A Π-C-system defined by a Π-universe in a locally Cartesian closed category[1]

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Abstract

This is the n-th paper in a series started in [**?**].

1 Introduction

The concept of a C-system was introduced in [**?**].

2 Π-C-systems

The notion of a Π-C-system is equivalent to the notion of a contextual category with products of families of types from [**?**]. We use the name Π-C-systems to emphasize the fact that we are dealing here with an additional structure on a C-system rather than with a property of such an object.

Let us recall first the following definition.

Definition 2.1 *[**2009.11.24.def2**] Let $\mathcal{C}$ be a category. Let $g : Z \to Y$, $f : Y \to X$ be a pair of morphisms such that for any $U \to X$ a fiber product $U \times_X Y$ exists. A pair*

$$(w : W \to X, h : W \times_X Y \to Z)$$

such that $g \circ h = pr$ is called a universal pair for (f, g) if for any $U \to X$ the map

$$Hom_X(U, W) \to Hom_Y(U \times_X Y, Z)$$

of the form $u \mapsto h \circ (u \times Id_Y)$ is a bijection.

If a universal pair exists then it is easily seen to be unique up to a unique isomorphism. We denote such a pair by $(\Pi(g, f), e_{g,f} : \Pi(g, f) \times_X Y \to Z)$. Note that if $f' : Y \to X$ and $pr : Y' \times_X Y \to Y$ is the projection then

$$(\Pi(pr, f), pr' \circ e_{pr,f} : \Pi(g, f) \times_X Y \to Y') = (\underline{Hom}_X(Y, Y'), ev : \underline{Hom}_X(Y, Y') \times_X Y \to Y')$$

so that relative internal Hom-objects are particular cases of universal pairs.

Definition 2.2 *[**2009.11.24.def1**] A Π-C-system is a C-system CC together with additional data of the form*

1. *for each $Y \in Ob(CC)_{\geq 2}$ an object $\Pi(Y) \in Ob(CC)$ such that $ft(\Pi(Y)) = ft^2(Y)$,*

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2. *for each* $Y \in Ob(CC)_{\geq 2}$ *a morphism* $eval : T(ft(Y), \Pi(Y)) = p^*_{ft(Y)}(\Pi(Y)) \to Y$ *over* $ft(Y)$,

such that

(i) *for any* $f : Z \to ft^2(Y)$ *one has* $f^*(\Pi(Y)) = \Pi(f^*(Y))$ *and* $f^*(eval_Y) = eval_{f^*(Y)}$,

(ii) $(\Pi(Y), eval_Y))$ *is a universal pair for* $(p_Y, p_{ft(Y)})$.

Let us now prove that this definition can be re-written in a less compact but purely equational form. As before let us write B_n for $Ob(CC)_n$, $\widetilde{B}_n$ for $\widetilde{Ob}(CC)_n$ etc.

The C-system is completely determined by the sets $B_n, \widetilde{B}_{n+1}$, $n \geq 0$ and maps $\partial : \widetilde{B}_{n+1} \to B_{n+1}$, $ft : B_{n+1} \to B_n$, $\delta : B_n \to \widetilde{B}_{n+1}$ and the maps $T_{n+1}, \widetilde{T}_{n+1}, S_{n+1}, \widetilde{S}_{n+1}$ considered above.

Suppose now that we are given a Π-C-system. Then we have maps

1. $\Pi : B_{n+2} \to B_{n+1}$, $n \geq 0$,
2. $\lambda : \widetilde{B}_{n+2} \to \widetilde{B}_{n+1}$, $n \geq 0$,
3. $ev : (\widetilde{B}_{n+1})_{\partial} \times_{ft} (B_{n+2})_{\Pi} \times_{\partial} (\widetilde{B}_{n+1}) \to \widetilde{B}_{n+1}$, $n \geq 0$

as follows. The map Π is the map from Definition 2.2. Since $(\Pi(Y), eval_Y)$ is a universal pair for $(p_Y, p_{ft(Y)})$ the mapping

$$\phi_Y : \{f \in \widetilde{B}_{n+1} \,|\, \partial(f) = \Pi(Y)\} \to \{s \in \widetilde{B}_{n+2} \,|\, \partial(s) = Y\}$$

given by the formula

$$\phi_Y(f) = eval_Y \circ \widetilde{T}(ft(Y), f)$$

is a bijection. One defines λ_Y as the inverse to this bijection.

