:*

1. *The TT-condition:*

$$\cfrac{\Gamma, T \rhd \quad \Gamma, \Delta, T' \rhd \quad \Gamma, \Delta \rhd \mathcal{J}}{\cfrac{\Gamma,T\rhd \quad \Gamma,\Delta,T'\rhd\mathcal{J}}{\Gamma,T,\Delta,T'\rhd\mathcal{J}} \qquad \cfrac{\Gamma,T,\Delta,T'\rhd \quad \Gamma,T,\Delta\rhd\mathcal{J}}{\Gamma,T,\Delta,T'\rhd\mathcal{J}}}$$

2. *The SS-condition:*

$$\cfrac{\Gamma \rhd s : T \quad \Gamma, T, \Delta \rhd s' : T' \quad \Gamma, T, \Delta, T' \rhd \mathcal{J}}{\cfrac{\Gamma\rhd s:T \quad \Gamma,T,\Delta\rhd\mathcal{J}[s]}{\Gamma,\Delta[s]\rhd\mathcal{J}[s'][s]} \qquad \cfrac{\Gamma,\Delta[s]\rhd s'[s]:T'[s] \quad \Gamma,\Delta[s],T'[s]\rhd\mathcal{J}[s]}{\Gamma,\Delta[s]\rhd\mathcal{J}[s][s']}}$$

3. *The TS-condition:*

$$\cfrac{\Gamma, T \rhd \quad \Gamma, \Delta \rhd s' : T' \quad \Gamma, \Delta, T' \rhd \mathcal{J}}{\cfrac{\Gamma,T\rhd \quad \Gamma,\Delta\rhd\mathcal{J}[s']}{\Gamma,T,\Delta\rhd\mathcal{J}[s']} \qquad \cfrac{\Gamma,T,\Delta\rhd s':T' \quad \Gamma,T,\Delta,T'\rhd\mathcal{J}}{\Gamma,T,\Delta\rhd\mathcal{J}[s']}}$$

4. *The ST-condition:*

$$\cfrac{\Gamma \rhd s : T \quad \Gamma, T, \Delta, T' \rhd \quad \Gamma, T, \Delta \rhd \mathcal{J}}{\cfrac{\Gamma\rhd s:T \quad \Gamma,T,\Delta,T'\rhd\mathcal{J}[s]}{\Gamma,\Delta[s],T'[s]\rhd\mathcal{J}[s]} \qquad \cfrac{\Gamma,\Delta[s],T'[s]\rhd \quad \Gamma,\Delta[s]\rhd\mathcal{J}[s]}{\Gamma,\Delta[s],T'[s]\rhd\mathcal{J}[s]}}$$

5. *The STid-condition:*

$$\cfrac{\Gamma \rhd s : T \quad \Gamma, T \rhd \quad \Gamma \rhd \mathcal{J}}{\cfrac{\Gamma\rhd s:T \quad \Gamma,T\rhd\mathcal{J}}{\Gamma\rhd\mathcal{J}[s]}}$$

6. *The δT-condition:*

$$\cfrac{\Gamma, T \rhd \quad \Gamma, \Delta, x : T' \rhd}{\cfrac{\Gamma,T\rhd \quad \Gamma,\Delta,x:T'\rhd x:T'}{\Gamma,T,\Delta,x:T'\rhd x:T'} \qquad \cfrac{\Gamma,T,\Delta,x:T'\rhd}{\Gamma,T,\Delta,x:T'\rhd x:T'}}$$

7. *The δS-condition:*

$$\cfrac{\Gamma \rhd s : T \quad \Gamma, T, \Delta, x : T' \rhd}{\cfrac{\Gamma\rhd s:T \quad \Gamma,T,\Delta,x:T'\rhd x:T'}{\Gamma,\Delta[s],x:T'[s]\rhd x:T'[s]} \qquad \cfrac{\Gamma,\Delta[s],x:T[s]'\rhd}{\Gamma,\Delta[s],x:T'[s]\rhd x:T'[s]}}$$

8. *The δSid-condition:*

$$\cfrac{\Gamma \rhd s : T \quad \Gamma, x : T \rhd}{\cfrac{\Gamma\rhd s:T \quad \Gamma,x:T\rhd x:T}{\Gamma\rhd s:T}}$$

9. *The $S\delta T$-condition:*

$$\cfrac{\Gamma, y : X, \Delta \rhd \mathcal{J}}{\cfrac{\Gamma,y_1:X,y:X,\Delta\rhd\mathcal{J} \quad \Gamma,y_1:X\rhd y_1:X}{\Gamma,y_1:X,\Delta[y_1/y]\rhd\mathcal{J}[y_1/y]}}$$

Definition 2.4 *[2014.10.10.def2a] [2014.10.20.def4] A non-unital B-system is a non-unital B0-system that satisfy the conditions TT, SS, TS, ST and STid of Definition 2.1.*

Definition 2.5 *[2014.10.10.def2b] [2014.10.20.def5] A unital B-system is a unital B0-system that satisfy the conditions TT, SS, TS, ST, STid of Definition 2.1 and the conditions δT, δS, δSid and $S\delta T$ of Definition 2.2.*

Equivalently, a unital B-system is non-unital B-system such that there exists a family of operations δ satisfying the conditions δT, δS, δSid and $S\delta T$ of Definition 2.2.

