B-systems[1]

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Abstract

B-systems are algebras (models) of an essentially algebraic theory that is expected to be constructively equivalent to the essentially algebraic theory of C-systems which is, in turn, constructively equivalent to the theory of contextual categories. The theory of B-systems is closer in its form to the structures directly modeled by contexts and typing judgements of (dependent) type theories and further away from categories than contextual categories and C-systems.

Contents

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1 Lft-sets, pre-B-systems and B0-systems

1 Lft-sets

Let us start with the definition of lft-sets. For two natural numbers m, n define

$$m -_{\mathbf{N}} n = max(m - n, 0).$$

Definition 1.1 *[**2016.01.27.def1**] An lft-set is a collection of data of the following form:*

1. *a set B,*
2. *a function $l : B \to \mathbf{N}$,*
3. *a function $ft : B \to B$*

such that for all $X \in B$ one has $l(ft(X)) = l(X) -_{\mathbf{N}} 1$.

An lft-set is called pointed if the set $\{X \in B, l(X) = 0\}$ is a one element set. In this case the only element of this set is usually denoted by pt.

Lemma 1.2 *[**2016.02.18.l2**] Let B be an lft-set, $X \in B$ and $n \in \mathbf{N}$. Then*

$$l(ft^n(X)) = l(X) -_{\mathbf{N}} n$$

Proof: Obvious induction on n.

For an lft-set B, define the relation $\geq$ on B by the condition that $Y \geq X$ if and only if $l(Y) \geq l(X)$ and

$$X = ft^{l(Y)-l(X)}(Y).$$

Define the relation $>$ on B by the condition that $Y > X$ if and only if $Y \geq X$ and $l(Y) > l(X)$.

Lemma 1.3 *[**2016.01.27.l1**] For any lft-set B one has:*

1. *the relation $\geq$ is a partial order relation, i.e., it is reflexive, transitive and antisymmetric,*
2. *the relation $>$ is a strict partial order relation, i.e., it is transitive and asymmetric.*

Proof: Straightforward using the corresponding properties of the relations $\geq$ and $>$ on $\mathbf{N}$ and properties of $-_{\mathbf{N}}$.

Lemma 1.4 *[**2016.02.22.l2**] Let B be an lft-set. The following mixed transitivities hold:*

1. *if $Z > Y$ and $Y \geq X$ then $Z > X$,*
2. *if $Z \geq Y$ and $Y > X$ then $Z > X$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ and $>$ on $\mathbf{N}$.

Lemma 1.5 *[**2016.02.22.l3**] Let B be an lft-set, $Y \geq X$ in B and $i \in \mathbf{N}$. Then one has:*

1. *if $l(Y) \geq i + l(X)$ then $ft^i(Y) \geq X$,*
2. *if $l(Y) > i + l(X)$ then $ft^i(Y) > X$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ and $>$ on $\mathbf{N}$.

Lemma 1.6 *[**2016.01.27.l6**] Let B be an lft-set and $Y > X$ in B. Then $ft(Y) \geq X$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ on $\mathbf{N}$.

Lemma 1.7 *[**2016.01.29.l3**] Let B be an lft-set, $X \in B$ and $n \in \mathbf{N}$. Then $X \geq ft^n(X)$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ on $\mathbf{N}$.

Lemma 1.8 *[**2016.01.29.l2**] Let B be an lft-set, $X \in B$, $n > 0$ and $l(ft^n(X)) > 0$. Then $X > ft^n(X)$.*

Proof: From Lemma 1.7 we know that $X \geq ft^n(X)$. It remains to show that $l(X) > l(ft^n(X))$. By Lemma 1.2, $l(ft^n(X)) = max(l(X) - n, 0)$ which implies that under the condition of the lemma $l(ft^n(X)) = l(X) - n$ and since $n > 0$ we have that $l(X) > l(ft^n(X))$.

Definition 1.9 *[**2016.01.27.def2**] Let B, B' be lft-sets. A morphism of lft-sets $f : B \to B'$ is a function $f : B \to B'$ such that for all $X \in B$ one has $l(f(X)) = l(X)$ and $l(ft(X)) = ft(l(X))$.*

We let $Mor_{lft}(B, B')$ denote the set of morphisms of lft-sets from B to B'.

Lemma 1.10 *[**2016.03.15.l5**] Let $f : B \to B'$ be a morphism of lft-sets, $X \in B$ and $j \in \mathbf{N}$. Then one has*

$$f(ft^j(X)) = ft^j(f(X))$$

Proof: By induction on j.

Lemma 1.11 *[**2017.01.27.l3**] Let $f : B \to B'$ be a morphism of lft-sets and $X, Y \in B$. Then one has:*

1. *if $Y \geq X$ then $f(Y) \geq f(X)$,*
2. *if $Y > X$ one has $f(Y) > f(X)$,*

Proof: Straightforward from Lemma 1.10.

Lemma 1.12 *[**2016.01.27.l2**] One has:*

1. *for any lft-set B the identity function $Id_B : B \to B$ is a morphism of lft-sets,*
2. *for any lft sets B, B', B'' and morphisms $f : B \to B'$, $f' : B' \to B''$ the composition of functions $f \circ f'$ is a morphism of lft-sets.*

Proof: Straightforward using the properties of $-_{\mathbf{N}}$.

Let $lft(U)$ be the set of lft-sets in the universe U.

Problem 1.13 *[***2016.01.27.prob1***] Let U be a universe. To construct a category $LFT(U)$ with the set of objects $lft(U)$.*

Construction 1.14 *[***2016.01.27.constr1a***]*We define

$$Ob(LFT(U)) = lft(U)$$

$$Mor(LFT(U)) = \amalg_{B,B' \in lft(U)} Mor_{lft}(B, B')$$

with the obvious domain and codomain functions and the identity function and the composition function being defined from the identity and composition of functions between sets using Lemma 1.12.

The proofs of the associativity and the identity axioms of a category are straightforward.

We can not use $\cup$ in this definition instead of $\amalg$ because the sets $Mor(B, B')$ need not be disjoint for different B, B'. For example, if B' has one element of each length then the set $Mor(B, B')$ depends on the set B and the length function l but is independent on the ft function on B.

Therefore there is no category with the set of objects $lft(U)$ and the set of morphisms between any two lft-sets being the set Mor_{lft} of Definition 1.9. Instead in our category the set of morphisms from B to B' is the set of iterated pairs of the form $((B, B'), f)$ where f is a function $B \to B'$ that satisfies the conditions of Definition 1.9. This set is in the obvious bijective correspondence with the set of morphisms from B to B' and we will use both directions of this bijection as coercions - if an element of $Mor_{LFT(U)}(B, B')$ occurs in a position where an element of $Mor_{lft}(B, B')$ should be it is replaced by its image in $Mor_{lft}(B, B')$ under the corresponding function of the bijection and vice versa.

This completes Construction 2.2.

In what follows we fix a universe and write LFT instead of $LFT(U)$ and lft instead of $lft(U)$.

2 Pre-B-systems

Definition 2.1 *[***2016.01.27.def7***] A B-system carrier is a triple $(B, \widetilde{B}, \partial)$ where B is an lft-set, $\widetilde{B}$ is a set and $\partial : \widetilde{B} \to B$ is a function such that for all $r \in \widetilde{B}$ one has $l(\partial(r)) > 0$.*

Remark 2.2 [**2016.03.31.rem2**] Elements of a B-system carrier $\mathbf{B} = (B, \widetilde{B}, \partial)$ and connecting them relation $\leq$ can be shown diagrammatically as follows:

$$\begin{array}{ccc} Y & \qquad & \partial(r) \\ \big\downarrow & & \big\downarrow \uparrow r \\ X & & ft(\partial(r)) \end{array}$$

where the first diagram shows a pair $X, Y \in B$ such that $X \leq Y$ and the second one an element $r \in \widetilde{B}$.

Definition 2.3 *[***2016.01.27.def3***] Let $(B, \widetilde{B}, \partial)$ be a B-system carrier. We set:*

$$T_{dom} = \{X, Y \in B, l(X) \geq 1, ft(X) < Y\} \quad \widetilde{T}_{dom} = \{X \in B, s \in \widetilde{B}, (X, \partial(s)) \in T_{dom}\}$$

$$S_{dom} = \{r \in \widetilde{B}, Y \in B, \partial(r) < Y\} \quad \widetilde{S}_{dom} = \{r, s \in \widetilde{B}, (r, \partial(s)) \in S_{dom}\}$$
$$\delta_{dom} = \{X \in B, l(X) \geq 1\}$$

Definition 2.4 *[**2014.10.10.def1**] A non-unital pre-B-system is a B-system carrier together with functions $T, \widetilde{T}, S$ and $\widetilde{S}$ of the form:*

$$T : T_{dom} \to B \quad \widetilde{T} : \widetilde{T}_{dom} \to \widetilde{B}$$

$$S : S_{dom} \to B \quad \widetilde{S} : \widetilde{S}_{dom} \to \widetilde{B}$$

Definition 2.5 *[**2014.10.20.def1**] A pre-B-system is a non-unital pre-B-system together with a function*

$$\delta : \delta_{dom} \to B$$

Definition 2.6 *[**2016.27.def8**] A morphism of B-system carriers $\mathbf{f} : (B, \widetilde{B}, \partial) \to (B', \widetilde{B}', \partial')$ is a pair $(\widetilde{f}, f)$ where $\widetilde{f} : \widetilde{B} \to \widetilde{B}'$ is a function, $f : B \to B'$ is a morphism of lft-sets and for any $s \in \widetilde{B}$ one has*

$$\partial'(\widetilde{f}(s)) = f(\partial(s))$$

Problem 2.7 *[**2016.01.27.prob9**] For a morphism of B-system carriers $\mathbf{f} : (B, \widetilde{B}, \partial) \to (B', \widetilde{B}', \partial')$ to construct functions*

$$f_T : T_{dom} \to T'_{dom} \quad f_{\widetilde{T}} : \widetilde{T}_{dom} \to \widetilde{T}'_{dom}$$

$$f_S : S_{dom} \to S'_{dom} \quad f_{\widetilde{S}} : \widetilde{S}_{dom} \to \widetilde{S}'_{dom}$$

$$f_\delta : \delta_{dom} \to \delta'_{dom}$$

Construction 2.8 [**2016.01.27.constr8**] For $(X, Y) \in T_{dom}$ we set $f_T(X, Y) = (f(X), f(Y))$. The condition that $f_T(X, Y) \in T'_{dom}$ follows immediately from the fact that f is an lft-set morphism and Lemma 1.11.

For $(X, s) \in \widetilde{T}_{dom}$ we set $f_{\widetilde{T}}(X, s) = (f(X), \widetilde{f}(s))$. The condition that $f_{\widetilde{T}}(X, s) \in \widetilde{T}'_{dom}$ follows immediately from the fact that $(f, \widetilde{f})$ is a morphism of B-system carriers and and Lemma 1.11.

The proofs for the remaining three subsets are equally easy corollaries of the definitions and Lemma 1.11.

Definition 2.9 *[**2016.01.27.def4**] Let $\mathbf{B}, \mathbf{B}'$ be pre-B-systems. A homomorphism of non-unital pre-B-systems $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ is a morphism $\mathbf{f} = (f, \widetilde{f})$ of the B-system carriers such that one has*

$$\text{for } (X, Y) \in T_{dom},\ f(T(X, Y)) = T'(f_T(X, Y)), \quad \text{for } (X, s) \in \widetilde{T}_{dom},\ \widetilde{f}(\widetilde{T}(X, s)) = \widetilde{T}'_{dom}(f_{\widetilde{T}}(X, s))$$

$$\text{for } (r, Y) \in S_{dom},\ f(S(r, Y)) = S'(f_S(r, Y)) \quad \text{for } (r, s) \in \widetilde{S}_{dom},\ \widetilde{f}(\widetilde{S}(r, s)) = \widetilde{S}'(f_{\widetilde{S}}(r, s))$$

A homomorphism of pre-B-systems is a morphism $\mathbf{f} = (f, \widetilde{f})$ of B-system carriers that is a homomorphism of non-unital pre-B-systems and such that one has:

$$\text{for } X \in \delta_{dom},\ \widetilde{f}(\delta(X)) = \delta'(f(X))$$

Lemma 2.10 *[***2016.01.27.l5***] One has:*

1. *Let* $\mathbf{B}$ *be a non unital pre-B-system (resp. pre-B-system) and* $(Id_{\widetilde{B}}, Id_B)$ *be the identity morphism of the underlying pre-B-system carries. Then* $(Id_{\widetilde{B}}, Id_B)$ *is a homomorphism of non-unital pre-B-systems (resp. pre-B-systems).*

2. *Let* $\mathbf{B}, \mathbf{B}', \mathbf{B}''$ *be non-unital pre-B-systems (resp. pre-B-systems) and* $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$*,* $\mathbf{f}' : \mathbf{B}' \to \mathbf{B}''$ *be two homomorphism of non-unital pre-B-systems (resp. pre-B-systems). Then the composition of the underlying homomorphisms of B-system carriers is a homomorphism of non-unital pre-B-systems (resp. pre-B-systems).*

Proof: The proof is straightforward but long since all five conditions of Definition 2.9 have to be verified.

2.1 Construction of the sets $T_{ext,dom}$ and operations T_{ext}

Let

$$T_{ext,dom} = \{X, Y \in B, l(X) \geq 1, Y \geq ft(X)\}$$

Given a function $T : T_{dom} \to B$ let us define the extended version of T as the function

$$T_{ext} : T_{ext,dom} \to B$$

given by the rule:

1. if $Y = ft(X)$ then $T_{ext}(X, Y) = X$,
2. if $Y > ft(X)$ then $T_{ext}(X, Y) = T(X, Y)$.

2.2 Operations T_{ext} and homomorphisms of B-system carriers

Lemma 2.11 *[***2016.03.15.l1***] Let* $f : B \to B'$ *be a morphism of lft-sets. Then for* $(X, Y) \in T_{ext,dom}$ *one has* $(f(X), f(Y)) \in T'_{ext,dom}$.

Proof: Immediate from definitions.

Lemma 2.12 *[***2016.03.15.l2***] Let* $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ *be a homomorphism of pre-B-systems. Then for* $(X, Y) \in T_{ext,dom}$ *one has*

$$f(T_{ext}(X, Y)) = T_{ext}(f(X), f(Y))$$

Proof: Straightforward by case.

2.3 Construction of the sets $S_{ext,dom}$ and operations S_{ext}

Let

$$S_{ext,dom} = \{r \in \widetilde{B}, Y \in B, \partial(r) \leq Y\}$$

Given a function $S : S_{dom} \to B$ let us define the extended version of S as the function

$$S_{ext} : S_{ext,dom} \to B$$

given by the rule:

1. if $\partial(r) = Y$ then $S_{ext}(r, Y) = ft(\partial(r))$,
2. if $\partial(r) < Y$ then $S_{ext}(r, Y) = S(r, Y)$.

2.4 Operations S_{ext} and homomorphisms of B-system carriers

Lemma 2.13 *[**2016.03.27.l3**] Let* $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ *be a homomorphism of pre-B-system carriers. Then for* $(r, Y) \in S_{ext,dom}$ *one has* $(\widetilde{f}(r), f(Y)) \in S'_{ext,dom}$.

Proof: Immediate from definitions.

Lemma 2.14 *[**2016.03.27.l4**] Let* $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ *be a homomorphism of pre-B-systems. Then for* $(r, Y) \in S_{ext,dom}$ *one has*

$$f(S_{ext}(r, Y)) = S_{ext}(\widetilde{f}(r), f(Y))$$

Proof: Straightforward by case.

3 B0-systems

The complex of axioms that define B0-systems among all pre-B-systems is as follows:

Definition 3.1 *[**2014.10.16.def1.fromold**] [**2014.10.16.def1**] [**2016.01.29.def1**] A non-unital pre-B-system is called a non-unital B0-system if the following conditions hold:*

1. *For* $(X, Y) \in T_{dom}$ *one has:*
   (a) $T(X, Y) > X$,
   (b) *if* $ft(Y) > ft(X)$ *then* $ft(T(X, Y)) = T(X, ft(Y))$,
   (c) $l(T(X, Y)) = l(Y) + 1$.
2. *For* $(X, s) \in \widetilde{T}_{dom}$ *one has:*
   (a) $\partial(\widetilde{T}(X, s)) = T(X, \partial(s))$,
   (b) $l(\partial(\widetilde{T}(X, s))) = l(\partial(s)) + 1$.
3. *For* $(r, Y) \in S_{dom}$ *one has:*
   (a) $S(r, Y) > ft(\partial(r))$,
   (b) *if* $ft(Y) > \partial(r)$ *then* $ft(S(r, Y)) = S(r, ft(Y))$,
   (c) $l(S(r, Y)) = l(Y) - 1$.

4. For $(r,s) \in \widetilde{S}_{dom}$ one has

(a) $\partial(\widetilde{S}(r,s)) = S(r, \partial(s))$,

(b) $l(\widetilde{S}(r,s)) = l(\partial(s)) - 1$.

Remark 3.2 [**2016.03.31.rem1**] Elements participating in the operations $T, \widetilde{T}, S$ and $\widetilde{S}$ on a B0-system can be shown by the following diagrams

$$\begin{array}{ccc} T(X,Y) & & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & ft(X) \end{array} \qquad \begin{array}{ccc} T(X,\partial(s)) & & \partial(s) \\ {\scriptstyle \widetilde{T}(X,s)\uparrow}\downarrow & & \downarrow{\scriptstyle \uparrow s} \\ ft(T(X,\partial(s))) & & ft(\partial(s)) \\ \downarrow & & \downarrow \\ X & \longrightarrow & ft(X) \end{array}$$

$$\begin{array}{ccc} S(r,Y) & & Y \\ \downarrow & & \downarrow \\ ft(\partial(r)) & \stackrel{\stackrel{r}{\longrightarrow}}{\longleftarrow} & \partial(r) \end{array} \qquad \begin{array}{ccc} S(r,\partial(s)) & & \partial(s) \\ {\scriptstyle \widetilde{S}(r,s)\uparrow}\downarrow & & \downarrow{\scriptstyle \uparrow s} \\ ft(S(r,\partial(s))) & & ft(\partial(s)) \\ \downarrow & & \downarrow \\ ft(\partial(r)) & \stackrel{\stackrel{r}{\longrightarrow}}{\longleftarrow} & \partial(r) \end{array}$$

Remark 3.3 [**2016.01.29.rem1**] The axioms of a B0-system given in Definition 3.1 are not independent. If the axioms 1(a) and 2(b) hold then the axiom 2(a) holds and if the axioms 3(a) and 4(b) hold then the axiom 4(a) holds. The axioms are presented there in this form to make it possible to prove facts about various operations in B0-systems independently from each other.

