B-systems[1]

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Abstract

B-systems are algebras (models) of an essentially algebraic theory that is expected to be constructively equivalent to the essentially algebraic theory of C-systems which is, in turn, constructively equivalent to the theory of contextual categories. The theory of B-systems is closer in its form to the structures directly modeled by contexts and typing judgements of (dependent) type theories and further away from categories than contextual categories and C-systems.

1 Introduction

In [?, Def. 2.2] we introduced the concept of a C-system. The type of the C-systems is constructively equivalent to the type of contextual categories defined by Cartmell in [?] and [?] but the definition of a C-system is slightly different from the Cartmell's foundational definition.

The concept of a B-system is introduced in this paper. It provides an abstract formulation of a structure formed by contexts and "typing judgements" of a type theory relative to the operations of context extensions, weakening and substitutions.

The important difference between B-systems and C-systems is that in B-systems there are no sorts for morphisms between contexts. There are only sorts for contexts of each lengths and for typing judgements, i.e., judgements whose meaning is that a given object has a given type in a given context. This gives us two infinite families of sorts B_n, for contexts of length n, and $\widetilde{B}_{n+1}$, for judgements of the form $\Gamma \vdash o : T$ where $l(\Gamma) = n$.

The operations on these sorts correspond to the empty context (pt), truncation of contexts (ft), taking extended context of a typing judgement (∂), weakening on contexts (T), weakening on typing judgements ($\widetilde{T}$), substitution on contexts (S), substitution on typing judgements ($\widetilde{S}$) and units, also known as projections, (δ).

Of these operations pt, ft, ∂ and δ are everywhere defined while $T, \widetilde{T}, S$ and $\widetilde{S}$ are partially defined with the domains of definition being given by equations that involve only everywhere defined operations ft and ∂.

We may say that operations pt, ft, ∂ and δ are of depth 0 while operations $T, \widetilde{T}, S$ and $\widetilde{S}$ are of depth 1.

We call the structures formed by these sorts and operations with no relations imposed on them pre-B-systems. We distinguish between unital and non-unital pre-B-systems depending on whether operations δ are considered or not. Pre-B-systems are models of an essentially algebraic theory of depth 1 with two infinite families of sorts. The importance of this concept is that while it

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is sufficiently easy to define, categorically it provides a lot of information since homomorphisms between models of essentially algebraic theories depend only on the operations of these theories but not on relations between them.

We next show (Theorem **??**) that the constructions of [**?**] define for every C-system a unital pre-B-system and that the first main theorem of [**?**] can be restated by saying that this theorem establishes a bijection between the C-subsystems of a C-system and sub-pre-B-systems of the corresponding unital pre-B-system.

Theorem **??** is saying that the construction of a pre-B-system from a C-system extends to a functor and that this functor is a full embedding. The sketch of a proof of this theorem that we give occupies the rest of Section **??**.

First we define the concept of a B0-system (Definition 4.1) that adds to the concept of a pre-B-system the axioms that include compositions of operations $T, \widetilde{T}, S, \widetilde{S}$ and δ with the everywhere defined operations ft and ∂. We again distinguish between the unital and non-unital cases.

In **??** we show that the pre-B-system defined by a C-system is a B0-system.

We then construct, for any (unital) B0-system BB and any two objects $X \in B_m$, $Y \in B_n$ a set $Mor(X,Y)$ in such a way that when BB is the B0-system that is defined by a C-system the sets $Mor(X,Y)$ are in natural bijection with the sets of morphisms in the C-system.

The construction of the Mor-sets is obviously functorial with respect to homomorphisms of B0-systems.

On the other hand we prove Proposition **??** which shows that for C-systems CC_1, CC_2 and a pair of functions $F_{Ob} : Ob(CC_1) \rightarrow Ob(CC_2)$ and $F_{Mor} : Mor(CC_1) \rightarrow Mor(CC_2)$ that commute with the source and target maps ∂_0, ∂_1 the condition that $F = (F_{Ob}, F_{Mor})$ is a functor is equivalent to the condition that F is compatible with a set of ??? operations, which does not include the composition operation.

Finally we show that F as above that arises from a homomorphism of the B0-systems corresponding to CC_1 and CC_2 commutes with operations from this list. This completes the proof of Theorem **??**.

??? Remind that we are using the diagrammatic ordering for compositions of morphisms and of maps between sets.

We next start looking for the set of axioms on a pre-B-system that will characterize the image of this functor. We introduce the candidate set of axioms in several layers.

These operations are subject to a number of axioms. We conjecture that the type of B-systems is constructively equivalent to the type of C-systems. A conjecture formulated in more traditional terms would say that the category of B-systems and their homomorphisms is equivalent to the category of C-systems and their homomorphisms. While these two conjectures are not equivalent the former expresses much of what the latter would be used for in practice.

Proving this conjecture is difficult because the definition of sets of morphisms between elements $X \in B_m$, $Y \in B_n$ of a B-system is based on an induction

We define B-systems in several steps. First we describe pre-B-systems that are models of an essentially algebraic theory with countable families of sorts and operations but no relations.

Already at this stage we start to distinguish between unital and non-unital (pre-)B-systems. This distinction continues throughout the paper. While non-unital B-systems have no direct connection to C-systems and therefore no direct connection to categories they have a definition with interesting symmetries and we believe that they are quite interesting in there own right.

Following the ideas of [**?**] we show how to construct a unital pre-B-system from a C-system. This construction is functorial with respect to homomorphisms of C-systems and unital pre-B-systems and moreover defines a full embedding of the category of C-systems to the category of unital pre-B-systems.

It is more or less clear from the proof of the full embedding theorem that the image of this full embedding consists of unital pre-B-systems whose operations satisfy some algebraic conditions. We suggest a form of these conditions in our definition of a non-unital and then unital B-system (Definitions 6.5 and 6.6).

We conclude the first part of the paper with a problem (essentially a conjecture) that the image of the full embedding from C-systems to unital pre-B-systems is precisely the class of unital B-systems. A constructive solution to this problem would also provide an explicit construction of a C-system from a unital B-system.

In the second part we describe an approach to the definition of non-unital B-systems that can be conveniently formalized in Coq and that provide a possible step towards the definition of higher B-systems that is B-systems whose component types are of higher h-levels.

The work on this paper, especially in the part where the axioms TT, SS, TS and ST of B-systems are introduced was influenced and facilitated by recent discussions with Richard Garner and Egbert Rijke. Many other ideas of this work go back to [**?**].

The subject of this paper is closely related to the subject of recent notes by John Cartmell [**?**]. The most important difference between our exposition and that of Cartmell is that we are using the formalism of *essentially algebraic* theories while Cartmell uses the formalism of *generalized algebraic* theories. While there are important connections between these two kinds of theories there are also important distinctions which we intend to discuss in a future paper.

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2 The lft-sets

Let us start with the definition of lft-sets. For two natural numbers m, n define

$$m -_{\mathbf{N}} n = max(m - n, 0).$$

Definition 2.1 *[**2016.01.27.def1**] An lft-set is a collection of data of the following form:*

1. *a set B,*
2. *a function $l : B \to \mathbf{N}$,*
3. *a function $ft : B \to B$*

such that for all $X \in B$ one has $l(ft(X)) = l(X) -_{\mathbf{N}} 1$.

An lft-set is called pointed if the set $\{X \in B, l(X) = 0\}$ is a one element set. In this case the only element of this set is usually denoted by pt.