3 Elementary properties of B-systems

Remark 3.1 In unital B-systems operations S and T can be expressed as follows.

$$[\textbf{2014.10.14.eq1}]T(X,Y) = \begin{cases} X & \text{if } l(Y) = l(X) - 1 \\ ft(\partial(\widetilde{T}(X, \delta(Y)))) & \text{if } l(Y) \geq l(X) \end{cases} \tag{11}$$

$$[\textbf{2014.10.14.eq2}]S(s,X) = \begin{cases} ft(\partial(s)) & \text{if } l(X) = l(\partial(s)) \\ ft(\partial(\widetilde{S}(s, \delta(X)))) & \text{if } l(X) > l(\partial(s)) \end{cases} \tag{12}$$

Lemma 3.2 *[2014.10.20.l1] [2014.10.16.l1] Let B be a non-unital B0-system and let δ_1, δ_2 be two operations of the form $\delta_{dom} \to \widetilde{B}$ satisfying the condition of Definition 3.4 and conditions δT, δSid and $S\delta T$ conditions. Then $\delta_1 = \delta_2$.*

Proof: We have:

$$\delta_1(X) = \widetilde{S}(\delta_2(X), \widetilde{T}(X, \delta_1(X))) = \widetilde{S}(\delta_2(X), \delta_1(T(X,X))) = \delta_2(X)$$

where the first equality is the $S\delta T$-condition for δ_2, the second equality is the δT-condition for δ_1 and the third equality is the δSid-condition for δ_1.

Example 3.3 [2014.10.20.eX] While being unital is a property of non-unital B-systems not every homomorphism of non-unital B-systems preserves units. Here is a sketch of an example of a homomorphism that does not preserve units.

Consider the following pairs of a monad and a left module over it. In both cases pt is the constant functor corresponding to the one point set $\{T\}$ that has a unique left module structure over any monad.

1. (R_1, pt) where R_1 is the monad corresponding to one unary operation $s_1(x)$ and the relation

$$s_1(s_1(x)) = s_1(x)$$

2. (R_2, pt) where R_2 is the monad corresponding to two unary operations $s_1(x)$ and $s_2(x)$ and relations:

$$s_1(s_1(x)) = s_1(x) \quad s_1(s_2(x)) = s_1(x) \quad s_2(s_1(x)) = s_1(x) \quad s_2(s_2(x)) = s_2(x)$$

Consider the unital B-systems $uB(R_1, pt)$ and $uB(R_2, pt)$. In $uB(R_1, pt)$ consider the non-unital sub-B-system nuB_1 generated by $(T \triangleright s_1(1) : T)$. In $uB(R_2, pt)$ consider the non-unital sub-B-system nuB_2 generated by $(T \triangleright s_1(1) : T)$ and $(T \triangleright s_2(1) : T)$.

Observe that both nuB_1 and nuB_2 are in fact unital with the unit in the first one given by $(T, \ldots, T \triangleright s_1(n) : T)$ and unit in the second one is given by $(T, \ldots, T \triangleright s_2(n) : T)$ where n is the number of T's before the turnstile $\triangleright$ symbol.

We also have an obvious (unital) homomorphism from $uB(R_1, pt)$ to $uB(R_2, pt)$ that defines a homomorphism $nuB_1 \to nuB_2$ and that latter homomorphism is not unital.

4 B-systems and C-systems

1 Some general results on C-systems

Let us recall (cf. [6]) that for a C-system CC an object Y is said to be an object over X if $Y \geq X$. In this case the composition of the canonical projections $Y \overset{p_Y}{\to} ft(Y) \overset{p_{ft(Y)}}{\to} \ldots \to X$ is denoted by $p_{Y,X}$. For a morphism $f : X' \to X$ one defines $f^*(Y)$ by induction using the f^* structure of the C-system. One also defines by induction a morphism $q(f, Y) : f^*(Y) \to Y$. For more detail see [8, Section 2].

Lemma 1.1 *[2016.02.18.l4] In the context introduced above one has $f^*(Y) \geq X'$,*

$$[\mathbf{2016.02.18.eq3}]l(f^*(Y)) - l(X') = l(Y) - l(X) \tag{13}$$

and the square

$$
\begin{array}{ccc}
f^*(Y) & \xrightarrow{q(f,Y)} & Y \\
{\scriptstyle [\mathbf{2016.02.18.eq2}]p_{f^*(Y),X'}}\Big\downarrow & & \Big\downarrow{\scriptstyle p_{Y,X}} \\
X' & \xrightarrow{\quad f \quad} & X
\end{array}
\tag{14}
$$

is a pull-back square.

Proof: By induction on $l(Y) - l(X)$ using the fact that vertical composition of pull-back squares is a pull-back square.

For $Y, Y' \geq X$ a morphism $g : Y \to Y'$ is said to be a morphism over X if $p_{Y,X} = g \circ p_{Y',X}$. For such a morphism g and a morphism $f : X' \to X$ there is a unique morphism $f^*(g) : f^*(Y) \to f^*(Y')$ over X' such that the square

$$
\begin{array}{ccc}
f^*(Y) & \xrightarrow{q(f,Y)} & Y \\
{\scriptstyle [\mathbf{2016.02.18.eq1}]f^*(g)}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\
f^*(Y') & \xrightarrow{q(f,Y')} & Y'
\end{array}
\tag{15}
$$

commutes (see [8, Lemma 2.1]).

We will also need the following lemmas.