Definition 3.4 *[**2014.10.20.def2**] A pre-B-system is called a B0-system if the underlying non-unital pre-B-system is a non-unital B0-system and for all $X \in \delta_{dom}$ one has*

$$[\mathbf{2009.12.27.eq1}]\partial(\delta(X)) = T(X,X) \tag{1}$$

Lemma 3.5 *[**2016.03.11.l1**] Let B be an lft-set, T_{dom} be the corresponding set of pairs (X,Y) in B and $T : T_{dom} \to B$ be a function satisfying the conditions of Definition 3.1(1). Let $(X,Y) \in T_{dom}$ be such that $ft(Y) = ft(X)$. Then*

$$ft(T(X,Y)) = X$$

Proof: We know that $T(X,Y) > X$. In particular, $ft^{l(T(X,Y))-l(X)}(T(X,Y)) = X$. On the other hand $l(T(X,Y)) = l(Y)+1$. Since $Y > ft(X)$ we know that $l(Y) \geq 1$. Therefore $l(ft(Y)) = l(Y)-1$ and similarly from $l(X) \geq 1$ we have $l(ft(X)) = l(X) - 1$. Therefore $ft(Y) = ft(X)$ implies that $l(Y) = l(X)$ and

$$l(T(X,Y)) - l(X) = l(Y) + 1 - l(X) = l(X) + 1 - l(X) = 1$$

We conclude that $ft(T(X,Y)) = X$.

Lemma 3.6 *[2016.03.19.l2] Let B be an lft-set, T_{dom} be the corresponding set of pairs (X,Y) in B and $T : T_{dom} \to B$ be a function satisfying the conditions of Definition 3.1(1). Then if $(X,Y),(X,Y') \in T_{dom}$ and $Y \leq Y'$ (resp. $Y < Y'$) then $T(X,Y) \leq T(X,Y')$ (resp. $T(X,Y) < T(X,Y')$).*

Proof: Note first that by Definition 3.1(1a) we have $l(T(X,Y)) \leq l(T(X,Y'))$ if $l(Y) \leq l(Y')$ and $l(T(X,Y)) < l(T(X,Y'))$ if $l(Y) < l(Y')$. In particular, it is sufficient to consider the $Y \leq Y'$ case. Then we need to prove that

$$[\mathbf{2016.03.19.eq1}] ft^{l(T(X,Y'))-l(T(X,Y))}(T(X,Y')) = T(X,Y) \tag{2}$$

We have $l(T(X,Y')) - l(T(X,Y)) = (l(Y')+1) - (l(Y)+1) = l(Y') - l(Y)$. By an easy induction one proves that $ft^i(T(X,Y')) = T(X, ft^i(Y'))$ if $(X, ft^i(Y')) \in T_{dom}$. In our case

$$ft^{l(T(X,Y'))-l(T(X,Y))}(Y') = ft^{(l(Y')+1)-(l(Y)+1)}(Y') = ft^{l(Y')-l(Y)}(Y') = Y$$

and since $(X,Y) \in T_{dom}$ by the assumption, (2) holds.

Lemma 3.7 *[2016.03.21.l3] Let **B** be a B-system carrier and $S : S_{dom} \to B$ an operation satisfying the conditions of Definition 3.1(3). Let $(r,Y) \in S_{dom}$ be such that $\partial(r) = ft(Y)$. Then*

$$ft(S(r,Y)) = ft(\partial(r)).$$

Proof: We know that $ft(\partial(r)) < S(r,Y)$. Therefore $ft^{l(S(r,Y))-l(ft(\partial(r)))}(S(r,Y)) = ft(\partial(r))$. From the definition of a B-system carrier we know that $l(\partial(r)) > 0$. Therefore $l(ft(\partial(r))) = l(\partial(r)) - 1$. We also have $l(S(r,Y)) = l(Y) - 1$. We also have that $l(ft(Y)) = l(\partial(Y)) > 0$ and therefore $l(\partial(r)) = l(ft(Y)) = l(Y) - 1 > 0$. Together this implies that

$$l(S(r,Y)) - l(ft(\partial(r))) = (l(Y)-1) - (l(\partial(r))-1) = (l(Y)-1) - ((l(Y)-1)-1) = 1$$

We conclude that $ft(S(r,Y)) = ft(\partial(r))$.

Lemma 3.8 *[2016.03.19.l5] Let **B** be a B-system carrier and $S : S_{dom} \to B$ an operation satisfying the conditions of Definition 3.1(3). Let $(r,Y),(r,Y') \in S_{dom}$ and suppose that $Y \leq Y'$ (resp. $Y < Y'$). Then $S(r,Y) \leq S(r,Y')$ (resp. $S(r,Y) < S(r,Y')$).*

Proof: Note first that $l(S(r,Y')) - l(S(r,Y)) = (l(Y')-1) - (l(Y)-1) = l(Y') - l(Y)$. In particular, if $S(r,Y) \leq S(r,Y')$ and $l(Y) < l(Y')$ then $S(r,Y) < S(r,Y')$ and so it is sufficient to prove the case of $\leq$. By an easy induction one proves that if $\partial(r) < ft^i(Y')$ then $ft^i(S(r,Y')) = S(r, ft^i(Y'))$. In our case

$$ft^{l(S(r,Y'))-l(S(r,Y))}(Y') = ft^{l(Y')-l(Y)}(Y') = Y$$

and since $\partial(r) < Y$ by the assumption we conclude that

$$ft^{l(S(r,Y'))-l(S(r,Y))}(S(r,Y')) = S(r,Y)$$

that is $S(r,Y) \leq S(r,Y')$.

3.1 Properties of T_{ext} when T satisfies the B0-system conditions

Lemma 3.9 *[***2016.03.09.l1***] Let* **B** *be a B-system carrier and let $T : T_{dom} \to B$ be an operation satisfying the conditions of Definition 3.1(1). Then the corresponding operation $T_{ext} : T_{ext,dom} \to B$ satisfies the following, similar, conditions:*

1. $T_{ext}(X, Y) \geq X$,
2. *if* $ft(Y) \geq ft(X)$ *and* $l(Y) \geq 1$ *then* $ft(T_{ext}(X, Y)) = T_{ext}(X, ft(Y))$,
3. $l(T_{ext}(X, Y)) = l(Y) + 1$.

Proof: Straightforward by case with the first branch being very easy to prove and the second branch being exactly the conditions of Definition 3.1(1).

3.2 Properties of S_{ext} when S satisfies the B0-system conditions

Lemma 3.10 *[***2016.03.27.l5***] Let* **B** *be a B-system carrier and let $S : S_{dom} \to B$ be an operation satisfying the conditions of Definition 3.1(3). Then the corresponding operation $S_{ext} : S_{ext,dom} \to B$ satisfies the following, similar, conditions:*

1. $S_{ext}(r, Y) \geq ft(\partial(r))$,
2. *if* $\partial(r) \leq ft(Y)$ *then* $ft(S_{ext}(r, Y)) = S_{ext}(r, ft(Y))$,
3. $l(S_{ext}(r, Y)) = l(Y) - 1$.

Proof: Straightforward by case with the first branch being very easy to prove and the second branch being exactly the conditions of Definition 3.1(3).

3.3 Construction of sets T^*_{dom} and operations T^*

Define the set T^*_{dom} by the formula:

$$T^*_{dom} = \{X, Y, Z \in B, X \geq Y, Z \geq Y\}.$$

Problem 3.11 *[***2016.02.18.prob3***] Let B be an lft-set and $T : T_{dom} \to B$ a function satisfying the conditions of Definition 3.1(1). To define a function*

$$T^* : T^*_{dom} \to B$$

such that:

1. $T^*(X, Y, Z) \geq X$,
2. *if* $ft(Z) \geq Y$ *and* $l(Z) \geq 1$ *then* $ft(T^*(X, Y, Z)) = T^*(X, Y, ft(Z))$,

3. $l(T^*(X,Y,Z)) - l(X) = l(Z) - l(Y)$.

Elements involved in operation T^* can be shown on a diagram as follows:

$$\begin{array}{ccc} T^*(X,Y,Z) & & Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Construction 3.12 [**2016.02.18.constr3**] We proceed by induction on $j = l(X) - l(Y)$.

For $j = 0$ we set $T^*(X,Y,Z) = Z$. The proofs of the conditions are obvious.

For $j = 1$ we set $T^*(X,Y,Z) = T_{ext}(X,Z)$. The fact that $(X,Z) \in T_{ext,dom}$ is easy to prove. The conditions are the conditions proved in Lemma 3.9.

For the successor of $j > 0$ we set

$$[\mathbf{2016.02.20.eq2}] T^*(X,Y,Z) = T_{ext}(X, T^*(ft(X),Y,Z)) \tag{3}$$

For the formula (3) to be well defined we need to show that for $(X,Y,Z) \in T^*_{dom}$ we have

$$[\mathbf{2016.02.20.eq4}] (ft(X),Y,Z) \in T^*_{dom} \tag{4}$$

and

$$[\mathbf{2016.02.20.eq5}] (X, T^*(ft(X),Y,Z)) \in T_{ext,dom} \tag{5}$$

The condition (4) follows from Lemma 1.5(1).

The condition (5) is equivalent to $l(X) \geq 1$ and

$$T^*(ft(X),Y,Z) \geq ft(X)$$

That $l(X) \geq 1$ follows from $l(X) \geq j+1$.

That $T^*(ft(X),Y,Z) \geq ft(X)$ follows from the inductive assumption.

Let us prove the conditions for $j+1$. The first is immediate from Lemma 3.13

$$T^*(X,Y,Z) = T_{ext}(X, T^*(ft(X),Y,Z)) \geq X$$

To prove the second condition let $ft(Z) \geq Y$ and $l(Z) \geq 1$. Then, by the inductive assumption we have

$$ft(T^*(ft(X),Y,Z)) = T^*(ft(X),Y,ft(Z))$$

Therefore,

$$ft(T^*(X,Y,Z)) = ft(T_{ext}(X, T^*(ft(X),Y,Z))) = T_{ext}(X, ft(T^*(ft(X),Y,Z))) =$$

$$T_{ext}(X, T^*(ft(X),Y,ft(Z))) = T^*(X,Y,ft(Z))$$

where the second equality follows from Lemma 3.9(3) whose condition is satisfied because

$$ft(T^*(ft(X),Y,Z)) = T^*(ft(X),Y,ft(Z)) \geq ft(X).$$

For the last condition we have

$$l(T^*(X,Y,Z)) - l(X) = l(T_{ext}(X, T^*(ft(X),Y,Z))) - l(X) = (l(T^*(ft(X),Y,Z)) + 1) - l(X) =$$

$$(l(T^*(ft(X),Y,Z)) + 1) - (l(ft(X)) + 1) = l(T^*(ft(X),Y,Z)) - l(ft(X)) = l(Z) - l(Y)$$

This completes Construction 3.12.

Lemma 3.13 *[**2016.03.09.l2**][**2016.04.08.l1**] In the context of Problem 3.11 let $X, Y, Z, W \in B$ be such that $X \geq Y$, $Y \leq Z \leq W$. Then one has*

1. *$T(X,Y,Z) \leq T(X,Y,W)$,*
2. *$l(T(X,Y,W)) - l(T(X,Y,Z)) = l(W) - l(Z)$,*
3. *if $Z < W$ then $T(X,Y,Z) < T(X,Y,W)$.*

Proof: To prove the first assertion we proceed by induction on $j = l(W) - l(Z)$.

For $j = 0$ we have $W = Z$ and $T(X,Y,Z) = T(X,Y,W)$.

For the successor of $j \geq 0$ we have $l(W) - l(Z) > 0$ and therefore $Z \leq ft(W)$ and $Y \leq Z \leq ft(W)$. Therefore $T(X,Y,ft(W))$ is defined and by property (2) of Problem 3.11 we have $ft(T(X,Y,W)) = T(X,Y,ft(W))$.

By the inductive assumption we have $T(X,Y,Z) \leq T(X,Y,ft(W))$ and since $ft(T(X,Y,W)) \leq T(X,Y,W)$ we conclude that $T(X,Y,Z) \leq T(X,Y,W)$.

For the second assertion consider that we have, from Problem 3.11(3), that

$$l(T^*(X,Y,W)) = l(W) + (l(X) - l(Y))$$

$$l(T^*(X,Y,Z)) = l(Z) + (l(X) - l(Y))$$

subtracting we obtain the second assertion.

The third assertion follows easily from the first and the second ones.

3.4 Construction of sets $\widetilde{T}^*_{dom}$ and operations $\widetilde{T}^*$

We will also require a similar construction for $\widetilde{T}$. Let

$$\widetilde{T}^*_{dom} = \{X, Y \in B, s \in \widetilde{B}, (X, Y, ft(\partial(s))) \in T^*_{dom}\}$$

Problem 3.14 *[**2016.02.20.prob1**] Let $(B, \widetilde{B}, \partial)$ be a B-system carrier and let*

$$T : T_{dom} \to B \quad \widetilde{T} : \widetilde{T}_{dom} \to \widetilde{B}$$

be functions satisfying the conditions of Definition 3.1(1,2). For $j \in \mathbf{N}$, define a function

$$\widetilde{T}^* : \widetilde{T}^*_{dom} \to \widetilde{B}$$

such that:

$$[\mathbf{2016.02.22.eq1}]\partial(\widetilde{T}^*(X,Y,s)) = T^*(X,Y,\partial(s)) \tag{6}$$

Elements involved in operation $\widetilde{T}^*$ can be shown on a diagram as follows:

$$\begin{array}{ccc}
T^*(X,Y,\partial(s)) & & \partial(s) \\
{\scriptstyle \widetilde{T}^*(X,Y,s)\uparrow}\big\downarrow & & \big\downarrow{\scriptstyle \uparrow s} \\
ft(T^*(X,Y,\partial(s))) & & ft(\partial(s)) \\
\big\downarrow & & \big\downarrow \\
X & \longrightarrow & Y
\end{array}$$

Construction 3.15 [**2016.02.20.constr1**] We proceed by induction on $j = l(X) - l(Y)$.

Observe first that the condition $(X, Y, ft(\partial(s))) \in T^*_{dom}$ is equivalent to the condition that $X \geq Y$ and $ft(\partial(s)) \geq Y$ and that since $l(\partial(s)) > 0$ the latter condition is equivalent to $\partial(s) > Y$.

For $j = 0$ we set $\widetilde{T}^*(X, Y, s) = s$. The proof of the conditions is obvious.

For $j = 1$ we set $\widetilde{T}^*(X, Y, s) = \widetilde{T}(X, s)$. For the right hand side to be defined we need $\partial(s) > ft(X)$ which is satisfied by the observation made above since $ft(X) = Y$. The condition (6) is the condition of Definition 3.1(2a).

For the successor of $j > 0$ we set

$$[\mathbf{2016.02.20.eq3}]\widetilde{T}^*(X, Y, s) = \widetilde{T}(X, \widetilde{T}^*(ft(X), Y, s)) \tag{7}$$

For the formula (7) to be well defined we need to show that assuming $(X, Y, ft(\partial(s))) \in T^*_{dom}$ we have:

$$[\mathbf{2016.02.20.eq7}](ft(X), Y, s) \in \widetilde{T}^*_{dom} \tag{8}$$

and

$$[\mathbf{2016.02.20.eq8}](X, \widetilde{T}^*(ft(X), Y, s)) \in \widetilde{T}_{dom} \tag{9}$$

The condition (8) is equivalent to $(ft(X), Y, ft(\partial(s))) \in T^*_{dom}$ and its proof is identical to the proof of (4) for $Z = ft(\partial(s))$.

The condition (9) is equivalent to $(X, \partial(\widetilde{T}^*(ft(X), Y, s))) \in T_{dom}$. By the inductive assumption we have $\partial(\widetilde{T}^*(ft(X), Y, s)) = T^*(ft(X), Y, \partial(s))$. Therefore we need to show that

$$(X, T^*(ft(X), Y, \partial(s))) \in T_{dom}$$

that is, that $l(X) \geq 1$ and $T^*(ft(X), Y, \partial(s)) > ft(X)$. The first condition follows from the fact that $l(X) = l(Y) + j + 1 \geq 1$. For the second condition we know that $T^*(ft(X), Y, \partial(s)) \geq ft(X)$ from the first condition of Problem 3.11. On the other hand

$$l(T^*(ft(X), Y, \partial(s))) - l(ft(X)) = l(\partial(s)) - l(Y)$$

Since $(ft(X), Y, ft(\partial(s))) \in T^*_{dom}$ we have $ft(\partial(s)) \geq Y$ and therefore $l(ft(\partial(s))) = l(\partial(s)) - 1 \geq l(Y)$. This implies that $l(\partial(s)) - l(Y) > 0$ and $T^*(ft(X), Y, \partial(s)) > ft(X)$.

Let us prove condition (6). We have

$$\partial(\widetilde{T}^*(X, Y, s)) = \partial(\widetilde{T}(X, \widetilde{T}^*(ft(X), Y, s))) = T(X, \partial(\widetilde{T}^*(ft(X), Y, s))) =$$

$$T(X, T^*(ft(X), Y, \partial(s))) = T^*(X, Y, \partial(s))$$

This completes Construction 3.15.

3.5 Operations T^* and $\widetilde{T}^*$ and homomorphisms of pre-B-systems

Lemma 3.16 *[**2016.03.15.l3**] Let $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ be a homomorphism of B0-systems. Then one has:*

1. *if $(X, Y, Z) \in T^*_{dom}$ then $(f(X), f(Y), f(Z)) \in (T')^*_{dom}$ and*
$$f(T^*(X, Y, Z)) = (T')^*(f(X), f(Y), f(Z))$$
2. *if $(X, Y, s) \in \widetilde{T}^*_{dom}$ then $(f(X), f(Y), \widetilde{f}(s)) \in (\widetilde{T}')^*_{dom}$ and*
$$\widetilde{f}(\widetilde{T}^*_{dom}(X, Y, s)) = (\widetilde{T}')^*(f(X), f(Y), \widetilde{f}(s))$$

Proof: The first parts of both assertions follow immediately from Lemma 1.11.

The second parts require proofs by induction using in the first case Lemma 2.12.

4 Sets $\widetilde{B}^*(X, Y)$ and associated with them operations

4.1 Construction of the sets $\widetilde{B}^*(X, Y)$

For a B-system carrier $(B, \widetilde{B}, \partial)$ and $X \in B$ denote by $\widetilde{B}(X)$ the subset of $\widetilde{B}$ of elements s such that $\partial(s) = X$.