Lemma 2.2 *[**2016.02.18.l2**] Let B be an lft-set, $X \in B$ and $n \in \mathbf{N}$. Then*

$$l(ft^n(X)) = l(X) -_{\mathbf{N}} n$$

Proof: Obvious induction n.

For an lft-set B, define the relation $\geq$ on B by the condition that $Y \geq X$ if and only if $l(Y) \geq l(X)$ and

$$X = ft^{l(Y) -_{\mathbf{N}} l(X)}(Y).$$

Define the relation $>$ on B by the condition that $Y > X$ if and only if $Y \geq X$ and $l(Y) > l(X)$.

Lemma 2.3 *[**2016.01.27.l1**] For any lft-set B one has:*

1. *the relation $\geq$ is a partial order relation, i.e., it is reflexive, transitive and antisymmetric,*
2. *the relation $>$ is a strict partial order relation, i.e., it is transitive and asymmetric.*

Proof: Straightforward using the corresponding properties of the relations $\geq$ and $>$ on $\mathbf{N}$ and properties of $-_{\mathbf{N}}$.

Lemma 2.4 *[**2016.02.22.l2**] Let B be an lft-set. The following mixed transitivities hold:*

1. *if $Z > Y$ and $Y \geq X$ then $Z > X$,*
2. *if $Z \geq Y$ and $Y > X$ then $Z > X$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ and $>$ on $\mathbf{N}$.

Lemma 2.5 *[**2016.02.22.l3**] Let B be an lft-set, $Y \geq X$ in B and $i \in \mathbf{N}$. Then one has:*

1. *if $l(Y) \geq i + l(X)$ then $ft^i(Y) \geq X$,*
2. *if $l(Y) > i + l(X)$ then $ft^i(Y) > X$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ and $>$ on $\mathbf{N}$.

Lemma 2.6 *[**2016.01.27.l6**] Let B be an lft-set and $Y > X$ in B. Then $ft(Y) \geq X$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ on $\mathbf{N}$.

Lemma 2.7 *[**2016.01.29.l3**] Let B be an lft-set, $X \in B$ and $n \in \mathbf{N}$. Then $X \geq ft^n(X)$.*

Proof: Straightforward from the properties of $-_{\mathbf{N}}$ and $>$ and $\geq$ on $\mathbf{N}$.

Lemma 2.8 *[***2016.01.29.l2***] Let B be an lft-set, $X \in B$, $n > 0$ and $l(ft^n(X)) > 0$. Then $X > ft^n(X)$.*

Proof: From Lemma 2.7 we know that $X \geq ft^n(X)$. It remains to show that $l(X) > l(ft^n(X))$. By Lemma 2.2, $l(ft^n(X)) = max(l(X) - n, 0)$ which implies that under the condition of the lemma $l(ft^n(X)) = l(X) - n$ and since $n > 0$ we have that $l(X) > l(ft^n(X))$.

Definition 2.9 *[***2016.01.27.def2***] Let B, B' be lft-sets. A morphism of lft-sets $f : B \to B'$ is a function $f : B \to B'$ such that for all $X \in B$ one has $l(f(X)) = l(X)$ and $l(ft(X)) = ft(l(X))$.*

We let $Mor_{lft}(B, B')$ denote the set of morphisms of lft-sets from B to B'.

Lemma 2.10 *[***2017.01.27.l3***] Let $f : B \to B'$ be a morphism of lft-sets and $X, Y \in B$. Then one has:*

1. *if $Y \geq X$ then $f(Y) \geq f(X)$,*
2. *if $Y > X$ one has $f(Y) > f(X)$,*

Proof: Straightforward.

Lemma 2.11 *[***2016.01.27.l2***] One has:*

1. *for any lft-set B the identity function $Id_B : B \to B$ is a morphism of lft-sets,*
2. *for any lft sets B, B', B'' and morphisms $f : B \to B'$, $f' : B' \to B''$ the composition of functions $f \circ f'$ is a morphism of lft-sets.*

Proof: Straightforward using the properties of $-_{\mathbf{N}}$.

Let $lft(U)$ be the set of lft-sets in the universe U.

Problem 2.12 *[***2016.01.27.prob1***] Let U be a universe. To construct a category $LFT(U)$ with the set of objects $lft(U)$.*

Construction 2.13 *[***2016.01.27.constr1a***]*We define

$$Ob(LFT(U)) = lft(U)$$

$$Mor(LFT(U)) = \amalg_{B,B' \in lft(U)} Mor_{lft}(B, B')$$

with the obvious domain and codomain functions and the identity function and the composition function being defined from the identity and composition of functions between sets using Lemma 2.11.

The proofs of the associativity and the identity axioms of a category are straightforward.

We can not use $\cup$ in this definition instead of $\amalg$ because the sets $Mor(B, B')$ need not be disjoint for different B, B'. For example, if B' has one element of each length then the set $Mor(B, B')$ depends on the set B and the length function l but is independent on the ft function on B.

Therefore there is no category with the set of objects $lft(U)$ and the set of morphisms between any two lft-sets being the set Mor_{lft} of Definition 2.9. Instead in our category the set of morphisms from B to B' is the set of iterated pairs of the form $((B, B'), f)$ where f is a function $B \to B'$ that satisfies the conditions of Definition 2.9. This set is in the obvious bijective correspondence with the set of morphisms from B to B' and we will use both directions of this bijection as coercions - if an element of $Mor_{LFT(U)}(B, B')$ occurs in a position where an element of $Mor_{lft}(B, B')$ should be it is replaced by its image in $Mor_{lft}(B, B')$ under the corresponding function of the bijection and vice versa.

This completes Construction 5.6.

In what follows we fix a universe and write LFT instead of $LFT(U)$ and lft instead of $lft(U)$.

3 pre-B-systems

Definition 3.1 *[***2016.01.27.def7***] A pre-B-system carrier is a triple $(B, \widetilde{B}, \partial)$ where B is an lft-set, $\widetilde{B}$ is a set and $\partial : \widetilde{B} \to B$ is a function such that for all $r \in \widetilde{B}$ one has $l(\partial(r)) > 0$.*

Definition 3.2 *[***2016.01.27.def3***] Let $(B, \widetilde{B}, \partial)$ be a pre-B-system carrier. We set:*

$$T_{dom} = \{X, Y \in B, l(X) \geq 1, Y > ft(X)\} \quad \widetilde{T}_{dom} = \{X \in B, s \in \widetilde{B}, (X, \partial(s)) \in T_{dom}\}$$

$$S_{dom} = \{r \in \widetilde{B}, Y \in B, Y > \partial(r)\} \quad \widetilde{S}_{dom} = \{r, s \in \widetilde{B}, (r, \partial(s)) \in S_{dom}\}$$

$$\delta_{dom} = \{X \in B, l(X) \geq 1\}$$

Definition 3.3 *[***2014.10.10.def1***] A non-unital pre-B-system is a pre-B-system carrier together with functions $T, \widetilde{T}, S$ and $\widetilde{S}$ of the form:*

$$T : T_{dom} \to B \quad \widetilde{T} : \widetilde{T}_{dom} \to \widetilde{B}$$

$$S : S_{dom} \to B \quad \widetilde{S} : \widetilde{S}_{dom} \to \widetilde{B}$$