The map ev sends a triple (r, Y, f) such that $\partial(r) = ft(Y)$ and $\partial(f) = \Pi(Y)$ to

$$ev(r, Y, f) = \widetilde{S}(r, eval \circ \widetilde{T}(ft(Y), f))$$

as partially illustrated by the following diagram:

$$\begin{array}{ccccc}
 & & Y & \longleftarrow & S(r, Y) \\
 & & {\scriptstyle p_Y}\downarrow & & \downarrow \\
p^*_{ft(Y)}(\Pi(Y)) & \longrightarrow & ft(Y) & \underset{r}{\longleftarrow} & ft^2(Y) \\
\downarrow & & \downarrow & & \\
\Pi(Y) & \xrightarrow{p_{\Pi(Y)}} & ft^2(Y) & &
\end{array}$$

Lemma 2.3 *[***2009.11.30.l1***] Let* $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$ *and* $f \in \widetilde{B}(\Pi(Y))$. *Then one has*

$$g^*(\phi_Y(f), i+2) = \phi_{g^*(Y, i+2)}(g^*(f, i+1))$$

Proof: Let $h_1 = q(g, ft(Y), i+1)$, $h_2 = q(g, ft^(Y), i+2)$. Then one has

$$g^*(\phi_Y(f), i+2) = h_1^*(\phi_Y(f)) = h_1^*(eval_Y \circ \widetilde{T}(ft(Y), f)) = h_1^*(eval_Y) \circ h_1^*(\widetilde{T}(ft(Y), f))$$

$$= eval_{h_1^*(Y)} p^*_{g^*(ft(Y),i+1)}(h_2^*(f)) = \phi_{h_1^*(Y)}(h_2^*(f)) = \phi_{g^*(Y,i+2)}(g^*(f, i+1)).$$

As an immediate corollary of Lemma 2.3 we have:

Lemma 2.4 *[***2009.11.30.l2***] Let $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$ and $r \in \widetilde{B}(Y)$. Then one has*

$$g^*(\lambda(r), i+1) = \lambda(g^*(r, i+2)).$$

Lemma 2.5 *[***2009.11.30.l3***] Let $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$, $r \in \widetilde{B}(ft(Y))$ and $f \in \widetilde{B}(\Pi(Y))$. Then one has*

$$g^*(ev(r, Y, f), i+1) = ev(g^*(r, i+2), g^*(Y, i+2), g^*(f, i+1))$$

Proof: Let $h_1 = q(g, ft(Y), i+1)$, $h_2 = q(g, ft^(Y), i+2)$. Then one has:

$$g^*(ev(r, Y, f), i+1) = h_2^*(\widetilde{S}(r, eval \circ \widetilde{T}(ft(Y), f))) = h_2^*(r^*(eval \circ \widetilde{T}(ft(Y), f))) =$$

$$= (h_2^*(r))^* h_1^*(eval \circ \widetilde{T}(ft(Y), f))) = (h_2^*(r))^*(h_1^*(eval) \circ h_1^* p^*_{ft(Y)}(f)) =$$

$$= (g^*(r, i+2))^*(eval \circ p^*_{g^*(ft(Y),i+1)}(h_2^*(f))) = ev(g^*(r, i+2), g^*(Y, i+2), g^*(f, i+1)).$$

Proposition 2.6 *[***2009.11.29.prop1***] Let $CC = (B_n, \widetilde{B}_n, ft, \partial, \delta)$ be a C-system. Let further $(\Pi, eval)$ be a Π-structure on CC. Then the maps Π, λ, ev defined by this structure satisfy the following conditions:*

1. *for $Y \in B_{n+2}$ one has*
   (a) *$ft\,\Pi(Y) = ft^2(Y)$,*
   (b) *for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}(Y)$, $T(Z, \Pi(Y)) = \Pi(T(Z, Y))$,*
   (c) *for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^{i+1}(Y)$, $S(t, \Pi(Y)) = \Pi(S(t, Y))$,*
2. *for $s \in \widetilde{B}_{n+2}$ one has*
   (a) *$\partial\,\lambda(s) = \Pi\,\partial(s)$,*
   (b) *for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}\,\partial(s)$, $\widetilde{T}(Z, \lambda(s)) = \lambda(\widetilde{T}(Z, s))$,*
   (c) *for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^{i+1}\,\partial(s)$, $\widetilde{S}(t, \lambda(s)) = \lambda(\widetilde{S}(t, s))$,*
3. *for $r \in \widetilde{B}_{n+1}$, $Y \in B_{n+2}$ and $f \in \widetilde{B}_{n+1}$ such that $\partial(r) = ft(Y)$ and $\partial(f) = \Pi(Y)$ one has*
   (a) *$\partial(ev(r, Y, f)) = S(r, Y)$,*
   (b) *for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}(Y)$,*

$$\widetilde{T}(Z, ev(r, Y, f)) = ev(\widetilde{T}(Z, r), T(Z, Y), \widetilde{T}(Z, f)),$$