Problem 4.1 *[**2016.01.29.prob2**] Let $(B, \widetilde{B}, \partial)$ be a B-system carrier and let $S : S_{dom} \to B$ be a function satisfying the conditions of Definition 3.1(3). To construct, for any $X, Y \in B$ such that $X \leq Y$ a set $\widetilde{B}^*(X, Y)$.*

Construction 4.2 [**2016.01.29.constr2**] We proceed by induction on $j = l(Y) - l(X)$ as follows:

1. for $j = 0$, $X = Y$ and we set $\widetilde{B}^*(Y, Y) = unit$ where $unit$ is our chosen set with one element tt,
2. for $j = 1$ we set $\widetilde{B}^*(ft(Y), Y) = \widetilde{B}(Y)$,
3. for the successor of $j > 0$ we need to define $\widetilde{B}^*(ft^{j+1}(Y), Y)$. We let it to be the set of pairs (r, s) where
$$r \in \widetilde{B}(ft^j(Y))$$
and
$$s \in \widetilde{B}^*(ft^{j+1}(Y), S(r, Y)).$$
By our condition we have $l(S(r, Y)) = l(Y) - 1$ and $S(r, Y) \geq ft(\partial(r)) = ft^{j+1}(Y)$. Therefore $\widetilde{B}^*(ft^{j+1}(Y), S(r, Y))$ is defined by the inductive assumption.

This completes Construction 4.2.

We will use the same diagrammatic notation $X \overset{\longrightarrow}{\longleftarrow} Y$ for elements of $\widetilde{B}^*(X, Y)$ we have been using for elements of $\widetilde{B}$.

4.2 Sets S^*_{dom} and operations S^*

Here we are going to construct the operations that in the B0-systems corresponding to C-systems correspond to the pull-back of objects Z over Y along elements of $\widetilde{B}^*(X,Y)$ and that generalize $S(r,Z)$ from elements r of $\widetilde{B}(Y)$ to elements s of $\widetilde{B}^*(X,Y)$.

Definition 4.3 *[**2016.03.29.def2**] Let* **B** *be a B0-system. Define*

$$S^*_{dom} = \{X \le Y \le Z \in B, s \in \widetilde{B}^*(X,Y)\}$$

We will often write elements (X,Y,s,Z) of S^*_{dom} as (s,Z) because X and Y can be recovered from the type of s.

Problem 4.4 *[**2016.03.27.prob1**] Let* **B** *be a B0-system. For $X,Y,Z \in B$ such that $X \le Y \le Z$ and $s \in \widetilde{B}^*(X,Y)$. To construct an element $S^*(s,Z) \in B$ such that one has:*

1. *$X \le S^*(s,Z)$,*
2. *if $Y \le ft(Z)$ then $ft(S^*(s,Z)) = S^*(s, ft(Z))$,*
3. *$l(S^*(s,Z)) - l(X) = l(Z) - l(Y)$.*

The digram for Problem 4.4 is as follows:

$$\begin{array}{ccc} S^*(s,Z) & & Z \\ \downarrow & & \downarrow \\ X & \overset{s}{\longleftarrow} & Y \end{array}$$

Construction 4.5 [**2016.03.27.constr1**]We proceed by induction on $j = l(Y) - l(X)$.

If $j = 0$ then $X = Y$, $s = tt$ and we set $S^*(s,Z) = Z$. Proofs of the conditions are straightforward.

If $j = 1$ then $X = ft(Y)$ and $\widetilde{B}^*(X,Y) = \widetilde{B}(Y)$, i.e., $s \in \widetilde{B}$ and $\partial(s) = Y$. In this case we set $S^*(s,Z) = S_{ext}(s,Z)$. The conditions are the conditions of Lemma 3.10.

For the successor of $j > 0$ we have $X = ft^{j+1}(Y)$ and $s = (r_1, s_1)$ where $r_1 \in \widetilde{B}(ft^j(Y))$ and $s_1 \in \widetilde{B}^*(ft^{j+1}(Y), S(r_1,Y))$. In this case we set

$$[\textbf{2016.03.29.eq1}] S^*(s,Z) = S^*(s_1, S(r_1,Z)) \qquad (10)$$

as can be seen on the diagram

$$[\textbf{2016.04.04.eq1}] \qquad \begin{array}{ccccc} S^*(s_1, S(r_1,Z)) & & S(r_1,Z) & & Z \\ \downarrow & & \downarrow & & \downarrow \\ X & \overset{s_1}{\longleftarrow} & S(r_1,Y) & & Y \\ & & \downarrow & & \downarrow \\ & & X & \overset{r_1}{\longleftarrow} & ft^j(Y) \end{array} \qquad (11)$$

Let us show that the right hand side of (10) is defined. Since $l(Y) - l(X) = j+1$ we have $l(Y) \geq j+1$. Since $j > 0$ we obtain that $\partial(r_1) = ft^j(Y) < Y$ and together with $Y \leq Z$ this gives us that $\partial(r_1) < Z$, i.e., $S(r_1, Z)$ is defined.

To prove that $S^*(s_1, S(r_1, Z))$ is defined by the inductive assumption we need to show that

$$l(S(r_1, Y)) - l(ft^{j+1}(Y)) \leq j$$

and that $S(r_1, Y) \leq S(r_1, Z)$. The first inequality follows from the B0-system axiom that $l(S(r_1, Y)) = l(Y) - 1$. The second one follows from the assumption $Y \leq Z$ and Lemma 3.8.

It remains to verify the conditions.

The first condition follows from the inequality $S(r_1, Y) \leq S(r_1, Z)$ by the inductive assumption.

For the second condition, if $Y \leq ft(Z)$ then by Lemma 3.8, $S(r_1, Y) \leq S(r_1, ft(Z))$ and by the B0-system axiom $S(r_1, ft(Z)) = ft(S_(r_1, Z))$. Therefore by the inductive assumption we have

$$ft(S^*(s_1, S(r_1, Z))) = S^*(s_1, ft(S(r_1, Z))) = S^*(s_1, S(r_1, ft(Z)))$$

For the third condition we have

$$l(S^*(s_1, S(r_1, Z))) - l(X) = l(S(r_1, Z)) - l(S(r_1, Y)) = (l(Z) - 1) - (l(Y) - 1) = l(Z) - 1(Y)$$

where the first equality is by the inductive assumption and the second by the axioms of B0-system.

This completes Construction 4.5.

Lemma 4.6 *[***2016.04.02.l1***] Let* **B** *be a B0-system. Then for any* $X \leq Y \leq Z \leq W$ *and* $s \in \widetilde{B}^*(X, Y)$ *one has*

1. $S(s, Z) \leq S(s, W)$,
2. $l(S(s, W)) - l(S(s, Z)) = l(W) - l(Z)$,
3. *if* $Z < W$ *then* $S(s, Z) < S(s, W)$.

Proof: Very similar to the proof of Lemma 3.13.

4.3 Sets $\widetilde{S}^*_{dom}$ and operations $\widetilde{S}^*$

Here we will construct operations $\widetilde{S}^*$ that correspond, in the B0-systems of C-systems, to the pull-back of elements of $\widetilde{B}$ along elements of $\widetilde{B}^*$.

Definition 4.7 *[***2016.04.04.def1***] For a B0-system* **B** *define:*

$$\widetilde{S}^*_{dom} = \{X, Y, s, r \,|\, X, Y \in B, X \leq Y, s \in \widetilde{B}^*(X, Y), r \in \widetilde{B}, Y \leq ft(\partial(r))\}$$

We will sometimes write elements of $\widetilde{S}^*_{dom}$ as (s, r) because X and Y can be recovered from the type of s.

Problem 4.8 *[***2016.04.04.prob1***] For a B0-system* **B** *and an element* $(X, Y, s, r) \in \widetilde{S}^*_{dom}$ *to construct an element*

$$\widetilde{S}^*(X, Y, s, r) \in \widetilde{B}$$

such that

$$[\mathbf{2016.04.04.eq5}]\partial(\widetilde{S}^*(X, Y, s, r)) = S^*(X, Y, s, \partial(r)) \tag{12}$$

where the right hand side is defined by the assumption $Y \leq ft(\partial(r))$*, the inequality* $ft(\partial(r)) \leq \partial(r)$ *and Lemma 1.3.*

Construction 4.9 [**2016.04.04.constr1**] We proceed by induction on $j = l(Y) - l(X)$.

If $j = 0$ then, since $X \leq Y$, we have $Y = X$, $s = tt$ and we set

$$\widetilde{S}^*(X, Y, tt, r) = r$$

If $j = 1$. Then $\widetilde{B}^*(X, Y) = \widetilde{B}(Y)$ and we set

$$\widetilde{S}^*(X, Y, s, r) = \widetilde{S}(s, r)$$

For the successor of $j > 0$ we have $s = (r_1, s_1)$ and we define

$$[\mathbf{2016.04.04.eq3}]\widetilde{S}^*(X, Y, (r_1, s_1), r) = \widetilde{S}^*(s_1, \widetilde{S}(r_1, r)) \tag{13}$$

The objects involved can be seen on the following diagram:

$$[\mathbf{2016.04.04.eq2}] \quad \begin{array}{ccccc} S^*(s_1, S(r_1, \partial(r))) & & S(r_1, \partial(r)) & & \partial(r) \\ {\scriptstyle \widetilde{S}^*(s_1, \widetilde{S}(r_1, r))\uparrow}\downarrow & & {\scriptstyle \widetilde{S}(r_1, r)\uparrow}\downarrow & & {\scriptstyle r\uparrow}\downarrow \\ ft(S^*(s_1, S(r_1, \partial(r)))) & & ft(S(r_1, \partial(r))) & & ft(\partial(r)) \\ \downarrow & & \downarrow & & \downarrow \\ X & \overset{s_1}{\longleftarrow} & S(r_1, Y) & & Y \\ & & \downarrow & & \downarrow \\ & & X & \overset{r_1}{\longleftarrow} & ft^j(Y) \end{array} \tag{14}$$

Let us show that the right hand side of (13) is defined and that

$$[\mathbf{2016.04.04.eq4}]\partial(\widetilde{S}^*(s_1, \widetilde{S}(r_1, r))) = S^*(s_1, \widetilde{S}(r_1, \partial(r))) \tag{15}$$

which will imply (12).

For $\widetilde{S}(r_1, r)$ to be defined we need the inequality $ft^j(Y) \leq ft(\partial(r))$. Since $Y \leq ft(\partial(r))$ it follows from the inequality $ft^j(Y) \leq Y$ and Lemma 1.3.

For $\widetilde{S}^*(s_1, \widetilde{S}(r_1, r))$ to be defined by the inductive assumption we need the inequality

$$S(r_1, Y) \leq ft(\partial(\widetilde{S}(r_1, r)))$$

We have $\partial(\widetilde{S}(r_1, r)) = S(r_1, \partial(r))$ and, since $Y \leq ft(\partial(r))$, we have $ft(S(r_1, \partial(r))) = S(r_1, ft(\partial(r)))$. The inequality

$$S(r_1, Y) \leq S(r_1, ft(\partial(r)))$$

follows from the assumption $Y \leq ft(\partial(r))$ and Lemma 3.8.

The equality (15) follows from the inductive assumption, the equality $\partial(\widetilde{S}(r_1, r)) = S(r_1, \partial(r))$.

This completes the construction of $\widetilde{S}^*(X, Y s, r)$.

4.4 Sets $\widetilde{B}^*(X,Y)$ and homomorphisms of B0-systems

Problem 4.10 *[**2016.03.15.prob1**] Let* $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ *be a homomorphism of B0-systems. For* $X \le Y$ *in* B *to define a function*

$$[\mathbf{2016.03.15.eq2}]\mathbf{f}_{\widetilde{B}^*(X,Y)} : \widetilde{B}^*(X,Y) \to \widetilde{B}'(f(X), f(Y)) \tag{16}$$

Construction 4.11 [**2016.03.15.constr1**] Note first that $f(X) \le f(Y)$ by Lemma 1.11 and therefore the right hand side of (16) is defined.

The construction of $\mathbf{f}_{\widetilde{B}^*(X,Y)}$ is by induction on $j = l(Y) - l(X)$. Since $l(f(Z)) = l(Z)$ for any $Z \in B$ we have that $j = l(f(Y)) - l(f(X))$.

For $j = 0$ both sides are one point sets and there is a unique function $\mathbf{f}_{\widetilde{B}^*(X,Y)}$.

For $j = 1$ we have $\widetilde{B}^*(X,Y) = \widetilde{B}(Y)$ and $\mathbf{f}_{\widetilde{B}^*(X,Y)}(r) = \widetilde{f}(r)$.

For the successor of $j > 0$ we have

$$\widetilde{B}^*(X,Y) = \{(r,s) \,|\, r \in \widetilde{B}(ft^j(Y)) \; s \in \widetilde{B}^*(ft^{j+1}(Y), S(r,Y))\}$$

We set

$$\mathbf{f}_{\widetilde{B}^*(X,Y)}(r,s) = (\widetilde{f}(r), \mathbf{f}_{\widetilde{B}^*(ft^{j+1}(Y), S(r,Y))}(s))$$

We have that $\widetilde{f}(r) \in \widetilde{B}'(ft^j(f(Y)))$ from Lemma 1.10.

We have that $\mathbf{f}_{\widetilde{B}^*(ft^{j+1}(Y), S(r,Y))}(s)$ is defined by the inductive assumption

$$\mathbf{f}_{\widetilde{B}^*(ft^{j+1}(Y), S(r,Y))}(s) \in \widetilde{B}'(f(ft^{j+1}(Y)), f(S(r,Y)))$$

By Lemma 1.10 we have that $f(ft^{j+1}(Y)) = ft^{j+1}(f(Y))$. Since f is a part of a homomorphism of B-system carriers we have that $f(S(r,Y)) = S(\widetilde{f}(r), f(Y))$. Therefore,

$$\mathbf{f}_{\widetilde{B}^*(ft^{j+1}(Y), S(r,Y))}(s) \in \widetilde{B}'(ft^{j+1}(f(Y)), S(\widetilde{f}(r), f(Y)))$$

and $\mathbf{f}_{\widetilde{B}^*(X,Y)}(r,s) \in \widetilde{B}^*(f(ft^{j+1}(Y)), f(Y))$.

2 B-systems

1 Preliminary lemmas

Lemma 1.1 *[**2016.03.19.l1**] Let* $\mathbf{B}$ *be a B0-system. Let* $X, Y, Z \in B$ *be such that* $l(X) \ge 1$*,* $ft(X) < Y$ *and* $ft(Y) < Z$ *then one has:*

1. $(X,Y), (X,Z), (Y,Z) \in T_{dom}$,
2. $(X, T(Y,Z)) \in T_{dom}$,
3. $(T(X,Y), T(X,Z)) \in T_{dom}$.

Proof: We have:

1. The inclusion $(X, Y) \in T_{dom}$ follows from the definitions.

   To show that $(X, Z) \in T_{dom}$ we need to prove that $ft(X) < Z$. We have $ft(X) < Y$ and therefore $ft(X) \leq ft(Y)$ by Lemma 1.6. Then $ft(X) \leq ft(Y) < Z$ implies $ft(X) < Z$ by Lemma 1.4(1).

   To show that $(Y, Z) \in T_{dom}$ we need only to prove $l(Y) \geq 1$ which follows from $Y > ft(X)$.

2. We have that $l(X) \geq 1$. We need to prove that $ft(X) < T(Y, Z)$. We have that $ft(X) < Y$ and $Y < T(Y, Z)$ which implies $ft(X) < T(Y, Z)$ by Lemma 1.3(2).

3. We have $T(X, Y) > X$ and therefore $l(T(X, Y)) \geq 1$. Next we need to prove that $ft(T(X, Y)) < T(X, Z)$. Since $ft(X) < Y$ by Lemma 1.6 we have $ft(X) \leq ft(Y)$. Therefore one of the two cases occurs. In the first case $ft(X) < ft(Y)$ in which case $ft(T(X, Y)) = T(X, ft(Y))$ and since $ft(Y) < Z$ the inequality $ft(T(X, Y)) < T(X, Z)$ follows from Lemma 3.6. In the second case $ft(X) = ft(Y)$. Then $ft(T(X, Y)) = X$ by Lemma 3.5 and $X < T(X, Z)$ by Definition 3.1(1b).

Lemma 1.2 *[**2016.03.19.l3**] Let* $\mathbf{B}$ *be a B0-system. Let* $X, Y \in B$, $s \in \widetilde{B}$ *be such that* $l(X) \geq 1$, $ft(X) < Y$ *and* $ft(Y) < \partial(s)$. *Then one has:*

1. $(X, Y) \in T_{dom}$, $(X, s), (Y, s) \in \widetilde{T}_{dom}$,
2. $(X, \widetilde{T}(Y, s)) \in \widetilde{T}_{dom}$,
3. $(T(X, Y), \widetilde{T}(X, s)) \in \widetilde{T}_{dom}$.

Proof: We have:

1. These assertions follow by applying Lemma 1.1(1) to $X, Y, \partial(s)$.

2. Since we know that $l(X) \geq 1$ we need to show that $ft(X) < \partial(\widetilde{T}(Y, s))$. We have $\partial(\widetilde{T}(Y, s)) = T(Y, \partial(s))$. From $ft(X) < Y$ and $Y < T(Y, \partial(s))$ we get that $ft(X) < T(Y, \partial(s))$ applying Lemma 1.3(2).

3. We have $l(T(X, Y)) \geq 1$ because $T(X, Y) > X$. It remains to show that $ft(T(X, Y)) < \partial(\widetilde{T}(X, s))$. We have $\partial(\widetilde{T}(X, s)) = T(X, \partial(s))$ and the required inequality follows from Lemma 1.1(3) applied to $X, Y, \partial(s)$.

Lemma 1.3 *[**2016.03.19.l4**] Let* $\mathbf{B}$ *be a B0-system. Let* $r, s \in \widetilde{B}$, $Y \in B$ *be such that* $\partial(r) < \partial(s)$ *and* $\partial(s) < Y$. *Then one has*

1. $(r, s) \in \widetilde{S}_{dom}$, $(r, Y), (s, Y) \in S_{dom}$,
2. $(r, S(s, Y)) \in S_{dom}$,
3. $(\widetilde{S}(r, s), S(r, Y)) \in S_{dom}$.

Proof: We have:

1. $(r,s) \in \widetilde{S}_{dom}$ and $(s,Y) \in S_{dom}$ is immediate from the assumptions. To show that $(r,Y) \in S_{dom}$ we need to prove that $\partial(r) < Y$. From $\partial(r) < \partial(s)$ and $\partial(s) < Y$ we obtain $\partial(r) < Y$ by Lemma 1.3(2).

2. We need to show that $\partial(r) < S(s,Y)$. We have $ft(\partial(s)) < S(s,Y)$ and from $\partial(r) < \partial(s)$ we have that $\partial(r) \leq ft(\partial(s))$ by Lemma 1.6. Using Lemma 1.4(1) we get $\partial(r) < S(s,Y)$.

3. We need to show that $\partial(\widetilde{S}(r,s)) < S(r,Y)$. We have $\partial(\widetilde{S}(r,s)) = S(r,\partial(s))$. It remains to show that $S(r,\partial(s)) < S(r,Y)$. It follows from our assumption $\partial(s) < Y$ and Lemma 3.8.