Definition 3.4 *[***2014.10.20.def1***] A pre-B-system is a non-unital pre-B-system together with a function*

$$\delta : \delta_{dom} \to B$$

Definition 3.5 *[***2016.27.def8***] A morphism of pre-B-system carriers $\mathbf{f} : (B, \widetilde{B}, \partial) \to (B', \widetilde{B}', \partial')$ is a pair $(\widetilde{f}, f)$ where $\widetilde{f} : \widetilde{B} \to \widetilde{B}'$ is a function, $f : B \to B'$ is a morphism of lft-sets and for any $s \in \widetilde{B}$ one has*

$$\partial'(\widetilde{f}(s)) = f(\partial(s))$$

Problem 3.6 *[***2016.01.27.prob9***] For a morphism of pre-B-system carriers $\mathbf{f} : (B, \widetilde{B}, \partial) \to (B', \widetilde{B}', \partial')$ to construct functions*

$$f_T : T_{dom} \to T'_{dom} \quad f_{\widetilde{T}} : \widetilde{T}_{dom} \to \widetilde{T}'_{dom}$$

$$f_S : S_{dom} \to S'_{dom} \quad f_{\widetilde{S}} : \widetilde{S}_{dom} \to \widetilde{S}'_{dom}$$
$$f_\delta : \delta_{dom} \to \delta'_{dom}$$

Construction 3.7 [**2016.01.27.constr8**] For $(X,Y) \in T_{dom}$ we set $f_T(X,Y) = (f(X), f(Y))$. The condition that $f_T(X,Y) \in T'_{dom}$ follows immediately from the fact that f is an lft-set morphism and Lemma 2.10.

For $(X,s) \in \widetilde{T}_{dom}$ we set $f_{\widetilde{T}}(X,s) = (f(X), \widetilde{f}(s))$. The condition that $f_{\widetilde{T}}(X,s) \in \widetilde{T}'_{dom}$ follows immediately from the fact that $(f,\widetilde{f})$ is a morphism of pre-B-system carriers and and Lemma 2.10.

The proofs for the remaining three subsets are equally easy corollaries of the definitions and Lemma 2.10.

Definition 3.8 *[**2016.01.27.def4**] Let $\mathbf{B}, \mathbf{B}'$ be pre-B-systems. A homomorphism of non-unital pre-B-systems $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ is a morphism $\mathbf{f} = (f, \widetilde{f})$ of the pre-B-system carriers such that one has*

$$\text{for } (X,Y) \in T_{dom},\ f(T(X,Y)) = T'(f_T(X,Y)), \quad \text{for } (X,s) \in \widetilde{T}_{dom},\ \widetilde{f}(\widetilde{T}(X,s)) = \widetilde{T}'_{dom}(f_{\widetilde{T}}(X,s))$$

$$\text{for } (r,Y) \in S_{dom},\ f(S(r,Y)) = S'(f_S(r,Y)) \quad \text{for } (r,s) \in \widetilde{S}_{dom},\ \widetilde{f}(\widetilde{S}(r,s)) = \widetilde{S}'(f_{\widetilde{S}}(r,s))$$

A homomorphism of pre-B-systems is a morphism $\mathbf{f} = (f, \widetilde{f})$ of pre-B-system carriers that is a homomorphism of non-unital pre-B-systems and such that one has:

$$\text{for } X \in \delta_{dom}, \widetilde{f}(\delta(X)) = \delta'(f(X))$$

Lemma 3.9 *[**2016.01.27.l5**] One has:*

1. *Let $\mathbf{B}$ be a non unital pre-B-system (resp. pre-B-system) and $(Id_{\widetilde{B}}, Id_B)$ be the identity morphism of the underlying pre-B-system carries. Then $(Id_{\widetilde{B}}, Id_B)$ is a homomorphism of non-unital pre-B-systems (resp. pre-B-systems).*
2. *Let $\mathbf{B}, \mathbf{B}', \mathbf{B}''$ be non-unital pre-B-systems (resp. pre-B-systems) and $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$, $\mathbf{f}' : \mathbf{B}' \to \mathbf{B}''$ be two homomorphism of non-unital pre-B-systems (resp. pre-B-systems). Then the composition of the underlying homomorphisms of pre-B-system carriers is a homomorphism of non-unital pre-B-systems (resp. pre-B-systems).*

Proof: The proof is straightforward but long since all five conditions of Definition 3.8 have to be verified.

For a pre-B-system carrier $(B, \widetilde{B}, \partial)$ and $X \in B$ denote by $\widetilde{B}(X)$ the subset of $\widetilde{B}$ of elements s such that $\partial(s) = X$.

Problem 3.10 *[**2016.01.29.prob2**] Let $(B, \widetilde{B}, \partial)$ be a pre-B-system carrier and let $S : S_{dom} \to B$ be a function. To construct, for any $n \in \mathbf{N}$ and X in B a set $\widetilde{B}(n, X)$.*

Construction 3.11 [**2016.01.29.constr2**] We proceed by induction on n as follows:

1. $\widetilde{B}(0, X) = unit$ where $unit$ is our chosen set with one element tt,

2. $\widetilde{B}(1,X) = \widetilde{B}(X)$,

3. For the successor of $n > 0$ we define $\widetilde{B}(n+1,X)$ to be the set of pairs (r,s) where

$$r \in \widetilde{B}(ft^n(X))$$

and

$$s \in \widetilde{B}(n, S(r,X)).$$

To show that this construction is well defined we need to prove that for $X \in B$ and $r \in \widetilde{B}(ft^n(X))$ we have $(r,X) \in S_{dom}$, i.e., that we have $X > \partial(r)$. We have $\partial(r) = ft^n(X)$. By Definition 3.1 we know that $l(ft^n(X)) > 0$. Therefore, by Lemma 2.8 we have that $X > ft^n(X)$.

This completes Construction 3.11.

Lemma 3.12 *[***2016.02.18.l1***] In the context of Problem 3.10 let $n > l(X)$ then $\widetilde{B}(n,X) = \emptyset$.*

Proof: Since $n > l(X)$ we have $n > 0$.

If $n = 1$ then $\widetilde{B}(n,X) = \widetilde{B}(X)$ and $l(X) = 0$. We have that $\widetilde{B}(X) = \emptyset$ by Definition 3.1.

For successor of $n > 0$ we have that any element of $\widetilde{B}(n,X)$ is a pair (r,s) where $r \in \widetilde{B}(ft^n(X))$. By Lemma 2.2 we have that $l(ft^n(X)) = l(X) -_{\mathbf{N}} n$ and since $n > l(X)$ we have that $l(ft^n(X)) = 0$. Then $\widetilde{B}(ft^n(X)) = \emptyset$ by Definition 3.1.

In the context of Problem 3.10 define

$$\widetilde{B}(n) = \coprod_{X \in B} \widetilde{B}(n,X)$$

and let

$$\partial : \widetilde{B}(n) \to B$$

be the function $(X,s) \mapsto X$. We will use the obvious bijections between the sets $\widetilde{B}(n)(X) = \{a \in \widetilde{B}(n) \,|\, \partial(a) = X\}$ and $\widetilde{B}(n,X)$ as coercions and may write an element of one of this sets in a position where an element of another one is required.