(c) for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ *such that* $\partial(t) = ft^{i+1}(Y)$,

$$\widetilde{S}(t, ev(r, Y, f)) = ev(\widetilde{S}(t, r), S(t, Y), \widetilde{S}(t, f)),$$

4. *for* $r \in \widetilde{B}_{n+1}$, $s \in \widetilde{B}_{n+2}$ *such that* $ft(\partial(s)) = \partial(r)$

$$ev(r, \partial\, s, \lambda(s)) = \widetilde{S}(r, s)$$

(β-reduction),

5. *for* $Y \in B_{n+2}$, $f \in \widetilde{B}_{n+1}$ *such that* $\partial(f) = \Pi(Y)$,

$$[\mathbf{2009.11.30.oldeq1}]\lambda(ev(\delta_{ft(Y)}, T(ft(Y), Y), \widetilde{T}(ft(Y), f))) = f \qquad (1)$$

(η-reduction).

Proof: (1a) Follows from Definition 2.2(1). (1b) Follows from Definition 2.2(i) applied to $f = q(p_Z, ft^2(Y), i-1)$. (1c) Follows from Definition 2.2(i) applied to $f = q(t, ft^2(Y), i-1)$.

(2a) Follows from the definition of λ. (2b) Follows from Lemma 2.4 applied to p_Z. (2c) Follows from Lemma 2.4 applied to t.

(3a) Follows from the definition of ev. (3b) Follows from Lemma 2.5 applied to p_Z. (3c) Follows from Lemma 2.5 applied to t.

(4) One has

$$ev(r, \partial\, s, \lambda(s)) = r^*(eval \circ (p^*_{ft(Y)}(\lambda(s)))) = r^*(\phi_Y(s)) = r^*(s) = \widetilde{S}(r, s).$$

(5) Let $T_1 = T(ft(Y), ft(Y))$ and $T_2 = T(ft(Y), Y)$. Then

$$ev(\delta_{ft(Y)}, T(ft(Y), Y), \widetilde{T}(ft(Y), f)) = \delta^*_{ft(Y)}(eval_{T_2} \circ p^*_{T_1}(p^*_{ft(Y)}(f))) =$$

$$= \delta^*_{ft(Y)}(eval_{T_2}) \circ \delta^*_{ft(Y)} p^*_{T_1} p^*_{ft(Y)}(f) = eval_{\delta^*_{ft(Y)}(T_2)} \circ p^*_{ft(Y)}(f) = eval_Y \circ p^*_{ft(Y)}(f) = \phi_Y(f)$$

which implies (1) by definition of λ.

The converse to Proposition 2.6 holds as well. Let $CC = (B_n, \widetilde{B}_n, ft, \partial, \delta)$ be a C-system and let

1. $\Pi : B_{n+2} \to B_{n+1}$, $n \geq 0$,
2. $\lambda : \widetilde{B}_{n+2} \to \widetilde{B}_{n+1}$, $n \geq 0$,
3. $ev : (\widetilde{B}_{n+1})_\partial \times_{ft} (B_{n+2})_\Pi \times_\partial (\widetilde{B}_{n+1}) \to \widetilde{B}_{n+1}$, $n \geq 0$

be maps satisfying the conclusion of Proposition 2.6. For each $Y \in \widetilde{B}_{n+2}$ define a morphism

$$eval_Y : T(ft(Y), \Pi(Y)) \to Y$$

by the formula

$$eval_Y = q(p_Z, Y) \circ ev(p^*_Z(\delta_{ft(Y)}), T_2(Z, Y), \delta_Z)$$

where $Z = p^*_{ft(Y)}(\Pi(Y))$.

Proposition 2.7 *[2009.11.30.prop2] Under the assumption made above the morphisms $eval_Y$ are well defined and* $(\Pi, eval)$ *is a* Π*-structure on* CC.