Lemma 1.4 *[**2016.03.21.l1**] Let* $\mathbf{B}$ *be a B0-system. Let $r,s,t \in \widetilde{B}$ be such that $\partial(r) < \partial(s)$ and $\partial(s) < \partial(t)$. Then one has*

1. $(r,s),(r,t),(s,t) \in \widetilde{S}_{dom}$,
2. $(r,\widetilde{S}(s,t)) \in \widetilde{S}_{dom}$,
3. $(\widetilde{S}(r,s),\widetilde{S}(r,t)) \in \widetilde{S}_{dom}$.

Proof: We have:

1. This follows by applying Lemma 1.3(1) to $r,s,\partial(t)$.

2. We need to show that $\partial(r) < \partial(\widetilde{S}(s,t))$. We have $\partial(\widetilde{S}(s,t)) = S(s,\partial(t))$. We can now apply the proof of Lemma 1.3(2) to $r,s,\partial(t)$.

3. We need to show that $\partial(\widetilde{S}(r,s)) < \partial(\widetilde{S}(r,t))$. We have $\partial(\widetilde{S}(r,t)) = \widetilde{S}(r,\partial(t))$ and can now apply the proof of Lemma 1.3(3) to $r,s,\partial(t)$.

Lemma 1.5 *[**2016.03.21.l2**] For any $r \in \widetilde{B}$, $Y,Z \in B$ such that $\partial(r) < Y$ and $ft(Y) < Z$ one has:*

1. $(r,Y),(r,Z) \in S_{dom}$ *and* $(Y,Z) \in T_{dom}$,
2. $(r,T(Y,Z)) \in S_{dom}$,
3. $(S(r,Y),S(r,Z)) \in T_{dom}$.

Proof: We have:

1. The inclusions $(r,Y) \in S_{dom}$ and $(Y,Z) \in T_{dom}$ are immediate from the definitions. It remains to show that $(r,Z) \in S_{dom}$, that is, $\partial(r) < Z$. We have $\partial(r) < Y$ and therefore by Lemma 1.6 we have $\partial(r) \leq ft(Y)$. Together with $ft(Y) < Z$, Lemma 1.4(1) gives us $\partial(r) < Z$.

2. We need to show that $\partial(r) < T(Y,Z)$. We have that $Y < T(Z,Y)$ and $\partial(r) < Y$ and applying Lemma 1.3(2) we obtain that $\partial(r) < T(Z,Y)$.

3. We need to show that $l(S(r,Y)) \geq 1$ and $ft(S(r,Y)) < S(r,Z)$. We have $l(S(r,Y)) = l(Y)-1$. We also have that $l(\partial(r)) > 0$ and $l(Y) > l(\partial(r))$. Therefore $l(Y) \geq 2$ and $l(Y) - 1 \geq 1$. Next we have, $\partial(r) < Y$ and therefore $\partial(r) \leq ft(Y)$ by Lemma 1.6. We have two cases. If $\partial(r) < ft(Y)$ then $ft(S(r,Y)) = S(r, ft(Y))$ and since $ft(Y) < Z$ we have that $S(r, ft(Y)) < S(r,Z)$ by Lemma 3.8. If $\partial(r) = ft(Y)$ then $ft(S(r,Y)) = ft(\partial(r))$ by Lemma 3.7 and we know that $ft(\partial(r)) < S(r,Z)$.

Lemma 1.6 *[**2016.03.21.l4**] For any $r, s \in \widetilde{B}$, $Y \in B$ such that $\partial(r) < Y$ and $ft(Y) < \partial(s)$ one has:*

1. $(r,Y) \in S_{dom}$, $(r,s) \in \widetilde{S}_{dom}$ *and* $(Y,s) \in \widetilde{T}_{dom}$,
2. $(r, \widetilde{T}(Y,s)) \in \widetilde{S}_{dom}$,
3. $(S(r,Y), \widetilde{S}(r,s)) \in \widetilde{T}_{dom}$.

Proof: We have:

1. This follows by applying Lemma 3.7(1) to $r, Y, \partial(s)$.
2. We need to show that $\partial(r) < \partial(\widetilde{T}(Y,s))$. We have that $\partial(\widetilde{T}(Y,s)) = T(Y, \partial(s))$ and the rest of the proof is the same as the proof of Lemma 3.7(2) for $r, Y, \partial(s)$.
3. We need to show that $l(S(r,Y)) \geq 1$ and $ft(S(r,Y)) < \partial(\widetilde{S}(r,s))$. We have $\partial(\widetilde{S}(r,s)) = S(r, \partial(s))$ and the rest of the proof is the same as the proof of Lemma 3.7(3) for $r, Y, \partial(s)$.

Lemma 1.7 *[**2016.03.21.l5a**] Let **B** be a B0-system. Then for any $X, Y \in B$, $r \in \widetilde{B}$ such that $l(X) \geq 1$, $ft(X) < \partial(r)$, $\partial(r) < Y$ one has:*

1. $(X,r) \in \widetilde{T}_{dom}$, $(X,Y) \in T_{dom}$, $(r,Y) \in S_{dom}$,
2. $(X, S(r,Y)) \in T_{dom}$,
3. $(\widetilde{T}(X,r), T(X,Y)) \in S_{dom}$.

Proof: We have:

1. The inclusions $(X,r) \in \widetilde{T}$ and $(r,Y) \in S_{dom}$ are immediate from the definitions. It remains to show that $(X,Y) \in T_{dom}$. We know that $l(X) \geq 1$ and need to prove that $ft(X) < Y$. We have $ft(X) < \partial(r)$ and $\partial(r) < Y$ and applying Lemma 1.3(2) we get $ft(X) < Y$.
2. We know that $l(X) \geq 1$. We need to show that $ft(X) < S(r,Y)$. We have $ft(X) < \partial(r)$. By Lemma 1.6 we get that $ft(X) \leq ft(\partial(r))$. We also have that $ft(\partial(r)) < S(r,Y)$. Combining these two inequalities and applying Lemma 1.4(1) we get that $ft(X) < S(r,Y)$.
3. We need to show that $\partial(\widetilde{T}(X,r)) < T(X,Y)$. We have $\partial(\widetilde{T}(X,r)) = T(X, \partial(r))$ and $\partial(r) < Y$. By Lemma 3.6 we conclude that $T(X, \partial(r)) < T(X,Y)$.

Lemma 1.8 *[**2016.03.21.l5**] Let* **B** *be a B0-system. Then for any $X \in B$, $r, s \in \widetilde{B}$ such that $l(X) \geq 1$, $ft(X) < \partial(r)$, $\partial(r) < \partial(s)$ one has:*

1. $(X, r), (X, s) \in \widetilde{T}_{dom}$, $(r, s) \in \widetilde{S}_{dom}$,
2. $(X, \widetilde{S}(r, s)) \in \widetilde{T}_{dom}$,
3. $(\widetilde{T}(X, r), \widetilde{T}(X, s)) \in \widetilde{S}_{dom}$.

Proof: We have:

1. The inclusions $(X, r) \in \widetilde{T}$ and $(r, s) \in \widetilde{S}_{dom}$ are immediate from the definitions. It remains to show that $(X, s) \in \widetilde{T}_{dom}$. This follows by applying Lemma 1.6(1) to $X, \partial(s), r$.
2. We need to show that $(X, \partial(\widetilde{S}(r, s))) \in T_{dom}$. We have $\partial(\widetilde{S}(r, s)) = S(s, \partial(s))$ and the required inclusion follows from Lemma 1.6(2) applied to $X, \partial(s), r$.
3. We need to show that $\partial(\widetilde{T}(X, r)) < \partial(\widetilde{T}(X, s))$. We have $\partial(\widetilde{T}(X, s)) = T(X, \partial(s))$ and the required inclusion follows from Lemma 1.6(3) applied to $X, \partial(s), r$.

Lemma 1.9 *[**2016.03.21.l6**] Let* **B** *be a B0-system. Then for any $r \in \widetilde{B}$ and $Y \in B$ such that $ft(\partial(r)) < Y$ one has*

1. $(\partial(r), Y) \in T_{dom}$,
2. $(r, T(\partial(r), Y)) \in S_{dom}$.

Proof: The first inclusion follows immediately from the definitions since $l(\partial(r)) \geq 1$ by the definition of a B-system carrier.

To prove the second inclusion we need to show that $\partial(r) < T(\partial(r), Y)$ which is immediate from the definition of a B0-system.

Lemma 1.10 *[**2016.03.21.l7**] Let* **B** *be a B0-system. Then for any $r, s \in \widetilde{B}$ such that $ft(\partial(r)) < \partial(s)$ one has*

1. $(\partial(r), s) \in \widetilde{T}_{dom}$,
2. $(r, \widetilde{T}(\partial(r), s)) \in \widetilde{S}_{dom}$.

Proof: The first inclusion follows immediately from the definitions since $l(\partial(r)) \geq 1$ by the definition of a B-system carrier.

To prove the second inclusion we need to show that $\partial(r) < \partial(\widetilde{T}(\partial(r), s))$. We have $\partial(\widetilde{T}(\partial(r), s)) = T(\partial(r), \partial(s))$ and $\partial(r) < T(\partial(r), \partial(s))$ by the definition of a B0-system.

Lemma 1.11 *[**2016.03.21.l8**] Let* **B** *be a (unital) B0-system. Then for any $X, Y \in B$ such that $l(X) \geq 1$ and $ft(X) < Y$ one has:*

1. $Y \in \delta_{dom}$,

2. $(X, \delta(Y)) \in \widetilde{T}_{dom}$,

3. $(X, Y) \in T_{dom}$,

4. $T(X, Y) \in \delta_{dom}$.

Proof: We have:

1. Since $ft(X) < Y$ we have $l(Y) \geq 1$, i.e., $Y \in \delta_{dom}$.
2. $l(X) \geq 1$ as one of our conditions. It remains to show that $ft(X) < \partial(\delta(Y))$. We have $\partial(\delta(Y)) = T(Y, Y)$. Next, we have $ft(X) < Y$ and $Y < T(Y, Y)$ and by Lemma 1.3(2) we conclude that $ft(X) < T(Y, Y)$.
3. This inclusion follows immediately from our conditions $l(X) \geq 1$ and $ft(X) < Y$.
4. We have $l(T(X, Y)) = l(Y) + 1 \geq 1$.

Lemma 1.12 *[***2016.03.21.l9***] Let* **B** *be a (unital) B0-system. Then for any* $r \in \widetilde{B}$, $Y \in B$ *such that* $\partial(r) < Y$ *one has:*

1. $Y \in \delta_{dom}$,

2. $(r, \delta(Y)) \in \widetilde{S}_{dom}$,

3. $(r, Y) \in S_{dom}$,

4. $S(r, Y) \in \delta_{dom}$.

Proof: We have:

1. Since $Y > \partial(s)$ we have $l(Y) \geq 1$.
2. We need to check that $\partial(r) < \partial(\delta(Y))$. We have $\partial(\delta(Y)) = T(Y, Y)$ and $Y < T(Y, Y)$. We have $\partial(r) < Y$ and $Y < T(Y, Y)$ and by Lemma 1.3(2) we get $\partial(r) < T(Y, Y)$.
3. We need to check that $\partial(r) < Y$ which is one of our conditions.
4. We need to check that $l(S(r, Y)) \geq 1$. We have $l(S(r, Y)) = l(Y) - 1$. Since $Y > \partial(r)$ and $l(\partial(r)) \geq 1$ we have $l(Y) \geq 2$ and $l(Y) - 1 \geq 1$.

Lemma 1.13 *[***2016.03.23.l1***] Let* **B** *be a B0-system. Then for any* $r \in \widetilde{B}$ *one has:*

1. $\partial(r) \in \delta_{dom}$,

2. $(r, \delta(\partial(r))) \in \widetilde{S}_{dom}$.

Proof: We have:

1. $l(\partial(r)) \geq 1$ by the definition of a B-system carrier and therefore $\partial(r) \in \delta_{dom}$.

2. We have to show that $\partial(r) < \partial(\delta(\partial(r)))$. By the definition of a B0-system we have $\partial(\delta(\partial(r))) = T(\partial(r), \partial(r))$ and $\partial(r) < T(\partial(r), \partial(r))$ by another part of the same definition.

Lemma 1.14 *[**2016.03.23.l2**] Let* **B** *be a B0-system. Then for any $X, Y \in B$ such that $l(X) \geq 1$ and $X < Y$ one has:*

1. $X \in \delta_{dom}$,

2. $(X, Y) \in T_{dom}$,

3. $(\delta(X), T(X, Y)) \in S_{dom}$.

Proof: We have:

1. The inclusion $X \in \delta_{dom}$ follows directly from the assumption.

2. Tor $(X, Y) \in T_{dom}$ we need to show that $l(X) \geq 1$ and $ft(X) < Y$. The first inequality is an assumption. For the second inequality we have $ft(X) \leq X$ by Lemma 1.7 and together with $X < Y$, Lemma 1.4(1) gives us $ft(X) < Y$.

3. We need to show that $\partial(\delta(X)) < T(X, Y)$. We have $\partial(\delta(X)) = T(X, X)$ and $T(X, X) < T(X, Y)$ by Lemma 3.6.

Lemma 1.15 *[**2016.03.23.l3**] Let* **B** *be a B0-system. Then for any $X \in B$, $s \in \widetilde{B}$ such that $l(X) \geq 1$ and $X < \partial(s)$ one has:*

1. $X \in \delta_{dom}$,

2. $(X, s) \in \widetilde{T}_{dom}$,

3. $(\delta(X), \widetilde{T}(X, s)) \in \widetilde{S}_{dom}$.

Proof: Using the fact that $\partial(\widetilde{T}(X, s)) = T(X, \partial(s))$ this lemma follows from Lemma 1.14 applied to X and $\partial(s)$.

2 Definition of B-systems

Definition 2.1 *[**2014.10.16.def2**] [**was.2014.06.18.eq2.to.eq11**] Let* **B** *be a non-unital B0-system. Define the following conditions on* **B***:*

1. The TT-condition. For all $X, Y \in B$ such that $l(X) \geq 1$ and $ft(X) < Y$ one has

   (a) for all $Z \in B$ such that $ft(Y) < Z$ one has

$$T(T(X, Y), T(X, Z)) = T(X, T(Y, Z))$$

   where the left and the right hand sides are defined in view of Lemma 1.1.

(b) *for all* $s \in \widetilde{B}$ *such that* $ft(Y) < \partial(s)$ *one has*

$$\widetilde{T}(T(X,Y), \widetilde{T}(X,s)) = \widetilde{T}(X, \widetilde{T}(Y,s))$$

where the left and the right hand sides are defined in view of Lemma 1.2.

2. *The SS-condition. For all* $r, s \in \widetilde{B}$ *such that* $\partial(r) < \partial(s)$ *one has*

   (a) *for all* $Y \in B$ *such that* $\partial(s) < Y$

   $$S(\widetilde{S}(r,s), S(r,Y)) = S(r, S(s,Y))$$

   where the left and the right hand sides are defined in view of Lemma 1.3.

   (b) *for all* $t \in \widetilde{B}$ *such that* $\partial(s) < \partial(t)$ *one has*

   $$\widetilde{S}(\widetilde{S}(r,s), \widetilde{S}(r,t)) = \widetilde{S}(r, \widetilde{S}(s,t))$$

   where the left and the right hand sides are defined in view of Lemma 1.4.

3. *The TS-condition. For any* $r \in \widetilde{B}$ *and* $Y \in B$ *such that* $\partial(r) < Y$ *one has*

   (a) *for all* $Z \in B$ *such that* $ft(Y) < Z$

   $$T(S(r,Y), S(r,Z)) = S(r, T(Y,Z))$$

   where the left and the right hand sides are defined in view of Lemma 1.5.

   (b) *for all* $r \in \widetilde{B}$ *such that* $ft(Y) < \partial(r)$

   $$\widetilde{T}(S(r,Y), \widetilde{S}(r,s)) = \widetilde{S}(r, \widetilde{T}(Y,s))$$

   where the left and the right hand sides are defined in view of Lemma 1.6.

4. *The ST-condition. For any* $X \in B$ *and* $r \in \widetilde{B}$ *such that* $l(X) \geq 1$ *and* $ft(X) < \partial(r)$ *one has*

   (a) *for all* $Y \in B$ *such that* $\partial(r) < Y$ *one has*

   $$S(\widetilde{T}(X,r), T(X,Y)) = T(X, S(r,Y))$$

   where the left and the right hand sides are defined in view of Lemma 1.7.

   (b) *for all* $s \in \widetilde{B}$ *such that* $\partial(r) < \partial(s)$ *one has*

   $$\widetilde{S}(\widetilde{T}(X,r), \widetilde{T}(X,s)) = \widetilde{T}(X, \widetilde{S}(r,s))$$

   where the left and the right hand sides are defined in view of Lemma 1.8.

5. *The STid-condition. For any* $r \in \widetilde{B}$ *one has*

   (a) *for all* $Y \in B$ *such that* $ft(\partial(r)) < Y$ *one has*

   $$S(r, T(\partial(r), Y)) = Y$$

   where the left and the right hand sides are defined in view of Lemma 1.9.

(b) *for all* $s \in \widetilde{B}$ *such that* $ft(\partial(r)) < \partial(s)$ *one has*

$$\widetilde{S}(r, \widetilde{T}(\partial(r), s)) = s$$

where the left and the right hand sides are defined in view of Lemma 1.10.

Definition 2.2 *[***2014.10.20.def3***] Let* $\mathbf{B}$ *be a unital B0-system. Define the following conditions on* $\mathbf{B}$*:*

1. *The* δT*-condition. For any* $X, Y \in B$ *such that* $l(X) \geq 1$ *and* $ft(X) < Y$ *one has*

$$\widetilde{T}(X, \delta(Y)) = \delta(T(X, Y))$$

where the left and the right hand sides are defined in view of Lemma 1.11.

2. *The* δS*-condition. For any* $r \in \widetilde{B}$ *and* $Y \in B$ *such that* $\partial(r) < Y$ *one has*

$$\widetilde{S}(r, \delta(Y)) = \delta(S(r, Y))$$

where the left and the right hand sides are defined in view of Lemma 1.12.

3. *The* δSid*-condition. For any* $r \in \widetilde{B}$ *one has*

$$\widetilde{S}(r, \delta(\partial(r))) = r$$

where the left hand sides is defined in view of Lemma 1.13.

4. *The* $S\delta T$*-condition. For any* $X \in B$ *such that* $l(X) \geq 1$ *one has*

   (a) *for* $Y \in B$ *such that* $X < Y$ *one has:*

$$S(\delta(X), T(X, Y)) = R$$

   where the left hand sides is defined in view of Lemma 1.14.