4 The B0-systems

The complex of axioms that define B0-systems among all pre-B-systems is as follows:

Definition 4.1 *[***2014.10.16.def1.fromold***] [***2014.10.16.def1***] [***2016.01.29.def1***] A non-unital pre-B-system is called a non-unital B0-system if the following conditions hold:*

1. *For $(X,Y) \in T_{dom}$ one has:*
   (a) *$l(T(X,Y)) = l(Y) + 1$,*
   (b) *$T(X,Y) > X$,*
   (c) *if $ft(Y) > ft(X)$ then $ft(T(X,Y)) = T(X, ft(Y))$.*

2. For $(X, s) \in \widetilde{T}_{dom}$ one has:

   (a) $l(\partial(\widetilde{T}(X, s))) = l(\partial(s)) + 1$,

   (b) $\partial(\widetilde{T}(X, s)) = T(X, \partial(s))$.

3. For $(r, Y) \in S_{dom}$ one has:

   (a) $l(S(r, Y)) = l(Y) - 1$,

   (b) $S(r, Y) > ft(\partial(r))$,

   (c) if $ft(Y) > \partial(r)$ then $ft(S(r, Y)) = S(r, ft(Y))$.

4. For $(r, s) \in \widetilde{S}_{dom}$ one has

   (a) $l(\widetilde{S}(r, s)) = l(\partial(s)) - 1$,

   (b) $\partial(\widetilde{S}(r, s)) = S(r, \partial(s))$.

Remark 4.2 [**2016.01.29.rem1**] The axioms of a B0-system given in Definition 4.1 are not independent. If the axioms 1(a) and 2(b) hold then the axiom 2(a) holds and if the axioms 3(a) and 4(b) hold then the axiom 4(a) holds. The axioms are presented there in this form to make it possible to prove facts about various operations in B0-systems independently from each other.

Definition 4.3 *[***2014.10.20.def2***] A pre-B-system is called a B0-system if the underlying non-unital pre-B-system is a non-unital B0-system and for all $X \in \delta_{dom}$ one has*

$$[\mathbf{2009.12.27.eq1}]\partial(\delta(X)) = T(X, X) \tag{1}$$

Consider the following sets for $j \in \mathbf{N}$

$$T^j_{dom} = \{X, Y \in B, l(X) \geq j, Y \geq ft^j(X)\} \quad \widetilde{T}^j_{dom} = \{X \in B, s \in \widetilde{B}, (X, ft(\partial(s))) \in T^j_{dom}\}$$

Note that T^1_{dom} is slightly larger than T_{dom} because we used $\geq$ in the definition of T^j_{dom} where $>$ is used in the definition of T_{dom}. However, $\widetilde{T}^1_{dom} = \widetilde{T}_{dom}$ since the condition that $X \geq ft(\partial(s))$ is equivalent to the condition that $X > \partial(s)$.

Problem 4.4 *[***2016.02.18.prob3***] Let B be an lft-set and $T : T_{dom} \to B$ a function satisfying the conditions of Definition 4.1(1). For $j \in \mathbf{N}$ to define a function*

$$T^j : T^j_{dom} \to B$$

such that:

1. $l(T^j(X, Y)) = l(Y) + j$,

2. $T^j(X, Y) \geq X$,

3. if $ft(Y) > ft^j(X)$ then $ft(T^j(X, Y)) = T^j(X, ft(Y))$.

Construction 4.5 [**2016.02.18.constr3**] We proceed by induction on j.

For $j=0$ we set $T^0(X,Y)=Y$. The proofs of the conditions are obvious.

For $j=1$ we set $T^1(X,Y)=T(X,Y)$ if $l(Y)>l(ft(X))$ and therefore $Y>ft(X)$ and $T^1(X,Y)=X$ if $l(Y)=l(ft(X))$ and therefore $Y=ft(X)$. The conditions in the first case are the conditions of Definition 4.1(1) and in the second case the first one follows from $l(X)\geq 1$, the second one is obvious and the third one is obvious since the premise $ft(Y)>ft^j(X)$ never occurs.

For the successor of $j>0$ also consider first the case $l(Y)>l(ft^j(X)$. Then we set

$$[\mathbf{2016.02.20.eq2}]T^{j+1}(X,Y)=T(X,T^j(ft(X),Y)) \tag{2}$$

For the formula (2) to be well defined we need to show that for $(X,Y)\in T^{j+1}_{dom}$ we have

$$[\mathbf{2016.02.20.eq4}](ft(X),Y)\in T^j_{dom} \tag{3}$$

and

$$[\mathbf{2016.02.20.eq5}](X,T^j(ft(X),Y))\in T_{dom} \tag{4}$$

The formula (3) is equivalent to $l(ft(X))\geq j$ and $Y>ft^{j+1}(X)$. The first statement follows from $l(X)\geq j+1$ the second coincides with the second part of the condition that $(X,Y)\in T^{j+1}_{dom}$.

The formula (4) is equivalent to $l(X)\geq 1$ and

$$T^j(ft(X),Y)>ft(X)$$

That $l(X)\geq 1$ follows from $l(X)\geq j+1$.

That $T^j(ft(X),Y)>ft(X)$ is one of our conditions.

Let us prove the conditions for $j+1$. The first two are straightforward:

$$l(T^{j+1}(X,Y))=l(T(X,T^j(ft(X),Y)))=l(T^j(ft(X),Y))+1=l(Y)+j+1$$

$$T^{j+1}(X,Y)=T(X,T^j(ft(X),Y))>X$$

To prove the third condition let $ft(Y)>ft^{j+1}(X)$. Consider

$$ft(T^{j+1}(X,Y))=ft(T(X,T^j(ft(X),Y))).$$

We have $ft(Y)>ft^j(ft(X))$ because $ft^{j+1}(X)=ft^j(ft(X))$. Therefore,

$$ft(T^j(ft(X),Y))=T^j(ft(X),ft(Y))>ft(X).$$

Therefore,

$$ft(T(X,T^j(ft(X),Y)))=T(X,ft(T^j(ft(X),Y)))=T(X,T^j(ft(X),ft(Y)))$$

We have $(X,ft(Y))\in T^{j+1}_{dom}$ since $ft(Y)>ft^{j+1}(X)$ by our assumption and therefore

$$T(X,T^j(ft(X),ft(Y)))=T^{j+1}(X,ft(Y)).$$

In the case when $Y=ft^j(X)$ we set

$$T^j(X,Y)=X$$

The first condition follows from the assumption $l(X)\geq j$, the second is obvious and the third is obvious because the premise $ft(Y)>ft^j(X)$ never occurs.

This completes Construction 4.5.

We will also require a similar construction for $\widetilde{T}$.

Problem 4.6 *[**2016.02.20.prob1**] Let $(B, \widetilde{B}, \partial)$ be a pre-B-system carrier and let*

$$T : T_{dom} \to B \quad \widetilde{T} : \widetilde{T}_{dom} \to \widetilde{B}$$

be functions satisfying the conditions of Definition 4.1(1,2). For $j \in \mathbf{N}$, define a function

$$\widetilde{T}^j : \widetilde{T}^j_{dom} \to \widetilde{B}$$

such that:

$$[\mathbf{2016.02.22.eq1}]\partial(\widetilde{T}^j(X, s)) = T^j(X, \partial(s)) \tag{5}$$

Construction 4.7 [**2016.02.20.constr1**] We proceed by induction on j. Observe first that the condition $(X, ft(\partial(s))) \in T^j_{dom}$ is equivalent to the condition that $l(X) \geq j$ and $ft(\partial(s)) \geq ft^j(X)$ and that since $l(\partial(s)) > 0$ the latter condition is equivalent to $\partial(s) > ft^j(X)$.

For $j = 0$ we set $\widetilde{T}^0(X, s) = s$. The proof of the conditions is obvious.