Proof: Let us show that eva_Y is well defined. This requires us to check the following conditions:

1. $ft^2(Y) = ft(\Pi(Y))$, therefore Z is defined,
2. $ft(Z) = ft\partial(\delta_{ft(Y)})$ since $ft(Z) = ft(Y)$, therefore $p_Z^*(\delta_{ft(Y)})$ is defined,
3. $ft^2(Z) = ft^2(Y)$, therefore $T_2(Z, Y)$ is defined,
4. $\partial(p_Z^*(\delta_{ft(Y)}))) = p_Z^* p_{ft(Y)}^*(ft(Y))$, $ft(T_2(Z, Y)) = T_2(Z, ft(Y)) = p_Z^* p_{ft(Y)}^*(ft(Y))$,
5. $\partial(\delta_Z) = p_Z^*(Z) = p_Z^* p_{ft(Y)}^*(\Pi(Y)) = \Pi_{T_2(Z,Y)}$, therefore $ev = ev(p_Z^*(\delta_{ft(Y)}), T_2(Z, Y), \delta_Z)$ is defined,
6. $$\partial(ev) = (p_Z^*(\delta_{ft(Y)}))^*(T_2(Z, Y)) = (p_Z^*(\delta_{ft(Y)}))^* T(Z, T(ft(Y), Y)) =$$
$$= (p_Z^*(\delta_{ft(Y)}))^*(p_Z)^*((p_{ft(Y)})^*(Y, 2), 2) = (p_Z^*(\delta_{ft(Y)}))^* q(p_Z, p_Y^*(ft(Y)))^*(p_{ft(Y)})^*(Y, 2) =$$
$$= (p_Z^*(\delta_{ft(Y)}))^* q(p_Z, p_Y^*(ft(Y)))^* q(p_{ft(Y)}, ft(Y))^*(Y) =$$
$$= (q(p_{ft(Y)}, ft(Y)) q(p_Z, p_Y^*(ft(Y))) p_Z^*(\delta_{ft(Y)}))^*(Y) = p_Z^*(Y)$$
and $q(p_Z, Y) : p_Z^*(Y) \to Y$. Therefore $eval_Y$ is defined and is a morphism from Z to Y as required by Definition 2.2(2).

We leave the verification of the conditions (i) of (ii) of Definition 2.2 for the later, more mechanized version of this paper.

3 Π-universes in lcc categories.

Recall that a category $\mathcal{C}$ is called a lcc (locally Cartesian closed) category if it has fiber products and all the over-categories $\mathcal{C}/X$ have internal Hom-objects.

Definition 3.1 *[2009.10.27.def1] Let* $\mathcal{C}$ *be an lcc category and let* $p_i : \widetilde{U}_i \to U_i$, $i = 1, 2, 3$ *be three morphisms in* $\mathcal{C}$*. A* Π*-structure on* (p_1, p_2, p_3) *is a Cartesian square of the form*

$$\text{[Pisq1]} \quad \begin{array}{ccc} \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times \widetilde{U_2}) & \xrightarrow{\widetilde{P}} & \widetilde{U}_3 \\ {\scriptstyle p_2'}\downarrow & & \downarrow{\scriptstyle p_3} \\ \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) & \xrightarrow{P} & U_3 \end{array} \qquad (2)$$

such that p_2' *is the natural morphism defined by* p_2*. A* Π*-structure on* $p : \widetilde{U} \to U$ *is a* Π*-structure on* (p, p, p).

Remark 3.2 A Π-structure on (p_1, p_2, p_3) corresponds to the rule

$$\frac{\Gamma, X : U_1, f : X \to U_2 \rhd}{\Gamma, X : U_1, f : X \to U_2 \vdash \prod x : X.ev(f, x) : U_3}$$

Let $\mathcal{C}$ be as above, $p : \widetilde{U} \to U$ and let $(\widetilde{P}, P)$ be a Π-structure on (p,p,p). Let us construct a structure of Π-C-system on $CC = CC(\mathcal{C}, p)$.

We start by recalling some constructions in $\mathcal{C}$.