   (b) *for* $s \in \widetilde{B}$ *such that* $X < \partial(s)$ *one has*

$$\widetilde{S}(\delta(X), \widetilde{T}(X, r)) = r$$

   where the left hand sides is defined in view of Lemma 1.15.

Remark 2.3 *[***2014.06.14.rem2***] In the case of a syntactic B0-system, the conditions defined above can be shown as follows:*

1. *The TT-condition:*

$$\frac{\Gamma, T \rhd \quad \Gamma, \Delta, T' \rhd \quad \Gamma, \Delta \rhd \mathcal{J}}{\frac{\Gamma, T \rhd \quad \Gamma, \Delta, T' \rhd \mathcal{J}}{\Gamma, T, \Delta, T' \rhd \mathcal{J}} \quad \frac{\Gamma, T, \Delta, T' \rhd \quad \Gamma, T, \Delta \rhd \mathcal{J}}{\Gamma, T, \Delta, T' \rhd \mathcal{J}}}$$

2. *The SS-condition:*

$$\frac{\Gamma \rhd s : T \quad \Gamma, T, \Delta \rhd s' : T' \quad \Gamma, T, \Delta, T' \rhd \mathcal{J}}{\frac{\Gamma \rhd s : T \quad \Gamma, T, \Delta \rhd \mathcal{J}[s]}{\Gamma, \Delta[s] \rhd \mathcal{J}[s'][s]} \quad \frac{\Gamma, \Delta[s] \rhd s'[s] : T'[s] \quad \Gamma, \Delta[s], T'[s] \rhd \mathcal{J}[s]}{\Gamma, \Delta[s] \rhd \mathcal{J}[s][s']}}$$

3. The TS-condition:

$$\frac{\Gamma, T \rhd \quad \Gamma, \Delta \rhd s' : T' \quad \Gamma, \Delta, T' \rhd \mathcal{J}}{\frac{\Gamma,T\rhd \quad \Gamma,\Delta\rhd\mathcal{J}[s']}{\Gamma,T,\Delta\rhd\mathcal{J}[s']} \quad \frac{\Gamma,T,\Delta\rhd s':T' \quad \Gamma,T,\Delta,T'\rhd\mathcal{J}}{\Gamma,T,\Delta\rhd\mathcal{J}[s']}}$$

4. The ST-condition:

$$\frac{\Gamma \rhd s : T \quad \Gamma, T, \Delta, T' \rhd \quad \Gamma, T, \Delta \rhd \mathcal{J}}{\frac{\Gamma\rhd s:T \quad \Gamma,T,\Delta,T'\rhd\mathcal{J}[s]}{\Gamma,\Delta[s],T'[s]\rhd\mathcal{J}[s]} \quad \frac{\Gamma,\Delta[s],T'[s]\rhd \quad \Gamma,\Delta[s]\rhd\mathcal{J}[s]}{\Gamma,\Delta[s],T'[s]\rhd\mathcal{J}[s]}}$$

5. The STid-condition:

$$\frac{\Gamma \rhd s : T \quad \Gamma, T \rhd \quad \Gamma \rhd \mathcal{J}}{\frac{\Gamma\rhd s:T \quad \Gamma,T\rhd\mathcal{J}}{\Gamma\rhd\mathcal{J}[s]}}$$

6. The δT-condition:

$$\frac{\Gamma, T \rhd \quad \Gamma, \Delta, x : T'\rhd}{\frac{\Gamma,T\rhd \quad \Gamma,\Delta,x:T'\rhd x:T'}{\Gamma,T,\Delta,x:T'\rhd x:T'} \quad \frac{\Gamma,T,\Delta,x:T'\rhd}{\Gamma,T,\Delta,x:T'\rhd x:T'}}$$

7. The δS-condition:

$$\frac{\Gamma \rhd s : T \quad \Gamma, T, \Delta, x : T'\rhd}{\frac{\Gamma\rhd s:T \quad \Gamma,T,\Delta,x:T'\rhd x:T'}{\Gamma,\Delta[s],x:T'[s]\rhd x:T'[s]} \quad \frac{\Gamma,\Delta[s],x:T[s]'\rhd}{\Gamma,\Delta[s],x:T'[s]\rhd x:T'[s]}}$$

8. The δSid-condition:

$$\frac{\Gamma \rhd s : T \quad \Gamma, x : T\rhd}{\frac{\Gamma\rhd s:T \quad \Gamma,x:T\rhd x:T}{\Gamma\rhd s:T}}$$

9. The SδT-condition:

$$\frac{\Gamma, y : X, \Delta \rhd \mathcal{J}}{\frac{\Gamma,y_1:X,y:X,\Delta\rhd\mathcal{J} \quad \Gamma,y_1:X\rhd y_1:X}{\Gamma,y_1:X,\Delta[y_1/y]\rhd\mathcal{J}[y_1/y]}}$$

Definition 2.4 *[***2014.10.10.def2a***] [***2014.10.20.def4***] A non-unital B-system is a non-unital B0-system that satisfy the conditions TT, SS, TS, ST and $STid$ of Definition 2.1.*

Definition 2.5 *[***2014.10.10.def2b***] [***2014.10.20.def5***] A unital B-system is a unital B0-system that satisfy the conditions TT, SS, TS, ST, $STid$ of Definition 2.1 and the conditions δT, δS, δSid and $S\delta T$ of Definition 2.2.*

Equivalently, a unital B-system is non-unital B-system such that there exists a family of operations δ satisfying the conditions δT, δS, δSid and $S\delta T$ of Definition 2.2.

3 Elementary properties of B-systems

Remark 3.1 In unital B-systems operations S and T can be expressed as follows.

$$[\mathbf{2014.10.14.eq1}]T(X,Y) = \begin{cases} X & \text{if } l(Y) = l(X) - 1 \\ ft(\partial(\widetilde{T}(X, \delta(Y)))) & \text{if } l(Y) \geq l(X) \end{cases} \tag{17}$$

$$[\mathbf{2014.10.14.eq2}]S(s,X) = \begin{cases} ft(\partial(s)) & \text{if } l(X) = l(\partial(s)) \\ ft(\partial(\widetilde{S}(s, \delta(X)))) & \text{if } l(X) > l(\partial(s)) \end{cases} \tag{18}$$

Lemma 3.2 *[**2014.10.20.l1**] [**2014.10.16.l1**] Let B be a non-unital B0-system and let δ_1, δ_2 be two operations of the form $\delta_{dom} \to \widetilde{B}$ satisfying the condition of Definition 3.4 and conditions δT, δSid and $S\delta T$ conditions. Then $\delta_1 = \delta_2$.*

Proof: We have:

$$\delta_1(X) = \widetilde{S}(\delta_2(X), \widetilde{T}(X, \delta_1(X))) = \widetilde{S}(\delta_2(X), \delta_1(T(X, X))) = \delta_2(X)$$

where the first equality is the $S\delta T$-condition for δ_2, the second equality is the δT-condition for δ_1 and the third equality is the δSid-condition for δ_1.

Example 3.3 [**2014.10.20.eX**] While being unital is a property of non-unital B-systems not every homomorphism of non-unital B-systems preserves units. Here is a sketch of an example of a homomorphism that does not preserve units.

Consider the following pairs of a monad and a left module over it. In both cases pt is the constant functor corresponding to the one point set $\{T\}$ that has a unique left module structure over any monad.

1. (R_1, pt) where R_1 is the monad corresponding to one unary operation $s_1(x)$ and the relation

$$s_1(s_1(x)) = s_1(x)$$

2. (R_2, pt) where R_2 is the monad corresponding to two unary operations $s_1(x)$ and $s_2(x)$ and relations:

$$s_1(s_1(x)) = s_1(x) \quad s_1(s_2(x)) = s_1(x) \quad s_2(s_1(x)) = s_1(x) \quad s_2(s_2(x)) = s_2(x)$$

Consider the unital B-systems $uB(R_1, pt)$ and $uB(R_2, pt)$. In $uB(R_1, pt)$ consider the non-unital sub-B-system nuB_1 generated by $(T \rhd s_1(1) : T)$. In $uB(R_2, pt)$ consider the non-unital sub-B-system nuB_2 generated by $(T \rhd s_1(1) : T)$ and $(T \rhd s_2(1) : T)$.

Observe that both nuB_1 and nuB_2 are in fact unital with the unit in the first one given by $(T, \ldots, T \rhd s_1(n) : T)$ and unit in the second one is given by $(T, \ldots, T \rhd s_2(n) : T)$ where n is the number of T's before the turnstile $\rhd$ symbol.

We also have an obvious (unital) homomorphism from $uB(R_1, pt)$ to $uB(R_2, pt)$ that defines a homomorphism $nuB_1 \to nuB_2$ and that latter homomorphism is not unital.

4 Operations $\widetilde{T}^{**}$ and $\widetilde{S}^{**}$

4.1 The $ST^*(a)$-property

Lemma 4.1 *[**2016.04.10.l1**] Let **B** be a B0-system. Then for any $X, Y, W \in B$, $r \in \widetilde{B}$ such that $X \geq Y$, $Y \leq ft(\partial(r))$ and $\partial(r) < W$ one has:*

1. *$(X, Y, r) \in \widetilde{T}^*_{dom}$,*
2. *$(X, Y, W) \in T^*_{dom}$,*

3. $(\widetilde{T}^*(X,Y,r), T^*(X,Y,W)) \in S_{dom}$,

4. $(r,W) \in S_{dom}$,

5. $(X,Y,S(r,W)) \in T^*_{dom}$.

Proof: We have:

1. The required conditions are $X \geq Y$, $Y \leq ft(\partial(r))$. Both are among our assumptions.

2. The required conditions are $X \geq Y$, $Y \leq W$. The first one is among our assumptions. To show the second one we first have $ft(\partial(r)) \leq \partial(r)$ and then $Y \leq W$ follows from $Y \leq ft(\partial(r)) \leq \partial(r) < W$ by Lemmas 1.3 and 1.4.

3. The required condition is $\partial(\widetilde{T}^*(X,Y,r)) < T^*(X,Y,W)$. By (6) we have $\partial(\widetilde{T}^*(X,Y,r)) = \widetilde{T}^*(X,Y,\partial(r))$ and the condition follows from the assumption $\partial(r) < W$ by Lemma 3.13.

4. The required condition is $\partial(r) < W$ and is among our assumptions.

5. The required conditions are $X \leq Y$, $Y \leq S(r,W)$. The first one is among our assumptions. By the axioms of a B0-system we have $ft(\partial(r)) < S(r,W)$ and the second condition follows this inequality and our assumption $Y \leq ft(\partial(r))$ by Lemma 1.4.

Lemma 4.2 *[***2016.03.27.l1***] Let* $\mathbf{B}$ *be a B0-system that satisfies the ST(a)-condition of Definition 2.1. Then for any* $X,Y,W \in B$, $r \in \widetilde{B}$ *such that* $X \geq Y$, $Y \leq ft(\partial(r))$ *and* $\partial(r) < W$ *one has:*

$$S(\widetilde{T}^*(X,Y,r), T^*(X,Y,W)) = T^*(X,Y,S(r,W))$$

where both sides are defined by Lemma 4.1.

Proof: We proceed by induction on $j = l(X) - l(Y)$ using Constructions 3.12 and 3.15.

For $j = 0$ we have

$$S(\widetilde{T}^*(X,Y,r), T^*(X,Y,W)) = S(r,W)$$

and

$$T^*(X,Y,S(r,W)) = S(r,W)$$

For $j = 1$ we need to show

$$S(\widetilde{T}(X,r), T_{ext}(X,W)) = T_{ext}(X,S(r,W))$$

if $l(X) \geq 1$, $ft(X) \leq ft(\partial(r))$, $\partial(r) < W$. We have $ft(X) < \partial(r) < W$ and in particular $ft(X) < W$. Therefore $T_{ext}(X,W) = T(X,W)$. We also have $ft(X) < S(r,W)$. Indeed, $ft(X) \leq ft(\partial(r)) < S(r,W)$. Therefore

$$T_{ext}(X,S(r,W)) = T(X,S(r,W))$$

It remains to show that

$$S(\widetilde{T}(X,r), T(X,W)) = T(X,S(r,W))$$

The objects (X, W, r) are in the domain of definition of the ST(a)-condition and the equality is the equality of the condition. This completes the proof of the $j = 1$ case.

For the successor of $j \geq 1$ we need to show that

$$[\mathbf{2016.03.21.eq2}] S(\widetilde{T}(X, \widetilde{T}^*(ft(X), Y, r)), T_{ext}(X, T^*(ft(X), Y, W))) = T_{ext}(X, T^*(ft(X), Y, S(r, W))) \tag{19}$$

assuming that $ft(X) > Y$, $Y \leq ft(\partial(r))$, $\partial(r) < W$. By the inductive assumption we may assume that

$$[\mathbf{2016.03.27.eq1}] S(\widetilde{T}(ft(X), Y, r), T(ft(X), Y, W)) = T(ft(X), Y, S(r, W)) \tag{20}$$

Let us first show that two T_{ext} in (19) can be replaced by T.

For the first one we need to show that $ft(X) < T^*(ft(X), Y, W)$. We have $ft(X) \leq T^*(ft(X), Y, W)$ by Problem 3.11(1). Next we have

$$l(T^*(ft(X), Y, W)) - l(ft(X)) = l(W) - l(Y)$$

Since $Y < \partial(r)$ and $\partial(r) < W$ we have $l(W) - l(Y) \geq 2$ and $l(T^*(ft(X), Y, W)) > l(ft(X))$. This shows that

$$T_{ext}(X, T^*(ft(X), Y, W)) = T(X, T^*(ft(X), Y, W))$$

Next we need to show that $ft(X) < T^*(ft(X), Y, S(r, Y))$. Again we use that by Problem 3.11(1) we have $ft(X) \leq T^*(ft(X), Y, S(r, Y))$. Next we have

$$l(T^*(ft(X), Y, S(r, Y))) - l(ft(X)) = l(S(r, Y)) - l(Y) = (l(W) - 1) - l(Y)$$

Here we use again that $Y < \partial(r)$ and $\partial(r) < W$ and therefore $(l(W) - 1) - l(Y) \geq 1$. This shows that

$$T_{ext}(X, T^*(ft(X), Y, S(r, W))) = T(X, T^*(ft(X), Y, S(r, W)))$$

It remains to prove that

$$[\mathbf{2016.03.27.eq3}] S(\widetilde{T}(X, \widetilde{T}^*(ft(X), Y, r)), T(X, T^*(ft(X), Y, W))) = T(X, T^*(ft(X), Y, S(r, W))) \tag{21}$$

assuming that $ft(X) > Y$, $Y \leq ft(\partial(r))$, $\partial(r) < W$.

Let us show that we can apply ST(a)-condition to the left hand side of (21).

We have $l(X) \geq 1$ because $X \geq ft(X) > Y$.

We have $ft(X) < \partial(\widetilde{T}(ft(X), Y, r))$ since

$$\partial(\widetilde{T}^*(ft(X), Y, r)) = T^*(ft(X), Y, \partial(r)) \geq ft(X)$$

and

$$l(T^*(ft(X), Y, \partial(r))) - l(ft(X)) = l(\partial(r)) - l(Y)$$

and since $Y < \partial(r)$, $l(\partial(r)) - l(Y) \geq 1$.

Last, we need that

$$\partial(\widetilde{T}^*(ft(X), Y, r)) < T^*(ft(X), Y, W)$$

Since $\partial(\widetilde{T}^*(ft(X), Y, r)) = T^*(ft(X), Y, \partial(r))$ and $\partial(r) < W$ it follows from Lemma 3.13.

Applying the ST(a)-condition to the left hand side of (21) and then applying (20) we get

$$S(\widetilde{T}(X, \widetilde{T}^*(ft(X), Y, r)), T(X, T^*(ft(X), Y, W))) = T(X, S(\widetilde{T}^*(ft(X), Y, r), T^*(ft(X), Y, W))) =$$

$$T(X, T^*(ft(X), Y, S(r, W)))$$

Finally, by Construction 3.12 we have

$$T(X, T^*(ft(X), Y, S(r, W))) = T^*(X, Y, S(r, W))$$

This completes the proof of Lemma 4.2.

4.2 Sets $\widetilde{T}^{**}_{dom}$ and operations $\widetilde{T}^{**}$

We will now construct the operations that correspond, in the B0-systems that correspond to C-systems, to the pull-back of sections of morphisms $p_{Y,ft^n(Y)}$ by morphisms $p_{X,ft^l(X)}$.

Definition 4.3 *[**2016.04.02.def1**] Let* **B** *be a B0-system. Define:*

$$\widetilde{T}^{**}_{dom} = \{X, Y, Z, W, s \,|\, X \geq Y \leq Z \leq W, s \in \widetilde{B}^*(Z, W)\}$$

We will sometimes write elements of $\widetilde{T}^{**}_{dom}$ as (X, Y, s) because Z and W can be recovered from the type of s.

Problem 4.4 *[**2016.02.22.prob1**] Let* **B** *be a B0-system carrier such that the operations $S, \widetilde{T}$ and T satisfy the ST(a)-condition of Definition 2.1.*

For each $(X, Y, Z, W, s) \in \widetilde{T}^{**}_{dom}$ to define an element

$$\widetilde{T}^{**}(X, Y, Z, W, s) \in \widetilde{B}^*(T(X, Y, Z), T(X, Y, W))$$

where the right hand side is well defined by Lemma 3.13.

The diagram for Problem 4.4 is as follows:

$$\begin{array}{ccc}
T^*(X,Y,W) & & W \\
{\scriptstyle \widetilde{T}^{**}(X,Y,s)\uparrow}\big\downarrow & & {\scriptstyle s\uparrow}\big\downarrow \\
T^*(X,Y,Z) & & Z \\
\big\downarrow & & \big\downarrow \\
X & \longrightarrow & Y
\end{array}$$

Construction 4.5 [**2016.02.22.constr1**] Proceed now by induction on $j = l(W) - l(Z)$.

For $j = 0$ we set

$$\widetilde{T}^{**}(X, Y, Z, Z, tt) = tt$$

where tt is the only element of $\widetilde{B}^*(Z, Z) = \widetilde{B}^*(T(X, Y, Z), T(X, Y, Z)) = unit$.