For $j = 1$ we set $\widetilde{T}^1(X, s) = \widetilde{T}(X, s)$. The conditions are the conditions of Definition 4.1(2).

For the successor of $j > 0$ we set

$$[\mathbf{2016.02.20.eq3}]\widetilde{T}^{j+1}(X, s) = \widetilde{T}(X, \widetilde{T}^j(ft(X), s)) \tag{6}$$

For the formula (6) to be well defined we need to show that assuming $(X, \partial(s)) \in T^{j+1}_{dom}$ we have:

$$[\mathbf{2016.02.20.eq7}](ft(X), s) \in \widetilde{T}^j_{dom} \tag{7}$$

and

$$[\mathbf{2016.02.20.eq8}](X, \widetilde{T}^j(ft(X), s)) \in \widetilde{T}_{dom} \tag{8}$$

The formula (7) is equivalent to $(ft(X), \partial(s)) \in T^j_{dom}$ and its proof is identical to the proof of (3) for $Y = \partial(s)$.

The formula (8) is equivalent to $(X, \partial(\widetilde{T}^j(ft(X), s))) \in T_{dom}$. By the inductive assumption we have $\partial(\widetilde{T}^j(ft(X), s)) = T^j(ft(X), \partial(s))$ and the rest of the proof is identical to the proof of (4) for $Y = \partial(s)$.

Let us prove condition (5). We have

$$\partial(\widetilde{T}^{j+1}(X, s)) = \partial(\widetilde{T}(X, \widetilde{T}^j(ft(X), s))) = T(X, \partial(\widetilde{T}^j(ft(X), s))) =$$
$$T(X, T^j(ft(X), \partial(s))) = T^{j+1}(X, \partial(s))$$

This completes Construction 4.7.

Problem 4.8 *[**2016.02.28.prob1**] For a pointed B0-system $(B, \widetilde{B}, \partial)$ and $X, Y \in B$ to define a set that will be denoted $BMor(X, Y)$.*

Construction 4.9 [**2016.02.28.constr1**][**2016.02.20.def1**] We define this set by the formula:

$$BMor(X, Y) = \widetilde{B}(l(Y), T^{l(X)}(X, Y))$$

Let us show that the right hand side is well defined. For that we need $T^{l(X)}(X, Y)$ to be well defined, i.e., to have $Y \geq ft^{l(X)}(X)$. We have

$$l(ft^{l(X)}(X)) = max(l(X) - l(X), 0) = 0$$

and since our B0-system is pointed we have $Y = ft^{l(X)}(X)$.

Remark 4.10 [**2016.02.28.def1**] To define $BMor(X,Y)$ we need less than the full set of B0-system structures and axioms. All we need is a pre-B-system carrier with operations T and S such that T satisfies conditions of Definition 4.1(1).

In the next section we will construct for any C-system CC a B-system $\mathbf{B}(CC)$ and for any $X,Y\in CC$ a bijection between $BMor_{\mathbf{B}(CC)}(X,Y)$ and $Mor_{CC}(X,Y)$.

To describe the operation on $BMor$ sets that will be related to the composition of morphisms in the case of a B-system of the form $\mathbf{B}(CC)$ we need further constructions.

First, we describe the operations that correspond in the B-systems that correspond to C-systems to the pull-back of sections of morphisms $p_{Y,ft^n(Y)}$ by morphisms $p_{X,ft^l(X)}$.

Definition 4.11 *[***2016.02.22.def2***] Let $(B,\widetilde{B},\partial)$ be a pre-B-system carrier. We define the sets $\widetilde{T}^l_{n,dom}$ by the formula*

$$\widetilde{T}^l_{n,dom}=\{X\in B, Y\in B, s\in\widetilde{B}(n,Y)\,|\,l(X)\geq l,\ ft^n(Y)\geq ft^l(X)\}$$

Problem 4.12 *[***2016.02.22.prob1***] Let $(B,\widetilde{B},\partial)$ be a pre-B-system carrier and let*

$$T:T_{dom}\rightarrow B\quad\widetilde{T}:\widetilde{T}_{dom}\rightarrow\widetilde{B}$$

be functions satisfying the conditions of Definition 4.1(1,2) and

$$S:S_{dom}\rightarrow B$$

be a function satisfying the conditions of Definition 4.1(3). For $l,n\in\mathbf{N}$, define a function

$$\widetilde{T}^l_n:\widetilde{T}^l_{n,dom}\rightarrow\widetilde{B}(n)$$

such that:

$$\partial(\widetilde{T}^l_n(X,Y,s))=T^l(X,Y)$$

Construction 4.13 [**2016.02.22.constr1**] Observe first that since $Y\geq ft^n(Y)$, for any l,n and $(X,Y,s)\in\widetilde{T}^l_{n,dom}$ we have $(X,Y)\in T^l_{dom}$.

Proceed now by induction on n.

For $n=0$ we set

$$\widetilde{T}^l_0(X,Y,s)=(T^l(X,Y),tt)$$

where tt is the only element of $\widetilde{B}(0,T^l(X,Y))=unit$.

For $n=1$ we set

$$\widetilde{T}^l_1(X,Y,s)=(T^l(X,Y),\widetilde{T}^l(X,s))$$

For the successor of $n>0$ we define $\widetilde{T}^l_{n+1}(X,Y,s)$ as follows. Recall that

$$\widetilde{B}(n+1,Y)=\{(r,s)\,|\,r\in\widetilde{B}(ft^n(Y)),\ s\in\widetilde{B}(n,S(r,Y))\}$$

We set

$$\widetilde{T}^l_{n+1}(X,Y,(r,s))=(T^l(X,Y),(\widetilde{T}^l(X,r),\widetilde{T}^l_n(X,S(r,Y),s)))$$

Let us check that the right hand side is well defined.

We know that $T^l(X,Y)$ is defined.

For $\widetilde{T}^l(X,r)$ to be defined we need $l(X) \geq l$ and $\partial(r) > ft^l(X)$. We have $\partial(r) = ft^n(Y)$. We also have that $ft^{n+1}(Y) > ft^l(X)$. Since $ft^n(Y) \geq ft^{n+1}(Y)$ we get $ft^n(Y) > ft^l(X)$ from Lemma 2.4.

For $\widetilde{T}^l_n(X, S(r,Y), s)$ to be defined we need $s \in \widetilde{B}(n, S(r,Y))$, $l(X) \geq l$ and $ft^n(S(r,Y)) > ft^l(X)$. The first two conditions are obvious. To prove the third one note that, from Definition 4.1(3), we have that $S(r,Y) > ft(\partial(r)) = ft^{n+1}(Y)$. We also have $ft^{n+1}(Y) > ft^l(X)$. By Lemma 2.4 it is sufficient to prove that $ft^n(S(r,Y)) \geq ft^{n+1}(Y)$. From Lemma 2.5(1) we see that it is sufficient to prove that $l(S(r,Y)) \geq n + l(ft^{n+1}(Y))$. By Definition 4.1(3) we have that $l(S(r,Y)) = l(Y) - 1$. Since $l(ft^{n+1}(Y)) \leq l(Y) - (n+1)$ we have $n + l(ft^{n+1}(Y)) \geq l(S(r,Y))$.

This completes Construction 4.13.

Next we are going to construct the operations that in the B0-systems corresponding to C-systems correspond to the pull-back of elements of $\widetilde{B}(m)$ along elements of $\widetilde{B}$.