Lemma 3.3 *[**2009.11.24.l5**] Consider a pair of pull back squares*

$$\begin{array}{ccc} I_2 & \xrightarrow{\widetilde{F}_1} & \widetilde{U}_1 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ I_1 & \xrightarrow{F_1} & U_1 \end{array} \qquad \begin{array}{ccc} I_3 & \xrightarrow{\widetilde{F}_2} & \widetilde{U}_2 \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ I_2 & \xrightarrow{F_2} & U_2 \end{array} \tag{3}$$

[**2009.11.24.eq3**]

Then there exists a unique morphism $f_{F_1,F_2} : I_1 \to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2)$ such that its composition with the natural morphism to U_1 is F_1 and the composition of its adjoint

$$ev \circ (f_{F_1,F_2} \times_{U_1} \widetilde{U}_1) : I_2 = I_1 \times_{U_1} \widetilde{U}_1 \to U_1 \times U_2$$

with the projection to U_2 is F_2.

Proof: Follows immediately from the definition of internal Hom-objects.

Lemma 3.4 *[**2009.11.24.l3**] In the notation of Lemma 3.3 let*

$$\begin{array}{ccc} J_2 & \xrightarrow{\phi_2} & I_2 \\ \downarrow & & \downarrow{\scriptstyle q_1} \\ J_1 & \xrightarrow{\phi_1} & I_1 \end{array} \qquad \begin{array}{ccc} J_3 & \xrightarrow{\phi_3} & I_3 \\ \downarrow & & \downarrow{\scriptstyle q_2} \\ J_2 & \xrightarrow{\phi_2} & I_2 \end{array}$$

be two pull-back squares. Then $f_{F_1\phi_1,F_2\phi_2} = f_{F_1,F_2} \circ \phi_1$.

Proof: Straightforward.

Let $p_1 : \widetilde{U}_1 \to U_1$, $p_2 : \widetilde{U}_2 \to U_2$ be a pair of morphisms in an lcc $\mathcal{C}$. Consider a pull-back square of the form

$$\begin{array}{ccc} Fam_2(p_1,p_2) & \longrightarrow & \widetilde{U}_2 \\ {\scriptstyle p_{12}}\downarrow & & \downarrow{\scriptstyle p_2} \\ \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} & \xrightarrow{pr \circ ev} & U_2 \end{array} \tag{4}$$

[**2009.11.24.eq4**]

where

$$ev : \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} \to U_1 \times U_2$$

is the canonical morphism.

Then for any two pull-back squares as in Lemma 3.3, the morphism f_{F_1,F_2} defines factorizations of the pull-back squares (3) of the form

$$\begin{array}{ccccc} I_2 & \xrightarrow{f_{F_1,F_2} \times_{U_1} \widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} & \xrightarrow{pr} & \widetilde{U}_1 \\ {\scriptstyle q_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_1} \\ I_1 & \xrightarrow{f_{F_1,F_2}} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) & \longrightarrow & U_1 \end{array}$$

and

$$\begin{array}{ccccc} I_3 & \longrightarrow & Fam_2(p_1,p_2) & \longrightarrow & \widetilde{U}_2 \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle p_{12}} & & \downarrow{\scriptstyle p_2} \\ I_2 & \xrightarrow{f_{F_1,F_2}\times_{U_1}\widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U_1} & \xrightarrow{pr\circ ev} & U_2 \end{array}$$

respectively and joining the left hand side squares of these diagrams we get a diagram with pull-back squares of the form

$$\begin{array}{ccc} I_3 & \longrightarrow & Fam_2(p_1,p_2) \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle p_{12}} \\ I_2 & \xrightarrow{f_{F_1,F_2}\times_{U_1}\widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U_1} \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle pr} \\ I_1 & \xrightarrow{f_{F_1,F_2}} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2) \end{array}$$

Let

$$g:\underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times\widetilde{U}_2)\times_{U_1}\widetilde{U_1}\rightarrow Fam_2(p_1,p_2)$$

be the morphism over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U_1}$ whose composition with the projection $Fam_2(p_1,p_2)\rightarrow \widetilde{U}_2$ equals $pr\circ\widetilde{ev}$ where

$$\widetilde{ev}:\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2)\times_{U_1}\widetilde{U_1}\rightarrow U_1\times\widetilde{U}_2$$

is the canonical morphism.

Lemma 3.5 *[***2009.11.24.l2***] The pair*

$$(\underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times\widetilde{U}_2)\rightarrow\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2), g)$$

is universal for (p_{12}, pr).