Lemma 1.2 [2016.03.15.l11] *Let Y be an object over X and $f' : X'' \to X'$, $f : X' \to X$ two morphisms. Then one has:*

$$[\mathbf{2016.03.15.eq4}](f')^*(f^*(Y)) = (f' \circ f)^*(Y) \tag{16}$$

and

$$[\mathbf{2016.03.15.eq5}]q(f' \circ f, Y) = q(f', f^*(Y)) \circ q(f, Y) \tag{17}$$

Proof: By induction on $j = l(Y) - l(X)$ using the definition given in [8, Section 2] and one of the axioms of the definition of a C-system, see [9, Definition 2.1(axiom 7)].

Lemma 1.3 [2016.03.15.l10] *Let $g : Y \to Y'$ be a morphism over X and $f' : X'' \to X'$, $f : X' \to X$ two morphisms. Then one has:*

$$[\mathbf{2016.03.15.eq4mor}](f')^*(f^*(g)) = (f' \circ f)^*(g) \tag{18}$$

Proof: Note first that both the right and the left hand sides of (1.3) are morphisms over X. The domains of the left and the right hand sides of (1.3) are $(f')^*(f^*(Y))$ and $(f' \circ f)^*(Y)$ coincide by Lemma 1.2. The codomains $(f')^*(f^*(Y'))$ and $(f' \circ f)^*(Y')$ coincide by the same lemma. In view of the uniqueness part of [8, Lemma 2.1] it remains to show that

$$(f')^*(f^*(g)) \circ q(f' \circ f, Y') = q(f' \circ f, Y) \circ g$$

This follows from the equalities

$$q(f' \circ f, Y') = q(f', f^*(Y')) \circ q(f, Y')$$

and

$$q(f' \circ f, Y) = q(f', f^*(Y)) \circ q(f, Y)$$

proved in Lemma 1.2 and the definition of $(f')^*(f^*(g))$.

Lemma 1.4 [2016.01.27.l8] *Let CC be a C-system, X an object over X' and X' an object over X'' then one has $p_{X,X''} = p_{X,X'} \circ p_{X',X''}$.*

Proof: By induction on $l(X') - l(X)$.

Lemma 1.5 [2016.01.27.l7] *Let CC be a C-system, $f : Y \to Y'$ be a morphism over X and X be an object over W. Then f is a morphism over W.*

Proof: Follows easily from Lemma 1.4.

Lemma 1.6 [2016.02.20.l9] *Let $Y > X$ and $f : X' \to X$. Then one has*

$$ft(f^*(Y)) = f^*(ft(Y))$$

Proof: It follows from the inductive definition of f^* since for $l(Y) - l(X) > 0$ we have $f^*(Y) = q(f, ft(Y))^*(Y)$ where $q(f, ft(Y)) : f^*(ft(Y)) \to ft(Y)$ and $q(f, ft(Y))^*(Y)$ is given by the C-system structure that satisfies the axiom $ft(a^*(Y)) = dom(a)$.

2 The B0-systems of C-systems

2.1 Construction of the pre-B-system $CB(CC)$

Problem 2.1 *[2016.01,27.prob2] Let CC be a C-system. To construct a pre-B-system*

$$CB(CC) = (B(CC), \widetilde{B}(CC), l, ft, \partial, T, \widetilde{T}, S, \widetilde{S})$$

Construction 2.2 *[2016.01.27.constr1]* Let $B(CC) = Ob(CC)$ and

$$\widetilde{B}(CC) = \{s \in Mor(CC) \,|\, dom(s) = ft(codom(s)) \text{ and } s \circ p_{codom(s)} = Id_{dom(s)}\}$$

(this set was previously denoted by $\widetilde{Ob}(CC)$). The triple $(B(CC), l, ft)$ is an lft-set and we have the function of sets

$$\partial = codom : \widetilde{B}(CC) \to B(CC)$$

obtaining a B-system carrier.

Starting with these data we can define the sets $T_{dom}, \widetilde{T}_{dom}, S_{dom}, \widetilde{S}_{dom}$ and δ_{dom}.

Next, we define operations $T, \widetilde{T}, S, \widetilde{S}, \delta$ as follows:

$$T(X,Y) = p_X^*(Y) \quad \widetilde{T}(X,s) = p_X^*(s)$$

$$S(r,Y) = r^*(Y) \quad \widetilde{S}(r,s) = r^*(s)$$

$$\delta(X) = s_{Id_X}$$

The first of these operations is defined because $Y > ft(X)$ and therefore Y is over $ft(X)$.

The second one is defined because $s : ft(\partial(s)) \to \partial(s)$ is a morphism over $ft(\partial(s))$ and since $\partial(s) > ft(X)$ we have that $ft(\partial(s)) \geq ft(X)$ by Lemma 1.6 and therefore $ft(\partial(s))$ is an object over $ft(X)$ and so the morphism s is a morphism over $ft(X)$ by Lemma 1.5.

The third of one is defined because Y is over $\partial(r)$.

The fourth one is defined because $s : ft(\partial(s)) \to \partial(s)$ is a morphism over $ft(\partial(s))$ while r is of the form $ft(\partial(r)) \to \partial(r)$ and since $\partial(s) > \partial(r)$ we have that $ft(\partial(s)) \geq \partial(r)$ by Lemma 1.6 and therefore s is a morphism over $\partial(r)$ by Lemma 1.5. . Finally δ is defined because s_f is defined for any morphism of the form $f : X \to Y$ where $l(Y) > 0$ (cf. [9, Definition 2.3]).