Definition 4.14 *[**2016.02.28.def1**] Let $(B, \widetilde{B})$ be a pre-B-system carrier. For $m \in \mathbf{N}$ define*

$$S_{m,dom} = \{r \in \widetilde{B}, s \in \widetilde{B}(m),\ ft^m(\partial(s)) \geq \partial(r)\}$$

Problem 4.15 *[**2016.02.28.prob1**] Let $(B, \widetilde{B})$ be a pre-B-system carrier and let*

$$S : S_{dom} \to B$$

$$\widetilde{S} : \widetilde{S}_{dom} \to \widetilde{B}$$

be two functions. To construct, for $m \in \mathbf{N}$ a function

$$\widetilde{S}_m : \widetilde{S}_{m,dom} \to \widetilde{B}(m)$$

such that for $m > 0$ one has

$$\partial(\widetilde{S}_m(r,s)) = S(r, \partial(s))$$

Construction 4.16 [**2016.02.28.constr1**] We proceed by induction on m.

For $m = 0$ we set, if $\partial(s) > \partial(r)$ then

$$\widetilde{S}_0(r, (\partial(s), tt)) = (S(r, \partial(s)), tt)$$

if $\partial(s) = \partial(r)$ then

$$\widetilde{S}_0(r, (\partial(s), tt)) = ft(\partial(r))$$

5 The pre-B-systems of C-systems

Let us recall (cf. [**?**]) that for a C-system CC an object Y is said to be an object over X if $Y \geq X$. In this case the composition of the canonical projections $Y \stackrel{p_Y}{\to} ft(Y) \stackrel{p_{ft(Y)}}{\to} \ldots \to X$ is denoted by $p_{Y,X}$. For a morphism $f : X' \to X$ one defines $f^*(Y)$ by induction using the f^* structure of the C-system. One also defines by induction a morphism $q(f,Y) : f^*(Y) \to Y$.

Lemma 5.1 *[**2016.02.18.l4**] In the context introduced above one has $f^*(Y) \geq X'$,*

$$[\mathbf{2016.02.18.eq3}]\, l(f^*(Y)) - l(X') = l(Y) - l(X) \tag{9}$$

and the square

$$[\mathbf{2016.02.18.eq2}]\quad \begin{array}{ccc} f^*(Y) & \xrightarrow{q(f,Y)} & Y \\ {\scriptstyle p_{f^*(Y),X'}}\downarrow & & \downarrow{\scriptstyle p_{Y,X}} \\ X' & \xrightarrow{f} & X \end{array} \tag{10}$$

is a pull-back square.

Proof: By induction on $l(Y) - l(X)$ using the fact that vertical composition of pull-back squares is a pull-back square.

For $Y, Y' \geq X$ a morphism $g : Y \to Y'$ is said to be a morphism over X if $p_{Y,X} = g \circ p_{Y',X}$. For such a morphism g and a morphism $f : X' \to X$ there is a unique morphism $f^*(g) : f^*(Y) \to f^*(Y')$ over X' such that the square

$$[\mathbf{2016.02.18.eq1}]\quad \begin{array}{ccc} f^*(Y) & \xrightarrow{q(f,Y)} & Y \\ {\scriptstyle f^*(g)}\downarrow & & \downarrow{\scriptstyle g} \\ f^*(Y') & \xrightarrow{q(f,Y')} & Y' \end{array} \tag{11}$$

commutes (see [**?**, Lemma 2.1]).

We will also need the following lemmas.

Lemma 5.2 *[**2016.01.27.l8**] Let CC be a C-system, X an object over X' and X' an object over X'' then one has $p_{X,X''} = p_{X,X'} \circ p_{X',X''}$.*

Proof: By induction on $l(X') - l(X)$.

Lemma 5.3 *[**2016.01.27.l7**] Let CC be a C-system, $f : Y \to Y'$ be a morphism over X and X be an object over W. Then f is a morphism over W.*

Proof: Follows easily from Lemma 5.2.

Lemma 5.4 *[**2016.02.20.l9**] Let $Y > X$ and $f : X' \to X$. Then one has*

$$ft(f^*(Y)) = f^*(ft(Y))$$

Proof: It follows immediately from the inductive definition of f^* since for $l(Y) - l(X) > 0$ we have $f^*(Y) = q(f, ft(Y))^*(Y)$ where $q(f, ft(Y))^*(Y)$ is given by the C-system structure.

Problem 5.5 *[**2016.01,27.prob2**] Let CC be a C-system. To construct a pre-B-system*

$$CB(CC) = (B(CC), \widetilde{B}(CC), l, ft, \partial, T, \widetilde{T}, S, \widetilde{S})$$

Construction 5.6 *[2016.01.27.constr1]* Let $B(CC) = Ob(CC)$ and

$$\widetilde{B}(CC) = \{s \in Mor(CC) \,|\, dom(s) = ft(codom(s)) \text{ and } s \circ p_{codom(s)} = Id_{dom(s)}\}$$

(this set was previously denoted by $\widetilde{B}(CC)$). The triple $(B(CC), l, ft)$ is an lft-set and we have the function of sets

$$\partial = codom : \widetilde{B}(CC) \to B(CC)$$

obtaining a pre-B-system carrier.

Starting with these data we can define the sets $T_{dom}, \widetilde{T}_{dom}, S_{dom}, \widetilde{S}_{dom}$ and δ_{dom}.

Next, we define operations $T, \widetilde{T}, S, \widetilde{S}, \delta$ as follows:

$$T(X,Y) = p_X^*(Y) \quad \widetilde{T}(X,s) = p_X^*(s)$$

$$S(r,Y) = r^*(Y) \quad \widetilde{S}(r,s) = r^*(s)$$

$$\delta(X) = s_{Id_X}$$

The first of these operations is defined because $Y > ft(X)$ and therefore Y is over $ft(X)$.

The second one is defined because $s : ft(\partial(s)) \to \partial(s)$ is a morphism over $ft(\partial(s))$ and since $\partial(s) > ft(X)$ we have that $ft(\partial(s)) \geq ft(X)$ by Lemma 2.6 and therefore $ft(\partial(s))$ is an object over $ft(X)$ and so the morphism s is a morphism over $ft(X)$ by Lemma 5.3.

The third of one is defined because Y is over $\partial(r)$.

The fourth one is defined because $s : ft(\partial(s)) \to \partial(s)$ is a morphism over $ft(\partial(s))$ while r is of the form $ft(\partial(r)) \to \partial(r)$ and since $\partial(s) > \partial(r)$ we have that $ft(\partial(s)) \geq \partial(r)$ by Lemma 2.6 and therefore s is a morphism over $\partial(r)$ by Lemma 5.3. . Finally δ is defined because s_f is defined for any morphism of the form $f : X \to Y$ where $l(Y) > 0$ (cf. [**?**, Definition 2.3]).

This completes Construction 5.6.

Lemma 5.7 *[2016.02.18.l6] Let CC be a C-system. Then $CB(CC)$ is a B0-system.*

Proof:

1. Let $(X,Y) \in T_{dom}$. Then one has:
   (a) We have $l(T(X,Y)) = l(p_X^*(Y))$. To define $p_X^*(Y)$ we consider Y as an object over $ft(X)$. Therefore by Lemma 5.1 we have

   $$l(p_X^*(Y)) = l(X) + (l(Y) - l(ft(X))) = (l(X) - l(ft(X))) + l(Y)$$

   Since $l(X) \geq 1$ we have $l(X) - l(ft(X)) = 1$. Therefore

   $$l(T(X,Y)) = l(Y) + 1$$

   (b) By Lemma 5.1 we have $T(X,Y) \geq X$ and

   $$l(T(X,Y)) - l(X) = l(Y) - l(ft(X)) > 0$$

   therefore $T(X,Y) > X$.