For $j = 1$ we set

$$\widetilde{T}^{**}(X, Y, ft(W), W, s) = \widetilde{T}^{*}(X, Y, s)$$

For the successor of $j > 0$ we define $\widetilde{T}^{**}(X, Y, ft^{j+1}(W), W, s)$ as follows. Recall that for $j > 0$ we have

$$\widetilde{B}^{*}(ft^{j+1}(W), W) = \{(r, s) \,|\, r \in \widetilde{B}(ft^{j}(W)),\ s \in \widetilde{B}^{*}(ft^{j+1}(W), S(r, W))\}$$

We set

$$[\mathbf{2016.03.09.eq1}]\widetilde{T}^{**}(X, Y, ft^{j+1}(W), W, (r, s)) = (\widetilde{T}^{*}(X, Y, r), \widetilde{T}^{**}(X, Y, ft^{j+1}(W), S(r, W), s) \tag{22}$$

The part of the diagram for this case that is over Y is as follows:

$$\begin{array}{ccc}
S(r,W) & & W \\
{\scriptstyle s\uparrow}\big\downarrow & & \big\downarrow \\
ft^{j+1}(W) & \overset{\overset{r}{\longrightarrow}}{\longleftarrow} & ft^{j}(W) \\
 & & \big\downarrow \\
 & & ft^{j+1}(W) \\
 & & \big\downarrow \\
 & & Y
\end{array}$$

and the part that is over X is as follows:

$$\begin{array}{cccc}
T^{*}(X,Y,S(r,W)) & & & T^{*}(X,Y,W) \\
\big\downarrow{\scriptstyle \uparrow \widetilde{T}^{**}(X,Y,ft^{j+1}(W),S(r,W),s)} & & & \big\downarrow \\
T^{*}(X,Y,ft^{j+1}(W)) & = & ft(T^{*}(X,Y,ft^{j}(W))) \overset{\overset{\widetilde{T}^{*}(X,Y,r)}{\longrightarrow}}{\longleftarrow} & T^{*}(X,Y,ft^{j}(W)) \\
 & & & \big\downarrow \\
 & & & ft(T^{*}(X,Y,ft^{j}(W))) \\
 & & & \big\downarrow \\
 & & & X
\end{array}$$

Let us check that the right hand side of (22) is well defined.

For $\widetilde{T}^{*}(X, Y, r)$ to be defined we need $X \geq Y$ and $ft(\partial(r)) \geq Y$. We know that $X \geq Y$. Next, we have $ft(\partial(r)) = ft^{j+1}(W)$ and $ft^{j+1}(W) \geq Y$. This proves that $\widetilde{T}^{*}(X, Y, r)$ is defined.

For $\widetilde{T}^{**}(X, Y, ft^{j+1}(W), S(r, W), s)$ to be defined we need $s \in \widetilde{B}^{*}(ft^{j+1}(W), S(r, W))$, $X \leq Y$ and $Y \leq ft^{j+1}(W) \leq S(r, W)$. The first two conditions as well as the condition that $Y \leq ft^{j+1}(W)$ are parts of our assumptions. To see that $ft^{j+1}(W) \leq S(r, W)$ we use the fact that $ft^{j+1}(W) = ft(\partial(r))$ and that $S(r, W) > ft(\partial(r))$ according to Definition 3.1(3).

It remains to show that

$$(\widetilde{T}^{*}(X, Y, r), \widetilde{T}^{**}(X, Y, ft^{j+1}(W), S(r, W), s)) \in \widetilde{B}^{*}(T(X, Y, ft^{j+1}(W)), T(X, Y, W))$$

To know what are the elements of the right hand side set we need to know $l(T^*(X,Y,ft^{j+1}(W)) - l(T^*(X,Y,W))$. By Lemma 3.13 we have

$$l(T^*(X,Y,W) - l(T^*(X,Y,ft^{j+1}(W))) = l(W) - l(ft^{j+1}(W)) = l(W) - l(Z) = j+1$$

and by the same lemma $T^*(X,Y,ft^{j+1}(W)) \leq T^*(X,Y,W)$. Therefore, by definition of $\leq$

$$T^*(X,Y,ft^{j+1}(W)) = ft^{j+1}(T^*(X,Y,W))$$

and elements of this set are pairs (r', s') where

$$r' \in \widetilde{B}(ft^j(T^*(X,Y,W)))$$

$$s' \in \widetilde{B}^*(ft^{j+1}(T^*(X,Y,W)), S(r', T^*(X,Y,W)))$$

Therefore, we need to prove that

$$[\mathbf{2016.04.06.eq1}]\widetilde{T}^*(X,Y,r) \in \widetilde{B}(ft^j(T^*(X,Y,W))) \tag{23}$$

and

$$[\mathbf{2016.04.06.eq2}]\widetilde{T}^{**}(X,Y,ft^{j+1}(W),S(r,W),s) \in \widetilde{B}^*(ft^{j+1}(T(X,Y,W)),S(\widetilde{T}^*(X,Y,r),T^*(X,Y,W))) \tag{24}$$

We already know that $ft^{j+1}(T^*(X,Y,W)) = T^*(X,Y,ft^{j+1}(W))$. Similar reasoning shows that $ft^j(T^*(X,Y,W)) = T^*(X,Y,ft^j(W))$. Together with (6) it gives us (23).

For (24) we have, by definition, that

$$\widetilde{T}^{**}(X,Y,ft^{j+1}(W),S(r,W),s) \in \widetilde{B}^*(T^*(X,Y,ft^{j+1}(W)),T^*(X,Y,S(r,W)))$$

Since we know that $ft^{j+1}(T^*(X,Y,W)) = T^*(X,Y,ft^{j+1}(W))$ it remains to show that

$$[\mathbf{2016.03.27.eq4}]S(\widetilde{T}^*(X,Y,r),T^*(X,Y,W)) = T^*(X,Y,S(r,W)) \tag{25}$$

Let us show that X, Y, W, r satisfy the conditions that allow us to apply Lemma 4.2. That $X \geq Y$ is one of our assumptions. We have shown that $Y \leq ft(\partial(r))$. We have $\partial(r) = ft^j(W)$, $j \geq 1$ and $l(W) \geq j \geq 1$. Therefore, $\partial(r) < W$. These conditions imply that Lemma 4.2 is applicable and (25) holds.

This completes Construction 4.5.

4.3 The $SS^*(a)$-property

Lemma 4.6 *[***2016.04.08.l3***] Let* **B** *be a B0-system. Then for any X, Y, s, r, Z where $X, Y, Z \in B$, $X \leq Y$, $s \in \widetilde{B}^*(X,Y)$, $r \in \widetilde{B}$ and $Y < \partial(r) < Z$ one has*

1. $(s, r) \in \widetilde{S}^*_{dom}$,
2. $(s, Z) \in S^*_{dom}$,
3. $(\widetilde{S}^*(s,r), S^*(s,Z)) \in S_{dom}$,
4. $(r, Z) \in S_{dom}$,

5. $(s, S(r, Z)) \in S^*_{dom}$.

Proof: We have:

1. the required condition is that $Y \leq ft(\partial(r))$ that follows by Lemma 1.6 from our assumption $Y < \partial(r)$,
2. the required condition is $Y \leq Z$ which follows from our assumptions by Lemma 1.3,
3. the required condition is $\partial(\widetilde{S}^*(s, r)) < S^*(s, Z)$ which follows by (12) and Lemma 4.6 from our assumption $\partial(r) < Z$,
4. the required condition is our assumption $\partial(r) < Z$,
5. the required condition is $Y \leq S(r, Z)$. By the definition of a B0-system we have $ft(\partial(r)) < S(r, Z)$. On the other hand, by part (1) we have $Y \leq ft(\partial(r))$ and by Lemma 1.4 we conclude that $Y < S(r, Z)$.

Lemma 4.7 *[**2016.04.08.l2**] Let* **B** *be a B0-system that satisfies the SS(a)-condition of Definition 2.1. Then for any X, Y, s, r, Z where $X, Y, Z \in B$, $X \leq Y$, $s \in \widetilde{B}^*(X, Y)$, $r \in \widetilde{B}$ and $Y < \partial(r) < Z$ one has*

$$S(\widetilde{S}^*(s, r), S^*(s, Z)) = S^*(s, S(r, Z))$$

Proof: We proceed by induction on $j = l(Y) - l(X)$.

For $j = 0$, $\widetilde{B}^*(X, Y) = unit$ and $\widetilde{S}^*(s, r) = r$, $S^*(s, Z) = Z$, $S^*(s, S(r, Z)) = S(r, Z)$. The required equality therefore becomes

$$S(r, Z) = S(r, Z)$$

For $j = 1$, $\widetilde{B}^*(X, Y) = \widetilde{B}(Y)$ and $\widetilde{S}^*(s, r) = \widetilde{S}(s, r)$,$S^*(s, Z) = S_{ext}(s, Z)$, $S^*(s, S(r, Z)) = S_{ext}(s, S(r, Z))$. Since $\partial(s) = Y$ we have $\partial(s) < \partial(r) < Z$ which implies that $\partial(s) < Z$ and $S_{ext}(s, Z) = S(s, Z)$. Moreover, since $\partial(s) < \partial(r)$ by Lemma 1.6 we have that $\partial(s) \leq ft(\partial(r))$ which together with $ft(\partial(r)) < S(r, Z)$ by a B0-systems axiom and Lemma 1.4 implies that $\partial(s) < S(r, Z)$ and $S_{ext}(s, S(r, Z)) = S(s, S(r, Z))$. Therefore the required equality becomes

$$S(\widetilde{S}(s, r), S(s, Z)) = S(s, S(r, Z))$$

which has the form of the SS(a)-condition for s, r, Z. Since $\partial(s) = Y$ we have $\partial(s) < \partial(r) < Z$ which implies that this condition is applicable.

For the successor of $j > 0$ one has $X = ft^{j+1}(Y)$ and elements of $\widetilde{B}(X, Y)$ are pairs of the form (r_1, s_1) where $r_1 \in \widetilde{B}(ft^j(Y))$, $s_1 \in \widetilde{B}(ft^{j+1}(Y), S(r_1, Y))$.

Next we have:

$$\widetilde{S}((r_1, s_1), r) = \widetilde{S}^*(s_1, \widetilde{S}(r_1, r))$$

$$S^*((r_1, s_1), Z) = S^*(s_1, S(r_1, Z))$$

$$S^*((r_1, s_1), S(r, Z)) = S^*(s_1, S(r_1, S(r, Z)))$$

and the required equality becomes

$$S(\widetilde{S}^*(s_1, \widetilde{S}(r_1, r)), S^*(s_1, S(r_1, Z))) = S^*(s_1, S(r_1, S(r, Z)))$$

Let us show that the inductive assumption for $ft^{j+1}(Y), S(r_1, Y), s_1, \widetilde{S}(r_1, r), S(r_1, Z)$ can be applied to the left hand side. We have $ft^{j+1}(Y) = ft(ft^j(Y)) = ft(\partial(r_1)) < S(r_1, Y)$. Next, $\partial(\widetilde{S}(r_1, r)) = S(r_1, \partial(r))$ by the axiom of a B0-system and $S(r_1, Y) < S(r_1, \partial(r)) < S(r_1, Z)$ by our assumptions $Y < \partial(r) < Z$ and Lemma 3.8. Therefore the inductive assumption is applicable and we have

$$S(\widetilde{S}^*(s_1, \widetilde{S}(r_1, r)), S^*(s_1, S(r_1, Z))) = S^*(s_1, S(\widetilde{S}(r_1, r), S(r_1, Z)))$$

It remains to show that $S(\widetilde{S}(r_1, r), S(r_1, Z)) = S(r_1, S(r, Z))$. This has the form of the SS(a)-condition for r_1, r, Z. Since $\partial(r_1) = ft^j(Y)$ we have $\partial(r_1) \leq Y < \partial(r)$ and therefore $\partial(r_1) < \partial(r)$ by Lemma 1.4 and $\partial(r) < Z$ be one of our assumptions. We conclude that SS(a)-condition is applicable to r_1, r, Z.

This completes the proof of Lemma 4.7.

4.4 Sets $\widetilde{S}^{**}_{dom}$ and operations $\widetilde{S}^{**}$

Here we will construct operations $\widetilde{S}^{**}$ that correspond, in the B-systems of C-systems, to the pull-back of elements of $\widetilde{B}^*$ along elements of $\widetilde{B}^*$.

Definition 4.8 *[***2016.03.29.def1***] Let* **B** *be a B0-system. Define*

$$\widetilde{S}^{**}_{dom} = \{X, Y, Z, W, s_1, s_2 \,|\, X \leq Y \leq Z \leq W, s_1 \in \widetilde{B}^*(X, Y), s_2 \in \widetilde{B}^*(Z, W)\}$$

We will sometimes write elements (X, Y, Z, W, s_1, s_2) of $\widetilde{S}^{**}_{dom}$ as (s_1, s_2) because X, Y, Z, W can be recovered from the type of s_1 and s_2.

Problem 4.9 *[***2016.03.29.prob1***] Let $(X, Y, Z, W, s, s') \in \widetilde{S}^{**}_{dom}$. To construct an element*

$$\widetilde{S}^{**}(X, Y, Z, W, s_1, s_2) \in \widetilde{B}^*(S^*(s_1, Z), S^*(s_1, W))$$

where the right hand side is defined by Lemma 4.6.

The diagram for Problem 4.9 looks as follows:

$$\begin{array}{ccc} S^*(s_1, W) & & W \\ {\scriptstyle \widetilde{S}^{**}(s_1,s_2)\uparrow}\downarrow & & {\scriptstyle s_2\uparrow}\downarrow \\ S^*(s_1, Z) & & Z \\ \downarrow & & \downarrow \\ X & \overset{s_1}{\underset{\longleftarrow}{\rightarrow}} & Y \end{array}$$

Construction 4.10 **[2016.03.29.constr1]** We will proceed by induction on $j' = l(W) - l(Z)$.

If $j' = 0$ then $W = Z$ and $S^*(s_1, Z) = S^*(s_1, W)$. Therefore

$$\widetilde{B}^*(S^*(s_1, Z), S^*(s_1, W)) = unit$$

and we define

$$\widetilde{S}^*(X, Y, Z, W, s_1, s_2) = tt$$

where, let us recall, tt is the notation for the unique element of the one element set $unit$.

If $j' = 1$ then $l(W) \geq 1$, $Z = ft(W)$ and $\widetilde{B}^*(Z, W) = \widetilde{B}(W)$.

Then we set

$$\widetilde{S}^{**}(X, Y, Z, W, s_1, s_2) = \widetilde{S}^*(X, Y, s_1, s_2)$$

It is easy to prove that the right hand side is defined based on Definition 4.7.

For the successor of $j' > 0$ we have $Z = ft^{j+1}(W)$ and, by Construction 4.2,

$$\widetilde{B}^*(ft^{j+1}(W), W) = \{(r_2, s_2) \,|\, r_2 \in \widetilde{B}(ft^j(W)), s_2 \in \widetilde{B}^*(ft^{j+1}(W), S(r, W))\}$$

We set

$$[\mathbf{2016.04.06.eq3}]\widetilde{S}^{**}(s_1, (r_2, s_2)) = (\widetilde{S}^*(s_1, r_2), \widetilde{S}^{**}(s_1, s_2)) \tag{26}$$

The diagram for this case is as follows. Its part over Y is of the same form as in the construction of $\widetilde{T}^{**}$:

$$\begin{array}{ccc}
S(r_2, W) & & W \\
{\scriptstyle s_2\uparrow}\downarrow & & \downarrow \\
ft^{j+1}(W) & \xleftarrow{\overset{r_2}{\to}} & ft^j(W) \\
 & & \downarrow \\
 & & ft^{j+1}(W) \\
 & & \downarrow \\
 & & Y
\end{array}$$

and the part that is over X is as follows:

$$\begin{array}{ccccc}
S^*(s_1, S(r_2, W)) & & & & S^*(s_1, W) \\
\downarrow{\scriptstyle \uparrow\widetilde{S}^{**}(s_1, ft^{j+1}(W), S(r_2, W), s_2)} & & & & \downarrow \\
S^*(s_1, ft^{j+1}(W)) & = & ft(S^*(s_1, ft^j(W))) & \xleftarrow{\overset{\widetilde{S}^*(s_1, r_2)}{\to}} & S^*(s_1, ft^j(W)) \\
 & & & & \downarrow \\
 & & & & ft(S^*(s_1, ft^j(W))) \\
 & & & & \downarrow \\
 & & & & X
\end{array}$$

Let us check that the right hand side of (26) is defined and belongs to $\widetilde{B}(S^*(s_1, ft^{j+1}(W)), S^*(s_1, W))$.

For $\widetilde{S}^*(s_1, r_2)$ to be defined we need to know that $Y \leq ft(\partial(r_2))$. This follows from:

$$ft(\partial(r_2)) = ft(ft^j(W)) = ft^{j+1}(W) = Z \geq Y$$

For $\widetilde{S}^{**}(s_1, s_2)$ to be defined we need to know that $Y \leq ft^{j+1}(W)$ which again follows from the equality $ft^{j+1}(W) = Z$ and the assumption $Y \leq Z$.

To check that the pair on the right hand side of (26) belongs to $\widetilde{B}^*(S^*(s_1, ft^{j+1}(W)), S^*(s_1, W))$ we need to know what are the elements of the latter set. By Lemma 4.6 we have

$$l(S^*(s_1, W)) - l(S^*(s_1, ft^{j+1}(W)) = l(W) - l(ft^{j+1}(W)) = l(W) - l(Z) = j + 1$$

Therefore,

$$\widetilde{B}^*(S^*(s_1, ft^{j+1}(W)), S^*(s_1, W)) =$$
$$\{(r', s') \,|\, r' \in \widetilde{B}(ft^j(S^*(s_1, W))), s' \in \widetilde{B}^*(ft^{j+1}(S^*(s_1, W)), S(r', S^*(s_1, W)))\}$$

Therefore, we have to prove that

$$[\mathbf{2016.04.08.eq1}]\widetilde{S}^*(s_1, r_2) \in \widetilde{B}(ft^j(S^*(s_1, W))) \tag{27}$$

and

$$[\mathbf{2016.04.08.eq2}]\widetilde{S}^{**}(s_1, s_2) \in \widetilde{B}^*(ft^{j+1}(S^*(s_1, W)), S(\widetilde{S}^*(s_1, r_2), S^*(s_1, W))) \tag{28}$$

By (12) we have

$$\partial(\widetilde{S}^*(s_1, r_2)) \in S^*(s_1, \partial(r_2)) = S^*(s_1, ft^j(W))$$

and by construction

$$\widetilde{S}^{**}(s_1, s_2) \in \widetilde{B}^*(S^*(s_1, ft^{j+1}(W)), S^*(s_1, S(r_2, W)))$$

Using Lemma 4.6 it is easy to prove that

$$ft^j(S^*(s_1, W)) = S^*(s_1, ft^j(W))$$

which implies (27). From the same lemma it follows that

$$ft^{j+1}(S^*(s_1, W)) = S^*(s_1, ft^{j+1}(W))$$

which reduces (28) to the proof of the equality

$$[\mathbf{2016.04.10.eq2}]S(\widetilde{S}^*(s_1, r_2), S^*(s_1, W)) = S^*(s_1, S(r_2, W)) \tag{29}$$

Let us show that it follows by application of Lemma 4.7 to X, Y, s_1, r_2, W.