Proof: For a given $w : Z\rightarrow\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)$, a morphism $Z\rightarrow\underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times\widetilde{U}_2)$ over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)$ is the same as a morphism $Z\times_{U_1}\widetilde{U}_1\rightarrow\widetilde{U}_2$ such that the adjoint of its composition with $p_2:\widetilde{U}_2\rightarrow U_2$ is w.

A morphism from Z to the universal pair for p_{12} over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)$ is a morphism $Z\times_{U_1}\widetilde{U_1}\rightarrow\widetilde{U}_2$ whose composition with p_2 is $(pr\circ ev)\circ(w\times_{U_1} Id_{\widetilde{U_1}})$ which coincides with the condition that the composition of its adjoint with p_2 is w. This can be also seen from the diagram

$$\begin{array}{ccccc} & & Fam_2(p_1,p_2) & \longrightarrow & \widetilde{U}_2 \\ & & {\scriptstyle p_{12}}\downarrow & & \downarrow{\scriptstyle p_2} \\ \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2)\times_{U_1}\widetilde{U}_1 & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U}_1 & \xrightarrow{pr\circ ev} & U_2 \\ \downarrow & & \downarrow{\scriptstyle pr} & & \\ \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2) & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times U_2) & & \end{array}$$

Lemma 3.6 *[2009.11.24.l4] For two pull back squares as in (3), consider a pull-back square of the form*

$$\begin{array}{ccc} R(F_1, F_2) & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times \widetilde{U}_2) \\ \downarrow & & \downarrow \\ I_1 & \xrightarrow{f_{F_1,F_2}} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \end{array}$$

and the morphism

$$g_{F_1,F_2} : R(F_1, F_2) \times_{I_1} I_2 \to I_3$$

whose composition with the morphism $I_3 \to \widetilde{U}_2$ coincides with the composition

$$R(F_1, F_2) \times_{I_1} I_2 = R(F_1, F_2) \times_{U_1} \widetilde{U}_1 \to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times \widetilde{U}_2) \times_{U_1} \widetilde{U}_1 \stackrel{proev}{\to} \widetilde{U}_2$$

Then $(R(F_1, F_2), g_{F_1,F_2})$ is a universal pair for (q_1, q_2).

Proof: It follows from Lemma 3.5 and the fact that in a lcc a pull-back of a universal pair is a universal pair.

Let us now construct a Π-C-system on $CC = CC(\mathcal{C}, p)$. Let $n \geq 2$ and $(F_1, \ldots, F_n) \in CC$. Denote $(pt, F_1, \ldots, F_{n-2})$ by I. Then we have two morphisms $F_{n-1} : I \to U$ and $F_n : (I, F_{n-1}) \to U$.

Applying Lemma 3.3 to the corresponding pull-back squares we get a morphism

$$f_{F_{n-1},F_n} : I \to \underline{Hom}_U(\widetilde{U}, U \times U)$$

Set $\Pi(F_1, \ldots, F_n) = (I, P \circ f_{F_{n-1},F_n}) = (F_1, \ldots, F_{n-2}, P \circ f_{F_{n-1},F_n})$. Since the square (2) is a pull-back square there is a unique morphism $\Pi(F_1, \ldots, F_n) \to \underline{Hom}_U(\widetilde{U}, U \times \widetilde{U})$ such that the diagram

$$\begin{array}{ccccc} \Pi(F_1, \ldots, F_n) & \longrightarrow & \underline{Hom}_U(\widetilde{U}, U \times \widetilde{U}) & \xrightarrow{\widetilde{P}} & \widetilde{U} \\ \downarrow & & \downarrow & & \downarrow \\ I & \xrightarrow{f_{F_{n-1},F_n}} & \underline{Hom}_U(\widetilde{U}, U \times U) & \xrightarrow{P} & U \end{array}$$

commutes and the composition of the two upper arrows is $Q(f_{F_{n-1},F_n})$. The left hand side square in this diagram is automatically a pull-back square. Applying to this square Lemma 3.6 we obtain a morphism

$$eval_{(F_1,\ldots,F_n)} : (I, F_{n-1}, (P \circ f_{F_{n-1},F_n}) \circ pr) \to (I, F_{n-1}, F_n)$$

over (I, F_{n-1}) (where $pr : (I, F_{n-1}) \to I$ is the projection).

The fact that this construction satisfies the first condition of Definition 2.2 follows from Lemma 3.4. The fact that it satisfies the second condition of this definition follows from Lemma 3.6.