(c) It follows from Lemma 5.4.

2. Let $(X, s) \in \widetilde{T}_{dom}$. The first condition follows from the second one and condition 1(a). The second condition follows immediately from the definitions.

3. Let $(r, Y) \in S_{dom}$. Then one has:

   (a) $l(S(r, Y)) = l(r^*(Y))$ were Y is considered as an object over $X = \partial(r)$. By Lemma 5.1 we have

   $$l(r^*(Y)) = l(ft(X)) + (l(Y) - l(X)) = l(Y) + (l(ft(X)) - l(X)) = l(Y) - 1$$

   where the last equality follows from the fact that $l(ft(X)) - l(X) = -1$ since $l(X) = l(\partial(r)) > 0$.

   (b) By Lemma 5.1 we have $r^*(Y) \geq ft(X)$ and since $Y > X$ the same lemma implies that $r^*(Y) > X$.

   (c) It follows from Lemma 5.4.

4. The first condition follows from the second one and condition 2(a). The second condition follows immediately from the definitions.

Problem 5.8 *[**2016.01.27.prob3**] Let $f : CC \to CC'$ be a homomorphism of C-systems. To construct a homomorphism of pre-B-systems $CB(f) : CB(CC) \to CB(CC')$.*

Construction 5.9 [**2016.01.27.constr3**] We need to construct a morphism of pre-B-system carriers and show that it is a homomorphism of pre-B-systems.

We already have the function $f : B(CC) \to B(CC')$ and by the definition of a homomorphism of C-systems (cf. [**?**, Definition 3.1]) it is a morphism of lft-sets.

By definition $\widetilde{B}(CC)$ is a subset of $Mor(CC)$. Therefore, by the morphism part of the functor f it is mapped to a subset of $Mor(CC')$. We need to verify that the image of $\widetilde{B}(CC)$ lies in $\widetilde{B}(CC')$. The subset $\widetilde{B}(CC)$ is the subset of elements s such that $dom(s) = ft(codom(s))$ and $s \circ p_{codom(s)} = Id_{dom(s)}$. It follows that it will be mapped to the subset defined by the same conditions by any functor that maps the p-morphisms of CC to p-morphisms of CC' and in particular by any homomorphism of C-systems. We obtain a function $\widetilde{B} \to \widetilde{B'}$ that we denote by $\widetilde{f}$.

It is immediate from the construction that the pair $\mathbf{f} = (f, \widetilde{f})$ is a morphism of pre-B-system carriers.

By Construction 3.7 we obtain functions $f_T, f_{\widetilde{T}}, f_S, f_{\widetilde{S}}, f_\delta$.

The fact that these functions commute, in the sense of Definition 3.8, with the pre-B-system operations follows from [**?**, Lemma 2.3(3,4,5)].

Lemma 5.10 *[**2016.01.29.l1**] Constructions 5.6 and 5.9 define a functor CB from the category of C-systems in U to the category of pre-B-systems in U.*

Proof: Since two homomorphisms of pre-B-systems are equal if and only if the underlying morphisms of the pre-B-system carriers are equal it is sufficient to prove the identity and composition axioms of a functor for the mappings from C-systems and their homomorphisms to the carriers of pre-B-systems and their morphisms. These axioms follow immediately from the fact that $\widetilde{B}(CC)$ is a subset of $Mor(CC)$ and the definition of composition of C-system homomorphisms.

We will sometimes call the functor CB defined by Constructions 5.6, 5.9 and Lemma 5.10, the B-sets functor.

Problem 5.11 *[***2016.02.18.prob1***] Let CC be a C-system, $X \in B(CC)$ and $l(X) \geq n$. To construct a bijection*

$$nt(n,X) : \widetilde{B}(n,X) \to \{f \in Mor(CC) \,|\, dom(f) = ft^n(X),\ codom(f) = X,\ f \circ p_{X,ft^n(X)} = Id_{ft^n(X)}\}$$

To provide a construction for this problem we need the following general lemma.

Lemma 5.12 *[***2016.02.18.l3***] Let $\mathcal{C}$ be a category and $X \xrightarrow{a} Y \xrightarrow{b} Z$ be a composable pair of morphisms in $\mathcal{C}$. Assume further that for any $r : Z \to Y$ such that $r \circ b = Id_Z$ we are given a pull-back square of the form:*

$$\begin{array}{ccc} r^*(X) & \xrightarrow{pr^r_X} & X \\ {\scriptstyle pr^r_Z}\downarrow & & \downarrow{\scriptstyle a} \\ Z & \xrightarrow{r} & Y \end{array}$$

Let $R(a,b)$ be the set of pairs (r,s) where $r : Z \to Y$ is such that $r \circ b = Id_Z$ and $s : Z \to r^*(X)$ is such that $s \circ pr^r_Z = Id_Z$.

Then the function $(r,s) \mapsto s \circ pr^r_X$ is a bijection from R to the set of morphisms $t : Z \to X$ such that $t \circ a \circ b = Id_Z$.

Proof: Note first that

$$s \circ pr^r_X \circ a \circ b = s \circ pr^r_Z \circ r \circ b = Id_Z \circ Id_Z = Id_Z$$

Let us show that our function is surjective. Let $f : Z \to X$ be a morphism such that $f \circ a \circ b = Id_Z$. Let $r = f \circ a$. Then $r \circ b = Id_Z$ and on the other hand $f = s \circ pr^r_X$ for some s such that $s \circ pr^r_Z = Id_Z$ by the universal property of the pull-backs.

Let us show that our function is injective. Let (r,s), (r',s') be two elements of $R(a,b)$ such that $s \circ pr^r_X = s' \circ pr^{r'}_X$. We have

$$s \circ pr^r_X \circ a = s \circ pr^r_Z \circ r = r$$

We conclude that $r = r'$. Then if $s \circ pr^r_X = s' \circ pr^r_X$ and $s \circ pr^r_Z = Id_Z = s' \circ pr^r_Z$ we have that $s = s'$ by the universal property of the pull-backs.

The lemma is proved.

We can now provide a construction for Problem 5.11.

Construction 5.13 *[***2016.02.18.constr1***]*If $n = 0$ then $p_{X,ft^n(X)} = Id_X$ and both sides are one element sets.

If $n = 1$ then $nt(X)$ is the identity by definition of $\widetilde{B}(X)$ and $\widetilde{B}(CC)$.

For the successor of $n > 0$ we have that $l(X) \geq n+1$ and assume by induction that $nt(n, X')$ is already constructed for all X' such that $l(X') \geq n$.