This completes Construction 2.2.

2.2 Pre-B-systems $CB(CC)$ are B0-systems

Lemma 2.3 *[2016.02.18.l6] Let CC be a C-system. Then $CB(CC)$ is a B0-system.*

Proof:

1. Let $(X,Y) \in T_{dom}$. Then one has:

(a) We have $l(T(X,Y)) = l(p_X^*(Y))$. To define $p_X^*(Y)$ we consider Y as an object over $ft(X)$. Therefore by Lemma 1.1 we have

$$l(p_X^*(Y)) = l(X) + (l(Y) - l(ft(X))) = (l(X) - l(ft(X))) + l(Y)$$

Since $l(X) \geq 1$ we have $l(X) - l(ft(X)) = 1$. Therefore

$$l(T(X,Y)) = l(Y) + 1$$

(b) By Lemma 1.1 we have $T(X,Y) \geq X$ and

$$l(T(X,Y)) - l(X) = l(Y) - l(ft(X)) > 0$$

therefore $T(X,Y) > X$.

(c) It follows from Lemma 1.6.

2. Let $(X,s) \in \widetilde{T}_{dom}$. The first condition follows from the second one and condition 1(a). The second condition follows immediately from the definitions.

3. Let $(r,Y) \in S_{dom}$. Then one has:

(a) $l(S(r,Y)) = l(r^*(Y))$ were Y is considered as an object over $X = \partial(r)$. By Lemma 1.1 we have

$$l(r^*(Y)) = l(ft(X)) + (l(Y) - l(X)) = l(Y) + (l(ft(X)) - l(X)) = l(Y) - 1$$

where the last equality follows from the fact that $l(ft(X)) - l(X) = -1$ since $l(X) = l(\partial(r)) > 0$.

(b) By Lemma 1.1 we have $r^*(Y) \geq ft(X)$ and since $Y > X$ the same lemma implies that $r^*(Y) > X$.

(c) It follows from Lemma 1.6.

4. The first condition follows from the second one and condition 2(a). The second condition follows immediately from the definitions.

2.3 Functoriality of the CB-construction

Problem 2.4 [2016.01.27.prob3] *Let $f : CC \to CC'$ be a homomorphism of C-systems. To construct a homomorphism of pre-B-systems $CB(f) : CB(CC) \to CB(CC')$.*

Construction 2.5 [2016.01.27.constr3] We need to construct a morphism of B-system carriers and show that it is a homomorphism of pre-B-systems.

We already have the function $f : B(CC) \to B(CC')$ and by the definition of a homomorphism of C-systems (cf. [7, Definition 3.1]) it is a morphism of lft-sets.

By definition $\widetilde{B}(CC)$ is a subset of $Mor(CC)$. Therefore, by the morphism part of the functor f it is mapped to a subset of $Mor(CC')$. We need to verify that the image of $\widetilde{B}(CC)$ lies in $\widetilde{B}(CC')$. The subset $\widetilde{B}(CC)$ is the subset of elements s such that $dom(s) = ft(codom(s))$ and $s \circ p_{codom(s)} = Id_{dom(s)}$. It follows that it will be mapped to the subset defined by the same conditions

by any functor that maps the p-morphisms of CC to p-morphisms of CC' and in particular by any homomorphism of C-systems. We obtain a function $\widetilde{B} \to \widetilde{B'}$ that we denote by $\widetilde{f}$.

It is immediate from the construction that the pair $\mathbf{f} = (f, \widetilde{f})$ is a morphism of B-system carriers.

By Construction 2.8 we obtain functions $f_T, f_{\widetilde{T}}, f_S, f_{\widetilde{S}}, f_\delta$.

The fact that these functions commute, in the sense of Definition 2.9, with the pre-B-system operations follows from [6, Lemma 2.3(3,4,5)].

Lemma 2.6 [2016.01.29.l1] *Let U be a universe. Then Constructions 2.2 and 2.5 define a functor CB_U from the category of C-systems in U to the category of pre-B-systems in U.*

Proof: Since two homomorphisms of pre-B-systems are equal if and only if the underlying morphisms of the B-system carriers are equal it is sufficient to prove the identity and composition axioms of a functor for the mappings from C-systems and their homomorphisms to the carriers of pre-B-systems and their morphisms. These axioms follow immediately from the fact that $\widetilde{B}(CC)$ is a subset of $Mor(CC)$ and the definition of composition of C-system homomorphisms.

We will sometimes call the functor CB defined by Constructions 2.2, 2.5 and Lemma 2.6, the B-sets functor.

2.4   Operations T_{ext} for B0-systems of the form $CB(CC)$

Lemma 2.7 [2016.03.15.l8] *Let CC be a C-system and $X, Y \in B(CC)$ be such that $l(X) \geq 1$, $l(Y) \geq 1$ and $ft(X) \leq Y$. Then one has*

$$T_{ext}(X, Y) = p^*_{X, ft(X)}(Y)$$

Proof: Straightforward by case analysis from the definition of the operation T in the B0-systems of the form $B(CC)$.