For this we need to verify the conditions $X \leq Y$, $Y < \partial(r_2)$ and $\partial(r_2) < W$. The first condition is a part of our assumptions. Since $\partial(r_2) = ft^j(W)$ we have $Z = ft(ft^j(W)) = ft(\partial(r_2)) < \partial(r_2)$ and together with $Y \leq Z$, Lemma 1.4 gives us that $Y < \partial(r_2)$. Finally, since $j > 0$ and $l(\partial(r_2)) = l(ft^j(W)) > 0$ we have $\partial(r_2) < W$. These conditions imply that Lemma 4.7 is applicable to X, Y, s_1, r_2, W and gives (29).

This completes Construction 4.10.

5 Sets $BMor(X, Y)$, composition operations and the identity elements

5.1 Construction of the set $BMor(X, Y)$

Problem 5.1 *[***2016.02.28.prob1***] For a pointed B0-system $(B, \widetilde{B}, \partial)$ and $X, Y \in B$ to define a set that will be denoted $BMor(X, Y)$.*

Construction 5.2 [**2016.02.28.constr1**][**2016.02.20.def1**] We define this set by the formula:

$$BMor(X,Y) = \widetilde{B}^*(X, T^*(X, pt, Y))$$

Let us show that the right hand side is well defined. For that we need $T^*(X, pt, Y)$ to be well defined, i.e., to have $X \geq pt$ and $Y \geq pt$. This is immediate from the definition of a pointed B0-system. We also need that $T^*(X, pt, Y) \geq X$ which is condition (1) of Problem 3.11.

Remark 5.3 [**2016.02.28.rem1**] To define $BMor(X,Y)$ we need less than the full set of B0-system structures and axioms. All we need is a B-system carrier with operations T and S such that T satisfies conditions of Definition 3.1(1) and S the conditions of Definition 3.1(3).

In the next section we will construct for any C-system CC a B0-system $\mathbf{B}(CC)$ and for any $X, Y \in CC$ a bijection between $BMor_{\mathbf{B}(CC)}(X,Y)$ and $Mor_{CC}(X,Y)$.

5.2 Sets $BMor(X,Y)$ and homomorphisms of pointed B0-systems

Problem 5.4 *[**2016.03.15.prob2**] Let $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ be a homomorphism of pointed B0-systems. For $X, Y \in B$ to define a function*

$$\mathbf{f}_{BMor,X,Y} : BMor(X,Y) \to BMor'(f(X), f(Y))$$

Construction 5.5 [**2016.03.15.constr3**] By Construction 5.2 we have

$$BMor(X,Y) = \widetilde{B}^*(X, T^*(X, pt, Y))$$

Applying Construction 4.11 we obtain the function

$$\mathbf{f}_{\widetilde{B}^*(X,T^*(X,pt,Y))} : \widetilde{B}^*(X, T^*(X, pt, Y)) \to \widetilde{B}'(f(X), f(T^*(X, pt, Y)))$$

Applying Lemma 3.16(1) we get that $f(T^*(X, pt, Y)) = (T')^*(f(X), f(pt), f(Y))$. Since $\mathbf{B}'$ is pointed we have $f(pt) = pt'$ and

$$\widetilde{B}'(f(X), (T')^*(f(X), f(pt'), f(Y))) = BMor'(f(X), f(Y))$$

This completes the construction.

5.3 Composition operation

To construct the operations on the $BMor$ sets that will be related to the composition of morphisms and the identity morphisms in the case of the B0-systems of the form $CB(CC)$ we need the underlying B0-system to satisfy some of the axioms of a B-system.

Everywhere below when we say that an equality involving partially defined operations holds for certain values of the arguments we mean that the left and the right hand side expressions are defined and equal.

3 B-systems and C-systems

1 Some general results on C-systems

Let us recall (cf. [**?**]) that for a C-system CC an object Y is said to be an object over X if $Y \geq X$. In this case the composition of the canonical projections $Y \stackrel{p_Y}{\to} ft(Y) \stackrel{p_{ft(Y)}}{\to} \ldots \to X$ is denoted by $p_{Y,X}$. For a morphism $f : X' \to X$ one defines $f^*(Y)$ by induction using the f^* structure of the C-system. One also defines by induction a morphism $q(f, Y) : f^*(Y) \to Y$. For more detail see [**?**, Section 2].

Lemma 1.1 *[**2016.02.18.l4**] In the context introduced above one has $f^*(Y) \geq X'$,*

$$[\mathbf{2016.02.18.eq3}]\, l(f^*(Y)) - l(X') = l(Y) - l(X) \tag{30}$$

and the square

$$[\mathbf{2016.02.18.eq2}]\quad \begin{array}{ccc} f^*(Y) & \xrightarrow{q(f,Y)} & Y \\ {\scriptstyle p_{f^*(Y),X'}}\downarrow & & \downarrow{\scriptstyle p_{Y,X}} \\ X' & \xrightarrow{f} & X \end{array} \tag{31}$$

is a pull-back square.

Proof: By induction on $l(Y) - l(X)$ using the fact that vertical composition of pull-back squares is a pull-back square.

For $Y, Y' \geq X$ a morphism $g : Y \to Y'$ is said to be a morphism over X if $p_{Y,X} = g \circ p_{Y',X}$. For such a morphism g and a morphism $f : X' \to X$ there is a unique morphism $f^*(g) : f^*(Y) \to f^*(Y')$ over X' such that the square

$$[\mathbf{2016.02.18.eq1}]\quad \begin{array}{ccc} f^*(Y) & \xrightarrow{q(f,Y)} & Y \\ {\scriptstyle f^*(g)}\downarrow & & \downarrow{\scriptstyle g} \\ f^*(Y') & \xrightarrow{q(f,Y')} & Y' \end{array} \tag{32}$$

commutes (see [**?**, Lemma 2.1]).

We will also need the following lemmas.

Lemma 1.2 *[**2016.03.15.l11**] Let Y be an object over X and $f' : X'' \to X'$, $f : X' \to X$ two morphisms. Then one has:*

$$[\mathbf{2016.03.15.eq4}]\, (f')^*(f^*(Y)) = (f' \circ f)^*(Y) \tag{33}$$

and

$$[\mathbf{2016.03.15.eq5}]\, q(f' \circ f, Y) = q(f', f^*(Y)) \circ q(f, Y) \tag{34}$$

Proof: By induction on $j = l(Y) - l(X)$ using the definition given in [**?**, Section 2] and one of the axioms of the definition of a C-system, see [**?**, Definition 2.1(axiom 7)].

Lemma 1.3 *[**2016.03.15.l10**] Let $g : Y \to Y'$ be a morphism over X and $f' : X'' \to X'$, $f : X' \to X$ two morphisms. Then one has:*

$$[\mathbf{2016.03.15.eq4mor}](f')^*(f^*(g)) = (f' \circ f)^*(g) \quad (35)$$

Proof: Note first that both the right and the left hand sides of (1.3) are morphisms over X. The domains of the left and the right hand sides of (1.3) are $(f')^*(f^*(Y))$ and $(f' \circ f)^*(Y)$ coincide by Lemma 1.2. The codomains $(f')^*(f^*(Y'))$ and $(f' \circ f)^*(Y')$ coincide by the same lemma. In view of the uniqueness part of [**?**, Lemma 2.1] it remains to show that

$$(f')^*(f^*(g)) \circ q(f' \circ f, Y') = q(f' \circ f, Y) \circ g$$

This follows from the equalities

$$q(f' \circ f, Y') = q(f', f^*(Y')) \circ q(f, Y')$$

and

$$q(f' \circ f, Y) = q(f', f^*(Y)) \circ q(f, Y)$$

proved in Lemma 1.2 and the definition of $(f')^*(f^*(g))$.

Lemma 1.4 *[**2016.01.27.l8**] Let CC be a C-system, X an object over X' and X' an object over X'' then one has $p_{X,X''} = p_{X,X'} \circ p_{X',X''}$.*

Proof: By induction on $l(X') - l(X)$.

Lemma 1.5 *[**2016.01.27.l7**] Let CC be a C-system, $f : Y \to Y'$ be a morphism over X and X be an object over W. Then f is a morphism over W.*

Proof: Follows easily from Lemma 1.4.

Lemma 1.6 *[**2016.02.20.l9**] Let $Y > X$ and $f : X' \to X$. Then one has*

$$ft(f^*(Y)) = f^*(ft(Y))$$

Proof: It follows from the inductive definition of f^* since for $l(Y) - l(X) > 0$ we have $f^*(Y) = q(f, ft(Y))^*(Y)$ where $q(f, ft(Y)) : f^*(ft(Y)) \to ft(Y)$ and $q(f, ft(Y))^*(Y)$ is given by the C-system structure that satisfies the axiom $ft(a^*(Y)) = dom(a)$.

2 The B0-systems of C-systems

2.1 Construction of the pre-B-system $CB(CC)$

Problem 2.1 *[**2016.01,27.prob2**] Let CC be a C-system. To construct a pre-B-system*

$$CB(CC) = (B(CC), \widetilde{B}(CC), l, ft, \partial, T, \widetilde{T}, S, \widetilde{S})$$

Construction 2.2 *[2016.01.27.constr1]* Let $B(CC) = Ob(CC)$ and

$$\widetilde{B}(CC) = \{s \in Mor(CC)\,|\,dom(s) = ft(codom(s)) \text{ and } s \circ p_{codom(s)} = Id_{dom(s)}\}$$

(this set was previously denoted by $\widetilde{Ob}(CC)$). The triple $(B(CC), l, ft)$ is an lft-set and we have the function of sets

$$\partial = codom : \widetilde{B}(CC) \to B(CC)$$

obtaining a B-system carrier.

Starting with these data we can define the sets $T_{dom}, \widetilde{T}_{dom}, S_{dom}, \widetilde{S}_{dom}$ and δ_{dom}.

Next, we define operations $T, \widetilde{T}, S, \widetilde{S}, \delta$ as follows:

$$T(X,Y) = p_X^*(Y) \quad \widetilde{T}(X,s) = p_X^*(s)$$

$$S(r,Y) = r^*(Y) \quad \widetilde{S}(r,s) = r^*(s)$$

$$\delta(X) = s_{Id_X}$$

The first of these operations is defined because $Y > ft(X)$ and therefore Y is over $ft(X)$.

The second one is defined because $s : ft(\partial(s)) \to \partial(s)$ is a morphism over $ft(\partial(s))$ and since $\partial(s) > ft(X)$ we have that $ft(\partial(s)) \geq ft(X)$ by Lemma 1.6 and therefore $ft(\partial(s))$ is an object over $ft(X)$ and so the morphism s is a morphism over $ft(X)$ by Lemma 1.5.

The third of one is defined because Y is over $\partial(r)$.

The fourth one is defined because $s : ft(\partial(s)) \to \partial(s)$ is a morphism over $ft(\partial(s))$ while r is of the form $ft(\partial(r)) \to \partial(r)$ and since $\partial(s) > \partial(r)$ we have that $ft(\partial(s)) \geq \partial(r)$ by Lemma 1.6 and therefore s is a morphism over $\partial(r)$ by Lemma 1.5. . Finally δ is defined because s_f is defined for any morphism of the form $f : X \to Y$ where $l(Y) > 0$ (cf. [**?**, Definition 2.3]).

This completes Construction 2.2.

2.2 Pre-B-systems $CB(CC)$ are B0-systems

Lemma 2.3 *[2016.02.18.l6] Let CC be a C-system. Then $CB(CC)$ is a B0-system.*

Proof:

1. Let $(X,Y) \in T_{dom}$. Then one has:
   (a) We have $l(T(X,Y)) = l(p_X^*(Y))$. To define $p_X^*(Y)$ we consider Y as an object over $ft(X)$. Therefore by Lemma 1.1 we have

   $$l(p_X^*(Y)) = l(X) + (l(Y) - l(ft(X))) = (l(X) - l(ft(X))) + l(Y)$$

   Since $l(X) \geq 1$ we have $l(X) - l(ft(X)) = 1$. Therefore

   $$l(T(X,Y)) = l(Y) + 1$$

   (b) By Lemma 1.1 we have $T(X,Y) \geq X$ and

   $$l(T(X,Y)) - l(X) = l(Y) - l(ft(X)) > 0$$

   therefore $T(X,Y) > X$.

(c) It follows from Lemma 1.6.

2. Let $(X,s) \in \widetilde{T}_{dom}$. The first condition follows from the second one and condition 1(a). The second condition follows immediately from the definitions.

3. Let $(r,Y) \in S_{dom}$. Then one has:

   (a) $l(S(r,Y)) = l(r^*(Y))$ were Y is considered as an object over $X = \partial(r)$. By Lemma 1.1 we have

   $$l(r^*(Y)) = l(ft(X)) + (l(Y) - l(X)) = l(Y) + (l(ft(X)) - l(X)) = l(Y) - 1$$

   where the last equality follows from the fact that $l(ft(X)) - l(X) = -1$ since $l(X) = l(\partial(r)) > 0$.

   (b) By Lemma 1.1 we have $r^*(Y) \geq ft(X)$ and since $Y > X$ the same lemma implies that $r^*(Y) > X$.

   (c) It follows from Lemma 1.6.

4. The first condition follows from the second one and condition 2(a). The second condition follows immediately from the definitions.

2.3 Functoriality of the CB-construction

Problem 2.4 *[***2016.01.27.prob3***] Let $f : CC \to CC'$ be a homomorphism of C-systems. To construct a homomorphism of pre-B-systems $CB(f) : CB(CC) \to CB(CC')$.*

Construction 2.5 [**2016.01.27.constr3**] We need to construct a morphism of B-system carriers and show that it is a homomorphism of pre-B-systems.

We already have the function $f : B(CC) \to B(CC')$ and by the definition of a homomorphism of C-systems (cf. [**?**, Definition 3.1]) it is a morphism of lft-sets.

By definition $\widetilde{B}(CC)$ is a subset of $Mor(CC)$. Therefore, by the morphism part of the functor f it is mapped to a subset of $Mor(CC')$. We need to verify that the image of $\widetilde{B}(CC)$ lies in $\widetilde{B}(CC')$. The subset $\widetilde{B}(CC)$ is the subset of elements s such that $dom(s) = ft(codom(s))$ and $s \circ p_{codom(s)} = Id_{dom(s)}$. It follows that it will be mapped to the subset defined by the same conditions by any functor that maps the p-morphisms of CC to p-morphisms of CC' and in particular by any homomorphism of C-systems. We obtain a function $\widetilde{B} \to \widetilde{B'}$ that we denote by $\widetilde{f}$.

It is immediate from the construction that the pair $\mathbf{f} = (f, \widetilde{f})$ is a morphism of B-system carriers.

By Construction 2.8 we obtain functions $f_T, f_{\widetilde{T}}, f_S, f_{\widetilde{S}}, f_\delta$.

The fact that these functions commute, in the sense of Definition 2.9, with the pre-B-system operations follows from [**?**, Lemma 2.3(3,4,5)].

Lemma 2.6 *[***2016.01.29.l1***] Let U be a universe. Then Constructions 2.2 and 2.5 define a functor CB_U from the category of C-systems in U to the category of pre-B-systems in U.*

Proof: Since two homomorphisms of pre-B-systems are equal if and only if the underlying morphisms of the B-system carriers are equal it is sufficient to prove the identity and composition axioms of a functor for the mappings from C-systems and their homomorphisms to the carriers of pre-B-systems and their morphisms. These axioms follow immediately from the fact that $\widetilde{B}(CC)$ is a subset of $Mor(CC)$ and the definition of composition of C-system homomorphisms.

We will sometimes call the functor CB defined by Constructions 2.2, 2.5 and Lemma 2.6, the B-sets functor.

2.4 Operations T_{ext} for B0-systems of the form $CB(CC)$

Lemma 2.7 *[**2016.03.15.l8**] Let CC be a C-system and $X, Y \in B(CC)$ be such that $l(X) \geq 1$, $l(Y) \geq 1$ and $ft(X) \leq Y$. Then one has*

$$T_{ext}(X, Y) = p^*_{X,ft(X)}(Y)$$

Proof: Straightforward by case analysis from the definition of the operation T in the B0-systems of the form $B(CC)$.

2.5 Operations T^* in the B0-systems of the form $CB(CC)$

Lemma 2.8 *[**2016.02.20.l10**] Let CC be a C-system. Let $(X, Y, Z) \in T^*_{dom}$, where the pre-B-system concepts refer to $CB(CC)$, then one has*

$$T^*(X, Y, Z) = p^*_{X,Y}(Z)$$

and in particular we have a pull-back square of the form

$$\begin{array}{ccc} T^*(X,Y,Z) & \xrightarrow{q(p_{X,Y},Z)} & Z \\ {\scriptstyle p_{T^*(X,Y,Z),X}}\downarrow & & \downarrow{\scriptstyle p_{Z,Y}} \\ X & \xrightarrow{p_{X,Y}} & Y \end{array}$$

Proof: By induction on $j = l(X) - l(Y)$, from Lemma 2.7, Lemma 1.2, the definition of $T^*(X, Y, Z)$ and the definition of $p_{X,Y}$.

The objects involved in the construction for the successor can be seen on the diagram:

[2016.02.20.eq1]

$$\begin{array}{ccccc} T_{ext}(X, T^*(ft(X), Y, Z)) & \longrightarrow & T^*(ft(X), Y, Z) & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p_{Z,Y}} \\ X & \xrightarrow{p_{X,ft(X)}} & ft(X) & \xrightarrow{p_{ft(X),Y}} & Y \end{array} \quad (36)$$

and one has to use the fact that

$$p^*_{X,ft(X)}(p^*_{ft(X),Y}(Z)) = (p_{X,ft(X)} \circ p_{ft(X),Y})^*(Z) = p^*_{X,Y}(Z)$$

which follows from Lemma 1.2.

2.6 Operations $\widetilde{T}^*$ in the B0-systems of the form $CB(CC)$

Lemma 2.9 *[**2016.03.15.l9**] Let CC be a C-system. Let $(X, Y, s) \in \widetilde{T}^*_{dom}$, where the B0-system concepts refer to $CB(CC)$, then one has*

$$\widetilde{T}^*(X, Y, s) = p^*_{X,Y}(s)$$

and in particular we have a pull-back square of the form

$$\begin{array}{ccc}
T^*_{ext}(X, Y, ft(\partial(s))) & \xrightarrow{q(p_{X,Y}, ft(\partial(s)))} & ft(\partial(s)) \\
\downarrow{\scriptstyle p^*_{X,Y}(s)} & & \downarrow{\scriptstyle s} \\
T^*(X, Y, \partial(s)) & \xrightarrow{q(p_{X,Y}, \partial(s))} & \partial(s) \\
{\scriptstyle p_{T^*(X,Y,\partial(s)),X}}\downarrow & & \downarrow{\scriptstyle p_{\partial(s),Y}} \\
X & \xrightarrow{p_{X,Y}} & Y
\end{array}$$

Proof: By induction on $j = l(X) - l(Y)$, from Lemma 2.7, Lemma 1.3, the definition of $\widetilde{T}$, the definition of $\widetilde{T}^*(X, Y, Z)$ and the definition of $p_{X,Y}$.