2.5   Operations T^* in the B0-systems of the form $CB(CC)$

Lemma 2.8 [2016.02.20.l10] *Let CC be a C-system. Let $(X, Y, Z) \in T^*_{dom}$, where the pre-B-system concepts refer to $CB(CC)$, then one has*

$$T^*(X, Y, Z) = p^*_{X,Y}(Z)$$

and in particular we have a pull-back square of the form

$$
\begin{array}{ccc}
T^*(X,Y,Z) & \xrightarrow{q(p_{X,Y},Z)} & Z \\
{\scriptstyle p_{T^*(X,Y,Z),X}}\downarrow & & \downarrow{\scriptstyle p_{Z,Y}} \\
X & \xrightarrow{p_{X,Y}} & Y
\end{array}
$$

Proof: By induction on $j = l(X) - l(Y)$, from Lemma 2.7, Lemma 1.2, the definition of $T^*(X,Y,Z)$ and the definition of $p_{X,Y}$.

The objects involved in the construction for the successor can be seen on the diagram:

$$T_{ext}(X, T^*(ft(X), Y, Z)) \longrightarrow T^*(ft(X), Y, Z) \longrightarrow Z$$

[2016.02.20.eq1]

$$\begin{array}{ccccc} & \downarrow & & \downarrow & \downarrow p_{Z,Y} \quad (19) \\ X & \xrightarrow{\;p_{X,ft(X)}\;} & ft(X) & \xrightarrow{\;p_{ft(X),Y}\;} & Y \end{array}$$

and one has to use the fact that

$$p^*_{X,ft(X)}(p^*_{ft(X),Y}(Z)) = (p_{X,ft(X)} \circ p_{ft(X),Y})^*(Z) = p^*_{X,Y}(Z)$$

which follows from Lemma 1.2.

2.6 Operations $\widetilde{T}^*$ in the B0-systems of the form $CB(CC)$

Lemma 2.9 [2016.03.15.l9] *Let CC be a C-system. Let $(X, Y, s) \in \widetilde{T}^*_{dom}$, where the B0-system concepts refer to $CB(CC)$, then one has*

$$\widetilde{T}^*(X, Y, s) = p^*_{X,Y}(s)$$

and in particular we have a pull-back square of the form

$$\begin{array}{ccc} T^*_{ext}(X, Y, ft(\partial(s))) & \xrightarrow{q(p_{X,Y}, ft(\partial(s)))} & ft(\partial(s)) \\ \downarrow p^*_{X,Y}(s) & & \downarrow s \\ T^*(X, Y, \partial(s)) & \xrightarrow{q(p_{X,Y}, \partial(s))} & \partial(s) \\ p_{T^*(X,Y,\partial(s)),X} \downarrow & & \downarrow p_{\partial(s),Y} \\ X & \xrightarrow{\;p_{X,Y}\;} & Y \end{array}$$

Proof: By induction on $j = l(X) - l(Y)$, from Lemma 2.7, Lemma 1.3, the definition of $\widetilde{T}$, the definition of $\widetilde{T}^*(X, Y, Z)$ and the definition of $p_{X,Y}$.

The objects involved in the construction for the successor can be seen on the diagram:

$$\begin{array}{ccccc} T^*_{ext}(X, Y, ft(\partial(s))) & \longrightarrow & T^*_{ext}(ft(X), Y, ft(\partial(s))) & \longrightarrow & ft(\partial(s)) \\ \downarrow & & \downarrow p^*_{ft(X),Y}(s) & & \downarrow s \\ T_{ext}(X, T^*(ft(X), Y, \partial(s))) & \longrightarrow & T^*(ft(X), Y, \partial(s)) & \longrightarrow & \partial(s) \qquad (20) \\ \downarrow & & \downarrow & & \downarrow p_{\partial(s),Y} \\ X & \xrightarrow{\;p_{X,ft(X)}\;} & ft(X) & \xrightarrow{\;p_{ft(X),Y}\;} & Y \end{array}$$

2.7 Sets $\widetilde{B}^*(X, Y)$ for B0-systems of the form $CB(CC)$

Let CC be a C-system. Let $X < Y$ in $B(CC)$. Set

$$Sec(X, Y) = \{f \in Mor(CC) \,|\, dom(f) = X, \; codom(f) = Y, \; f \circ p_{Y,X} = Id_X\}$$

that is, elements of $Sec(X, Y)$ are sections of the canonical morphism $p_{Y,X} : Y \to X$.

Problem 2.10 [2016.02.18.prob1] *Let CC be a C-system, $X, Y \in B(CC)$ and $X \leq Y$. To construct a bijection*

$$[\textbf{2016.03.11.eq2}]nt(X,Y) : \widetilde{B}^*(X,Y) \to Sec(X,Y) \tag{21}$$

To provide a construction for this problem we need the following general lemma.

Lemma 2.11 [2016.02.18.l3] *Let $\mathcal{C}$ be a category and $X \overset{a}{\to} Y \overset{b}{\to} Z$ be a composable pair of morphisms in $\mathcal{C}$. Assume further that for any $r : Z \to Y$ such that $r \circ b = Id_Z$ we are given a pull-back square of the form:*

$$
\begin{array}{ccc}
r^*(X) & \xrightarrow{pr_X^r} & X \\
{\scriptstyle pr_Z^r}\Big\downarrow & & \Big\downarrow{\scriptstyle a} \\
Z & \xrightarrow{\quad r \quad} & Y
\end{array}
$$

Let $R(a, b)$ be the set of pairs (r, s) where $r : Z \to Y$ is such that $r \circ b = Id_Z$ and $s : Z \to r^*(X)$ is such that $s \circ pr_Z^r = Id_Z$.

Then the function $(r, s) \mapsto s \circ pr_X^r$ is a bijection from R to the set of morphisms $t : Z \to X$ such that $t \circ a \circ b = Id_Z$.

Proof: Note first that

$$s \circ pr_X^r \circ a \circ b = s \circ pr_Z^r \circ r \circ b = Id_Z \circ Id_Z = Id_Z$$

Let us show that our function is surjective. Let $f : Z \to X$ be a morphism such that $f \circ a \circ b = Id_Z$. Let $r = f \circ a$. Then $r \circ b = Id_Z$ and on the other hand $f = s \circ pr_X^r$ for some s such that $s \circ pr_Z^r = Id_Z$ by the universal property of the pull-backs.

Let us show that our function is injective. Let (r, s), (r', s') be two elements of $R(a, b)$ such that $s \circ pr_X^r = s' \circ pr_X^{r'}$. We have

$$s \circ pr_X^r \circ a = s \circ pr_Z^r \circ r = r$$

We conclude that $r = r'$. Then if $s \circ pr_X^r = s' \circ pr_X^r$ and $s \circ pr_Z^r = Id_Z = s' \circ pr_Z^r$ we have that $s = s'$ by the universal property of the pull-backs.

The lemma is proved.

We can now provide a construction for Problem 2.10.

Construction 2.12 [2016.02.18.constr1] We proceed by induction on $j = l(Y) - l(X)$.

If $j = 0$ then $p_{Y,X} = Id_X$ and both sides are one element sets.

If $j = 1$ then we can define $nt(X, Y)$ as the identity function because by definition of $\widetilde{B}(X)$ and $\widetilde{B}(CC)$ the left and the right hand sides of (21) are equal.

For the successor of $j > 0$ we have that $l(Y) \geq j+1$ and $X = ft^{j+1}(Y)$. By the inductive assumption $nt(X', Y')$ is already constructed for all pairs X', Y' such that $X' \leq Y'$ and $l(Y') - l(X') \leq j$.

By Construction **??** the set $\widetilde{B}^*(ft^{j+1}(Y), Y)$ is the set of pairs (r, s) where $r \in \widetilde{B}(ft^j(Y))$ and $s \in \widetilde{B}^*(ft^{j+1}(Y), S(r, Y))$.

Since r is a morphism of the form $r : ft^{j+1}(Y) \to ft^j(Y)$, by Lemma 1.1 we have that $S(r, Y)$ is over $ft^{j+1}(Y)$ and that the square

$$
\begin{array}{ccc}
S(r, Y) & \xrightarrow{q(r,Y)} & Y \\
{\scriptstyle p_{S(r,Y),ft^{j+1}(Y)}}\downarrow & & \downarrow{\scriptstyle p_{Y,ft^j(Y)}} \\
ft^{j+1}(Y) & \xrightarrow{r} & ft^j(Y)
\end{array}
\tag{22}
$$

[**2016.02.18.eq4**]

is a pull-back square. Since $l(Y) \geq j + 1$ we have that

$$l(Y) - l(ft^j(Y)) = j$$

and therefore by the equation (13) of the same lemma we have that

$$ft^j(S(r, Y)) = ft^{j+1}(Y)$$

Consider the set $R(p_{Y,ft^j(Y)}, p_{ft^j(Y),ft^{j+1}(Y)})$ where R is as in Lemma 2.11 relative to the pull-back squares (22). Then the function $(r, s) \mapsto (r, nt(ft^{j+1}(Y), S(r, Y))(s))$ is a bijection of the form

$$\widetilde{B}^*(ft^{j+1}(Y), Y) \to R(p_{Y,ft^j(Y)}, p_{ft^j(Y),ft^{j+1}(Y)}).$$

Composing this bijection with the bijection of Lemma 2.11 and using the fact that

$$p_{Y,ft^j(Y)} \circ p_{ft^j(Y),ft^{j+1}(Y)} = p_{Y,ft^{j+1}(Y)}$$

we obtain the bijection that is the goal of the construction.

This completes Construction 2.12.

2.8   Sets $Sec(X, Y)$ and homomorphisms of C-systems

Problem 2.13 [2016.03.17.prob1] *Let $F : CC \to CC'$ be a homomorphism of C-systems. To construct a function*

$$[\textbf{2016.03.17.eq1}] F_{Sec(X,Y)} : Sec(X, Y) \to Sec(F(X), F(Y)) \tag{23}$$

Construction 2.14 [2016.03.17.constr1] Note first that the right hand side of (23) is well defined by Lemma 1.11. By definition, $Sec(X, Y)$ is a subset of $Mor(X, Y)$ therefore it is sufficient to show that for $s \in Sec(X, Y)$ we have $F_{Mor}(s) \in Sec(F(X), F(Y))$. This follows immediately from the fact that F commutes with compositions and identities and takes p-morphisms of CC to p-morphisms of CC'.

2.9   Functions $\widetilde{B}^*(X, Y) \to Sec(X, Y)$ and homomorphisms of C-systems

Lemma 2.15 [2016.03.17.l1] *Let $F : CC \to CC'$ be a homomorphism of C-systems. Let $X \leq Y$ in $B(CC)$. Let*

$$CB(F)_{\widetilde{B}^*(X,Y)} : \widetilde{B}^*(X, Y) \to \widetilde{B}^*(F(X), F(Y))$$

be the function of Construction **??** *and*

$$F_{Sec(X,Y)} : Sec(X, Y) \to Sec(F(X), F(Y))$$

the function of Construction 2.14.