The objects involved in the construction for the successor can be seen on the diagram:

$$\begin{array}{ccccc}
T^*_{ext}(X, Y, ft(\partial(s))) & \longrightarrow & T^*_{ext}(ft(X), Y, ft(\partial(s))) & \longrightarrow & ft(\partial(s)) \\
\downarrow & & \downarrow{\scriptstyle p^*_{ft(X),Y}(s)} & & \downarrow{\scriptstyle s} \\
T_{ext}(X, T^*(ft(X), Y, \partial(s))) & \longrightarrow & T^*(ft(X), Y, \partial(s)) & \longrightarrow & \partial(s) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{\partial(s),Y}} \\
X & \xrightarrow{p_{X,ft(X)}} & ft(X) & \xrightarrow{p_{ft(X),Y}} & Y
\end{array} \tag{37}$$

2.7 Sets $\widetilde{B}^*(X, Y)$ for B0-systems of the form $CB(CC)$

Let CC be a C-system. Let $X < Y$ in $B(CC)$. Set

$$Sec(X, Y) = \{f \in Mor(CC) \,|\, dom(f) = X,\ codom(f) = Y,\ f \circ p_{Y,X} = Id_X\}$$

that is, elements of $Sec(X, Y)$ are sections of the canonical morphism $p_{Y,X} : Y \to X$.

Problem 2.10 *[**2016.02.18.prob1**] Let CC be a C-system, $X, Y \in B(CC)$ and $X \leq Y$. To construct a bijection*

$$[\mathbf{2016.03.11.eq2}]nt(X, Y) : \widetilde{B}^*(X, Y) \to Sec(X, Y) \tag{38}$$

To provide a construction for this problem we need the following general lemma.

Lemma 2.11 *[**2016.02.18.l3**] Let $\mathcal{C}$ be a category and $X \stackrel{a}{\to} Y \stackrel{b}{\to} Z$ be a composable pair of morphisms in $\mathcal{C}$. Assume further that for any $r : Z \to Y$ such that $r \circ b = Id_Z$ we are given a pull-back square of the form:*

$$\begin{array}{ccc} r^*(X) & \xrightarrow{pr^r_X} & X \\ {\scriptstyle pr^r_Z}\downarrow & & \downarrow{\scriptstyle a} \\ Z & \xrightarrow{r} & Y \end{array}$$

Let $R(a,b)$ be the set of pairs (r,s) where $r : Z \to Y$ is such that $r \circ b = Id_Z$ and $s : Z \to r^*(X)$ is such that $s \circ pr^r_Z = Id_Z$.

Then the function $(r,s) \mapsto s \circ pr^r_X$ is a bijection from R to the set of morphisms $t : Z \to X$ such that $t \circ a \circ b = Id_Z$.

Proof: Note first that

$$s \circ pr^r_X \circ a \circ b = s \circ pr^r_Z \circ r \circ b = Id_Z \circ Id_Z = Id_Z$$

Let us show that our function is surjective. Let $f : Z \to X$ be a morphism such that $f \circ a \circ b = Id_Z$. Let $r = f \circ a$. Then $r \circ b = Id_Z$ and on the other hand $f = s \circ pr^r_X$ for some s such that $s \circ pr^r_Z = Id_Z$ by the universal property of the pull-backs.

Let us show that our function is injective. Let (r,s), (r',s') be two elements of $R(a,b)$ such that $s \circ pr^r_X = s' \circ pr^{r'}_X$. We have

$$s \circ pr^r_X \circ a = s \circ pr^r_Z \circ r = r$$

We conclude that $r = r'$. Then if $s \circ pr^r_X = s' \circ pr^r_X$ and $s \circ pr^r_Z = Id_Z = s' \circ pr^r_Z$ we have that $s = s'$ by the universal property of the pull-backs.

The lemma is proved.

We can now provide a construction for Problem 2.10.

Construction 2.12 [**2016.02.18.constr1**]We proceed by induction on $j = l(Y) - l(X)$.

If $j = 0$ then $p_{Y,X} = Id_X$ and both sides are one element sets.

If $j = 1$ then we can define $nt(X,Y)$ as the identity function because by definition of $\widetilde{B}(X)$ and $\widetilde{B}(CC)$ the left and the right hand sides of (38) are equal.

For the successor of $j > 0$ we have that $l(Y) \geq j+1$ and $X = ft^{j+1}(Y)$. By the inductive assumption $nt(X',Y')$ is already constructed for all pairs X',Y' such that $X' \leq Y'$ and $l(Y') - l(X') \leq j$.

By Construction 4.2 the set $\widetilde{B}^*(ft^{j+1}(Y),Y)$ is the set of pairs (r,s) where $r \in \widetilde{B}(ft^j(Y))$ and $s \in \widetilde{B}^*(ft^{j+1}(Y), S(r,Y))$.

Since r is a morphism of the form $r : ft^{j+1}(Y) \to ft^j(Y)$, by Lemma 1.1 we have that $S(r,Y)$ is over $ft^{j+1}(Y)$ and that the square

[**2016.02.18.eq4**]
$$\begin{array}{ccc} S(r,Y) & \xrightarrow{q(r,Y)} & Y \\ {\scriptstyle p_{S(r,Y),ft^{j+1}(Y)}}\downarrow & & \downarrow{\scriptstyle p_{Y,ft^j(Y)}} \\ ft^{j+1}(Y) & \xrightarrow{r} & ft^j(Y) \end{array} \quad (39)$$

is a pull-back square. Since $l(Y) \geq j+1$ we have that

$$l(Y) - l(ft^j(Y)) = j$$

and therefore by the equation (30) of the same lemma we have that

$$ft^j(S(r,Y)) = ft^{j+1}(Y)$$

Consider the set $R(p_{Y,ft^j(Y)}, p_{ft^j(Y),ft^{j+1}(Y)})$ where R is as in Lemma 2.11 relative to the pull-back squares (39). Then the function $(r,s) \mapsto (r, nt(ft^{j+1}(Y), S(r,Y))(s))$ is a bijection of the form

$$\widetilde{B}^*(ft^{j+1}(Y), Y) \to R(p_{Y,ft^j(Y)}, p_{ft^j(Y),ft^{j+1}(Y)}).$$

Composing this bijection with the bijection of Lemma 2.11 and using the fact that

$$p_{Y,ft^j(Y)} \circ p_{ft^j(Y),ft^{j+1}(Y)} = p_{Y,ft^{j+1}(Y)}$$

we obtain the bijection that is the goal of the construction.

This completes Construction 2.12.

2.8 Sets $Sec(X,Y)$ and homomorphisms of C-systems

Problem 2.13 *[**2016.03.17.prob1**] Let $F : CC \to CC'$ be a homomorphism of C-systems. To construct a function*

$$[\mathbf{2016.03.17.eq1}] F_{Sec(X,Y)} : Sec(X,Y) \to Sec(F(X), F(Y)) \tag{40}$$

Construction 2.14 [**2016.03.17.constr1**] Note first that the right hand side of (40) is well defined by Lemma 1.11. By definition, $Sec(X,Y)$ is a subset of $Mor(X,Y)$ therefore it is sufficient to show that for $s \in Sec(X,Y)$ we have $F_{Mor}(s) \in Sec(F(X), F(Y))$. This follows immediately from the fact that F commutes with compositions and identities and takes p-morphisms of CC to p-morphisms of CC'.

2.9 Functions $\widetilde{B}^*(X,Y) \to Sec(X,Y)$ and homomorphisms of C-systems

Lemma 2.15 *[**2016.03.17.l1**] Let $F : CC \to CC'$ be a homomorphism of C-systems. Let $X \leq Y$ in $B(CC)$. Let*

$$CB(F)_{\widetilde{B}^*(X,Y)} : \widetilde{B}^*(X,Y) \to \widetilde{B}^*(F(X), F(Y))$$

be the function of Construction 4.11 and

$$F_{Sec(X,Y)} : Sec(X,Y) \to Sec(F(X), F(Y))$$

the function of Construction 2.14.

Let

$$nt(X,Y) : \widetilde{B}^*(X,Y) \to Sec(X,Y)$$
$$nt(F(X),F(Y)) : \widetilde{B}^*(F(X),F(Y)) \to Sec(F(X),F(Y))$$

be the functions of Construction 2.12. Then the square

$$\begin{array}{ccc} \widetilde{B}^*(X,Y) & \xrightarrow{nt(X,Y)} & Sec(X,Y) \\ {\scriptstyle CB(F)_{\widetilde{B}^*(X,Y)}}\downarrow & & \downarrow{\scriptstyle F_{Sec(X,Y)}} \\ \widetilde{B}^*(F(X),F(Y)) & \xrightarrow{nt(F(X),F(Y))} & Sec(F(X),F(Y)) \end{array}$$

commutes.

Proof: Since the construction of $\widetilde{B}^*(X,Y)$ is by induction on $j = l(Y) - l(X)$ so is the proof of the lemma. Note that $l(F(Y)) - l(F(X)) = l(Y) - l(X)$ because homomorphisms of C-systems preserve the lengths of objects.

For $j = 0$ the target set is a one element set and any two functions to a one element set with the same domain coincide.

For $j = 1$ we have

$$\widetilde{B}^*(X,Y) = \widetilde{B}(Y) = \{s \in Mor(CC)\,|\,dom(s) = X, codom(s) = Y, s \circ p_{Y,X} = Id\}$$

and $nt(X,Y)$ is in this case the identity function. The same is true for $nt(F(X),F(Y))$ and it remains to check that the definitions of $CB(F)_{\widetilde{B}^*(X,Y)}$ and $F_{Sec(X,Y)}$ in this case agree, which they do.

For the successor of $j > 0$ we have

$$\widetilde{B}^*(X,Y) = \{(r,s)\,|\,r \in \widetilde{B}(ft^j(Y))\ s \in \widetilde{B}^*(ft^{j+1}(Y), S(r,Y))\}$$

and

$$CB(F)_{\widetilde{B}^*(X,Y)}(r,s) = (F_{Mor}(r), CB(F)_{\widetilde{B}^*(ft^{j+1}(Y),S(r,Y))}(s))$$

The function $nt(X,Y)$ is of the form

$$nt(X,Y)(r,s) = (r, nt(ft^{j+1}(Y), S(r,Y))(s)) = nt(ft^{j+1}(Y), S(r,Y))(s) \circ q(r,Y)$$

where $nt(ft^{j+1}(Y), S(r,Y)) \in Sec(ft^{j+1}(Y), S(r,Y))$ and $q(r,Y)$ is a part of the pull-back square

$$\begin{array}{ccc} S(r,Y) & \xrightarrow{q(r,Y)} & Y \\ {\scriptstyle p_{S(r,Y),ft^{j+1}(Y)}}\downarrow & & {\scriptstyle p_{Y,ft^j(Y)}}\downarrow \\ ft^{j+1}(Y) & \xrightarrow{r} & ft^j(Y) \end{array}$$

When we apply $F_{Sec(X,Y)}$, which is just the restriction of F_{Mor} to the subset $Sec(X,Y)$ of $Mor(CC)$, to $nt(X,Y)(r,s)$ we get

$$F_{Mor}(nt(X,Y)(r,s)) = F_{Mor}(nt(ft^{j+1}(Y), S(r,Y))(s) \circ q(r,Y)) =$$

$$F_{Mor}(nt(ft^{j+1}(Y), S(r,Y))(s)) \circ F_{Mor}(q(r,Y)) =$$

$$nt(F(ft^{j+1}(Y)), F(S(r,Y)))(CB(F)_{\widetilde{B}^*(nt(ft^{j+1}(Y),S(r,Y))}(s)) \circ F_{Mor}(q(r,Y)) =$$

$$nt(ft^{j+1}(F(Y)), S(F_{Mor}(r), F(Y)))(CB(F)_{\widetilde{B}^*(nt(ft^{j+1}(Y),S(r,Y))}(s))) \circ q(F_{Mor}(r), F(Y)) =$$

$$nt(F(X),F(Y))(F_{Mor}(r),CB(F)_{\widetilde{B}^*(nt(ft^{j+1}(Y),S(r,Y))}(s))) =$$
$$nt(F(X),F(Y))(CB(F)_{\widetilde{B}^*(F(X),F(Y))}(r,s)).$$

where the third equality is by the inductive assumption, the fourth by [**?**, Lemma 2.3(3)] that implies that for a homomorphism of C-systems F one has $F(r^*(Y)) = F_{Mor}(r)^*(F(Y))$ and $F_{Mor}(q(r,Y)) = q(F_{Mor}(r),F(Y))$.

This completes proof of Lemma 2.15.

2.10 Bijections $bmor(X,Y) : BMor_{CB(CC)}(X,Y) \to Mor_{CC}(X,Y)$

Problem 2.16 *[**2016.02.20.prob3**] For a C-system CC and $X,Y \in CC$ to construct a bijection*

$$bmor(X,Y) : BMor(X,Y) \to Mor_{CC}(X,Y)$$

Construction 2.17 [**2016.02.20.constr3**] By Construction 2.12 to Problem 2.10 we have a bijection

$$[\mathbf{2016.02.20.eq5b}] nt(X,T^*(X,pt,Y)) : BMor(X,Y) \to Sec(X,T^*(X,pt,Y)) \quad (41)$$

By Lemma 2.8 we have a pull-back square

$$[\mathbf{2016.03.17.eq2}] \quad \begin{array}{ccc} T^*(X,pt,Y) & \xrightarrow{q(p_{X,pt},Y)} & Y \\ {\scriptstyle p_{T^*(X,pt,Y),X}}\downarrow & & \downarrow{\scriptstyle p_{Y,pt}} \\ X & \xrightarrow{p_{X,pt}} & pt \end{array} \quad (42)$$

Therefore, the function $s \mapsto s \circ q(p_{X,pt},Y)$ is a bijection between $Sec(X,T^*(X,pt,Y))$ and $Mor_{CC}(X,Y)$. Composing this bijection with $nt(X,T^*(X,pt,Y))$ we obtain a bijection

$$BMor(X,Y) \to Mor_{CC}(X,Y)$$

This completes Construction 2.17.

2.11 Bijections $bmor(X,Y)$ and homomorphisms of C-systems

Theorem 2.18 *[**2016.03.15.th1**] Let $F = (F_{Ob},F_{Mor}) : CC \to CC'$ be a homomorphism of C-systems. Then for any $X,Y \in B(CC)$ the square*

$$[\mathbf{2016.03.17.eq3}] \quad \begin{array}{ccc} BMor(X,Y) & \xrightarrow{CB(F)_{BMor,X,Y}} & BMor(F(X),F(Y)) \\ {\scriptstyle bmor(X,Y)}\downarrow & & \downarrow{\scriptstyle bmor(F_{Ob}(X),F_{Ob}(Y))} \\ Mor_{CC}(X,Y) & \xrightarrow{F_{Mor,X,Y}} & Mor_{CC'}(F_{Ob}(X),F_{Ob}(Y)) \end{array} \quad (43)$$

commutes.

Proof: By definition

$$BMor(X,Y) = \widetilde{B}^*(X, T^*(X, pt, Y)).$$

The function $bmor(X,Y)$ is the composition of two functions:

$$bmor(X,Y) = nt(X, T^*(X, pt, Y)) \circ \Phi(X,Y)$$

where for $s \in Sec(X, T^*(X, pt, Y))$, $\Phi(X,Y)(s) = s \circ q(p_{X,pt}, Y)$, see the square (42).

Therefore the square (43) is the vertical composition of two squares:

$$\begin{array}{ccc} BMor(X,Y) & \xrightarrow{CB(F)_{BMor,X,Y}} & BMor(F(X),F(Y)) \\ {\scriptstyle nt(X,T^*(X,pt,Y))}\downarrow & & \downarrow{\scriptstyle nt(F(X),T^*(F(X),pt,F(Y)))} \\ Sec(X,T^*(X,pt,Y)) & \xrightarrow{F_{Sec(X,T^*(X,pt,Y))}} & Sec(F(X),T^*(F(X),pt,F(Y))) \end{array}$$

where we used Lemma 3.16 to identify $F(T^*(X, pt, Y))$ with $T^*(F(X), pt, F(Y))$, and the square

$$\begin{array}{ccc} Sec(X,T^*(X,pt,Y)) & \xrightarrow{F_{Sec(X,T^*(X,pt,Y))}} & Sec(F(X),T^*(F(X),pt,F(Y))) \\ {\scriptstyle \Phi(X,Y)}\downarrow & & \downarrow{\scriptstyle \Phi(F(X),F(Y))} \\ Mor(X,Y) & \xrightarrow{F_{Mor,X,Y}} & Mor(F(X),F(Y)) \end{array}$$

The commutativity of the first square follows from Lemma 2.15.

For the commutativity of the second square we have:

$$F_{Mor,X,Y}(\Phi(X,Y)(s)) = F_{Mor}(s \circ q(p_{X,pt}, Y)) = F_{Mor}(s) \circ F_{Mor}(q(p_{X,pt}, Y)) =$$

$$F_{Mor}(s) \circ q(F_{Mor}(p_{X,pt}), F(Y)) = F_{Mor}(s) \circ q(p_{F(X),pt}, F(Y)) = \Phi(F(X), F(Y))(F_{Mor}(s)) =$$

$$\Phi(F(X), F(Y))(F_{Sec(X,T^*(X,pt,Y))}(s))$$

where the third equality is by [**?**, Lemma 2.3(3)] and the fourth equality is by [**?**, Lemma 2.3(1)] and the fact that $F(pt_{CC}) = pt_{CC'}$.

Corollary 2.19 *[***2016.03.11.cor1***] For any universe U the functor*

$$CB_U : CSys(U) \to BSys(U)$$

is faithful.

Proof: Let $F_1, F_2 : CC \to CC'$ be two homomorphisms of C-systems such that $CB(F_1) = CB(F_2)$. Since $CB(CC) = (B(CC), \widetilde{B}(CC), \partial)$ where $B(CC)$ is the set of objects of CC and similarly for CC' we know that the object components of F_1 and F_2 coincide. To show that $F_1 = F_2$ it remains to show that their morphism components coincide.

In the diagrams (43) for the functors F_1 and F_2 the upper horizontal arrows coincide because of the assumptions that $CB(F_1) = CB(F_2)$. The vertical arrows coincide because they depend only on the domain and codomain of F_1 and F_2. Since the vertical arrows are bijections we conclude that the lower horizontal arrows coincide. Since they coincide for all pairs of objects X, Y we conclude that $F_{1,Mor} = F_{2,Mor}$.