Let

$$nt(X, Y) : \widetilde{B}^*(X, Y) \to Sec(X, Y)$$

$$nt(F(X), F(Y)) : \widetilde{B}^*(F(X), F(Y)) \to Sec(F(X), F(Y))$$

be the functions of Construction 2.12. Then the square

$$
\begin{array}{ccc}
\widetilde{B}^*(X, Y) & \xrightarrow{\ nt(X,Y)\ } & Sec(X, Y) \\
CB(F)_{\widetilde{B}^*(X,Y)} \downarrow & & \downarrow F_{Sec(X,Y)} \\
\widetilde{B}^*(F(X), F(Y)) & \xrightarrow{\ nt(F(X),F(Y))\ } & Sec(F(X), F(Y))
\end{array}
$$

commutes.

Proof: Since the construction of $\widetilde{B}^*(X, Y)$ is by induction on $j = l(Y) - l(X)$ so is the proof of the lemma. Note that $l(F(Y)) - l(F(X)) = l(Y) - l(X)$ because homomorphisms of C-systems preserve the lengths of objects.

For $j = 0$ the target set is a one element set and any two functions to a one element set with the same domain coincide.

For $j = 1$ we have

$$\widetilde{B}^*(X, Y) = \widetilde{B}(Y) = \{ s \in Mor(CC) \,|\, dom(s) = X, codom(s) = Y, s \circ p_{Y,X} = Id \}$$

and $nt(X, Y)$ is in this case the identity function. The same is true for $nt(F(X), F(Y))$ and it remains to check that the definitions of $CB(F)_{\widetilde{B}^*(X,Y)}$ and $F_{Sec(X,Y)}$ in this case agree, which they do.

For the successor of $j > 0$ we have

$$\widetilde{B}^*(X, Y) = \{ (r, s) \,|\, r \in \widetilde{B}(ft^j(Y)) \ s \in \widetilde{B}^*(ft^{j+1}(Y), S(r, Y)) \}$$

and

$$CB(F)_{\widetilde{B}^*(X,Y)}(r, s) = (F_{Mor}(r), CB(F)_{\widetilde{B}^*(ft^{j+1}(Y), S(r,Y))}(s))$$

The function $nt(X, Y)$ is of the form

$$nt(X, Y)(r, s) = (r, nt(ft^{j+1}(Y), S(r, Y))(s)) = nt(ft^{j+1}(Y), S(r, Y))(s) \circ q(r, Y)$$

where $nt(ft^{j+1}(Y), S(r, Y)) \in Sec(ft^{j+1}(Y), S(r, Y))$ and $q(r, Y)$ is a part of the pull-back square

$$
\begin{array}{ccc}
S(r, Y) & \xrightarrow{\ q(r,Y)\ } & Y \\
p_{S(r,Y),ft^{j+1}(Y)} \downarrow & & \downarrow p_{Y,ft^j(Y)} \\
ft^{j+1}(Y) & \xrightarrow{\ r\ } & ft^j(Y)
\end{array}
$$

When we apply $F_{Sec(X,Y)}$, which is just the restriction of F_{Mor} to the subset $Sec(X, Y)$ of $Mor(CC)$, to $nt(X, Y)(r, s)$ we get

$$F_{Mor}(nt(X, Y)(r, s)) = F_{Mor}(nt(ft^{j+1}(Y), S(r, Y))(s) \circ q(r, Y)) =$$

$$F_{Mor}(nt(ft^{j+1}(Y), S(r,Y))(s)) \circ F_{Mor}(q(r,Y)) =$$

$$nt(F(ft^{j+1}(Y)), F(S(r,Y)))(CB(F)_{\widetilde{B}^*(nt(ft^{j+1}(Y),S(r,Y)))}(s)) \circ F_{Mor}(q(r,Y)) =$$

$$nt(ft^{j+1}(F(Y)), S(F_{Mor}(r), F(Y)))(CB(F)_{\widetilde{B}^*(nt(ft^{j+1}(Y),S(r,Y)))}(s))) \circ q(F_{Mor}(r), F(Y)) =$$

$$nt(F(X), F(Y))(F_{Mor}(r), CB(F)_{\widetilde{B}^*(nt(ft^{j+1}(Y),S(r,Y)))}(s))) =$$

$$nt(F(X), F(Y))(CB(F)_{\widetilde{B}^*(F(X),F(Y))}(r,s)).$$

where the third equality is by the inductive assumption, the fourth by [8, Lemma 2.3(3)] that implies that for a homomorphism of C-systems F one has $F(r^*(Y)) = F_{Mor}(r)^*(F(Y))$ and $F_{Mor}(q(r,Y)) = q(F_{Mor}(r), F(Y))$.

This completes proof of Lemma 2.15.

2.10 Bijections $bmor(X,Y) : BMor_{CB(CC)}(X,Y) \to Mor_{CC}(X,Y)$

Problem 2.16 *[**2016.02.20.prob3**] For a C-system CC and $X, Y \in CC$ to construct a bijection*

$$bmor(X,Y) : BMor(X,Y) \to Mor_{CC}(X,Y)$$

Construction 2.17 [**2016.02.20.constr3**] By Construction 2.12 to Problem 2.10 we have a bijection

$$[\mathbf{2016.02.20.eq5b}]nt(X, T^*(X, pt, Y)) : BMor(X,Y) \to Sec(X, T^*(X, pt, Y)) \qquad (24)$$

By Lemma 2.8 we have a pull-back square

$$[\mathbf{2016.03.17.eq2}]\begin{array}{ccc} T^*(X, pt, Y) & \xrightarrow{q(p_{X,pt}, Y)} & Y \\ {\scriptstyle p_{T^*(X,pt,Y),X}}\big\downarrow & & \big\downarrow{\scriptstyle p_{Y,pt}} \\ X & \xrightarrow{p_{X,pt}} & pt \end{array} \qquad (25)$$

Therefore, the function $s \mapsto s \circ q(p_{X,pt}, Y)$ is a bijection between $Sec(X, T^*(X, pt, Y))$ and $Mor_{CC}(X, Y)$. Composing this bijection with $nt(X, T^*(X, pt, Y))$ we obtain a bijection

$$BMor(X,Y) \to Mor_{CC}(X,Y)$$

This completes Construction 2.17.

2.11 Bijections $bmor(X,Y)$ and homomorphisms of C-systems

Theorem 2.18 *[**2016.03.15.th1**] Let $F = (F_{Ob}, F_{Mor}) : CC \to CC'$ be a homomorphism of C-systems. Then for any $X, Y \in B(CC)$ the square*

$$[\mathbf{2016.03.17.eq3}]\begin{array}{ccc} BMor(X,Y) & \xrightarrow{CB(F)_{BMor,X,Y}} & BMor(F(X), F(Y)) \\ {\scriptstyle bmor(X,Y)}\big\downarrow & & \big\downarrow{\scriptstyle bmor(F_{Ob}(X), F_{Ob}(Y))} \\ Mor_{CC}(X,Y) & \xrightarrow{F_{Mor,X,Y}} & Mor_{CC'}(F_{Ob}(X), F_{Ob}(Y)) \end{array} \qquad (26)$$

commutes.

Proof: By definition
$$BMor(X,Y) = \widetilde{B}^*(X, T^*(X, pt, Y)).$$

The function $bmor(X,Y)$ is the composition of two functions:
$$bmor(X,Y) = nt(X, T^*(X, pt, Y)) \circ \Phi(X,Y)$$

where for $s \in Sec(X, T^*(X, pt, Y))$, $\Phi(X,Y)(s) = s \circ q(p_{X,pt}, Y)$, see the square (25).

Therefore the square (26) is the vertical composition of two squares:

$$
\begin{array}{ccc}
BMor(X,Y) & \xrightarrow{\ CB(F)_{BMor,X,Y}\ } & BMor(F(X),F(Y)) \\
{\scriptstyle nt(X,T^*(X,pt,Y))}\Big\downarrow & & \Big\downarrow{\scriptstyle nt(F(X),T^*(F(X),pt,F(Y)))} \\
Sec(X,T^*(X,pt,Y)) & \xrightarrow{\ F_{Sec(X,T^*(X,pt,Y))}\ } & Sec(F(X),T^*(F(X),pt,F(Y)))
\end{array}
$$

where we used Lemma 3.16 to identify $F(T^*(X, pt, Y))$ with $T^*(F(X), pt, F(Y))$, and the square

$$
\begin{array}{ccc}
Sec(X,T^*(X,pt,Y)) & \xrightarrow{\ F_{Sec(X,T^*(X,pt,Y))}\ } & Sec(F(X),T^*(F(X),pt,F(Y))) \\
{\scriptstyle \Phi(X,Y)}\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi(F(X),F(Y))} \\
Mor(X,Y) & \xrightarrow{\ F_{Mor,X,Y}\ } & Mor(F(X),F(Y))
\end{array}
$$

The commutativity of the first square follows from Lemma 2.15.

For the commutativity of the second square we have:

$$F_{Mor,X,Y}(\Phi(X,Y)(s)) = F_{Mor}(s \circ q(p_{X,pt}, Y)) = F_{Mor}(s) \circ F_{Mor}(q(p_{X,pt}, Y)) =$$

$$F_{Mor}(s) \circ q(F_{Mor}(p_{X,pt}), F(Y)) = F_{Mor}(s) \circ q(p_{F(X),pt}, F(Y)) = \Phi(F(X), F(Y))(F_{Mor}(s)) =$$

$$\Phi(F(X), F(Y))(F_{Sec(X,T^*(X,pt,Y))}(s))$$

where the third equality is by [8, Lemma 2.3(3)] and the fourth equality is by [8, Lemma 2.3(1)] and the fact that $F(pt_{CC}) = pt_{CC'}$.

Corollary 2.19 *[2016.03.11.cor1] For any universe U the functor*
$$CB_U : CSys(U) \to BSys(U)$$

is faithful.

Proof: Let $F_1, F_2 : CC \to CC'$ be two homomorphisms of C-systems such that $CB(F_1) = CB(F_2)$. Since $CB(CC) = (B(CC), \widetilde{B}(CC), \partial)$ where $B(CC)$ is the set of objects of CC and similarly for CC' we know that the object components of F_1 and F_2 coincide. To show that $F_1 = F_2$ it remains to show that their morphism components coincide.

In the diagrams (26) for the functors F_1 and F_2 the upper horizontal arrows coincide because of the assumptions that $CB(F_1) = CB(F_2)$. The vertical arrows coincide because they depend only on the domain and codomain of F_1 and F_2. Since the vertical arrows are bijections we conclude that the lower horizontal arrows coincide. Since they coincide for all pairs of objects X, Y we conclude that $F_{1,Mor} = F_{2,Mor}$.

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