By Lemma 5.1 we have that $S(r, X)$ is over $ft^{n+1}(X)$ and that the square

[2016.02.18.eq4]
$$\begin{array}{ccc} S(r,X) & \xrightarrow{q(r,X)} & X \\ {\scriptstyle p_{S(r,X),ft^{n+1}(X)}}\downarrow & & \downarrow{\scriptstyle p_{X,ft^n(X)}} \\ ft^{n+1}(X) & \xrightarrow{r} & ft^n(X) \end{array} \tag{12}$$

is a pull-back square for any section r of $p_{X,ft^n(X)}$. Since $l(X) \geq n+1$ we have that

$$l(X) - l(ft^n(X)) = n$$

and therefore by equation (9) of the same lemma we have that

$$ft^n(S(r,X)) = ft^{n+1}(X)$$

Consider the set $R(p_{X,ft^n(X)}, p_{ft^n(X)})$ where R is as in Lemma 5.12 relative to the pull-back squares (12). Then the function $(r, f) \mapsto (r, nt(n, S(r,X))(f))$ is a bijection of the form

$$\widetilde{B}(n+1, X) \to R(p_{X,ft^n(X)}, p_{ft^n(X)}).$$

Composing this bijection with the bijection of Lemma 5.12 and using the fact that

$$p_{X,ft^n(X)} \circ p_{ft^n(X)} = p_{X,ft^{n+1}(X)}$$

we obtain the bijection that is the goal of the construction.

This completes Construction 5.13.

Lemma 5.14 *[***2016.02.20.l10***] Let CC be a C-system. Let $(X, Y) \in T^j_{dom}$, where the pre-B-system concepts refer to $CB(CC)$, then one has*

$$T^j(X,Y) = p^*_{X,ft^j(X)}(Y)$$

and in particular we have a pull-back square of the form

$$\begin{array}{ccc} T^j(X,Y) & \xrightarrow{q(p_{X,ft^j(X)},Y)} & Y \\ {\scriptstyle p_{T^j(X,Y),X}}\downarrow & & \downarrow{\scriptstyle p_{Y,X}} \\ X & \xrightarrow{p_{X,ft^j(X)}} & ft^j(X) \end{array}$$

Proof: Immediate, by induction on j, from the definition of $T^j(X,Y)$ and the definition of $p_{X,ft^j(X)}$.

The objects involved in the construction for the successor can be seen on the diagram:

[2016.02.20.eq1]
$$\begin{array}{ccccccc} T(X, T^j(ft(X),Y)) & \longrightarrow & T^j(ft(X),Y) & \longrightarrow & \dots & \longrightarrow & Y \\ \downarrow & & \downarrow & & & & \downarrow{\scriptstyle p_Y} \\ \vdots & & \vdots & & & & \vdots \\ \downarrow & & \downarrow & & & & \downarrow \\ X & \xrightarrow{p_X} & ft(X) & \xrightarrow{p_{ft(X)}} & \dots & \longrightarrow & ft^j(X) \end{array} \tag{13}$$

Problem 5.15 *[2016.02.20.prob3] For a C-system CC and $X, Y \in CC$ to construct a bijection*

$$bmor(X,Y) : BMor(X,Y) \to Mor_{CC}(X,Y)$$

Construction 5.16 [**2016.02.20.constr3**] By Construction 5.13 to Problem 5.11 we have a bijection

$$[\mathbf{2016.02.20.eq5b}] nt(l(Y), T^{l(X)}(X,Y)) : BMor(X,Y) \to \{f : T^j(X,Y) \to X \mid f \circ p_{T^j(X,Y),X} = Id_X\} \quad (14)$$

By Lemma 5.14 we have a pull-back square

$$\begin{array}{ccc} T^{l(X)}(X,Y) & \xrightarrow{q(p_{X,pt},Y)} & Y \\ {\scriptstyle p_{T^j(X,Y),X}}\downarrow & & \downarrow{\scriptstyle p_{Y,pt}} \\ X & \xrightarrow{p_{X,pt}} & pt \end{array}$$

Therefore, the function $f \mapsto f \circ q(p_{X,pt}, Y)$ is a bijection between the right hand side of (14) and $Mor_{CC}(X,Y)$. Composing this bijection with $nt(l(Y), T^{l(X)}(X,Y))$ we obtain a bijection

$$BMor(X,Y) \to Mor_{CC}(X,Y)$$

This completes Construction 5.16.

Lemma 5.17 *[2014.12.17.l1] Let BB be a non-unital pre-B-system. Then one has:*

1. *for $X \in B_{n+1}$, $Y \in B_{m+1}$ such that $ft(X) = ft^{m+1-n}(Y)$ and $m \geq n \geq 0$ one has:*

$$ft^k(T(X,Y)) = \begin{cases} T(X, ft^k(Y)) & \text{if } m-n \geq k \\ ft^{(k-1)-(m-n)}X & \text{if } m-n < k \end{cases} \quad (15)$$

2. *for $s \in \widetilde{B}_{n+1}$, $X \in B_{m+2}$ such that $\partial(s) = ft^{m+1-n}(X)$ and $m \geq n \geq 0$ one has:*

$$ft^k(S(s,X)) = \begin{cases} S(s, ft^k(X)) & \text{if } m-n \geq k \\ ft^{k-(m-n)}(\partial(s)) & \text{if } m-n < k \end{cases} \quad (16)$$

Proof: See T_l1 and S_l1 in [**?**].

In a B0-system let us denote by

$$T^j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}} (B_{m+1}) \to B_{m+1+j}$$

$$\widetilde{T}_j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \to \widetilde{B}_{m+1+j}$$

the maps which are defined inductively by

$$T^j(X,Y) = \begin{cases} Y & \text{if } j = 0 \\ T(X, T_{j-1}(ft(X), Y)) & \text{if } j > 0 \end{cases} \quad (17)$$

$$\widetilde{T}_j(X,s) = \begin{cases} s & \text{if } j = 0 \\ \widetilde{T}(X, \widetilde{T}_{j-1}(ft(X), s)) & \text{if } j > 0 \end{cases} \quad (18)$$

Note that for any $i = 0, \ldots, j$ we have

$$T^j(X,Y) = T_i(X, T_{j-i}(ft^i(X), Y))$$

and

$$\widetilde{T}_j(X,s) = \widetilde{T}_i(X, \widetilde{T}_{j-i}(ft^i(X), s))$$

Lemma 5.18 *[**Tnft**] One has*

$$T^j(X, ft(Y)) = ft(T^j(X,Y))$$

Proof: For $j = 0$ the statement is obvious. For $j > 0$ we have by induction on j

$$T_j(X, ft(Y)) = T(X, T_{j-1}(ft(X), ft(Y))) = T(X, ft(T_{j-1}(ft(X), Y))) =$$

$$= ft(T(X, T_{j-1}(ft(X), Y))) = ft(T^j(X,Y)).$$

Lemma 5.19 *[**l2014.10.10.l1**] Let B be a unital pre-B-system of the form $uB(CC)$. Then B is a unital B0-system.*

Proof: Straightforward.

Given a sequence of sets and maps such as (B_i, ft_i) and two elements $X \in B_m$, $Y \in B_n$ let us write $X \le Y$ if $m \le n$ and $X = ft^{n-m}Y$. This defines a reflexive transitive relation on $\amalg_{n \ge 0} B_n$.

Let us also denote by $\widetilde{B}(Y)$ the subset in $\widetilde{B}$ of elements r such that $\partial(r) = Y$.

Given a B0-system and two elements $X \in B_m$, $Y \in B_n$ let us define a set that we will denote later $Mor(X,Y)$. For the purpose of this construction we fix X and proceed to construct for each n and each $Y \in B_n$ a pair

1. a set $M_X(Y)$,
2. for any $f \in M_X(Y)$, $i \ge 0$ and $Y' \in B_{n+i}$ such that $Y' \ge Y$, an element $f^*(Y', i) \in B_{m+i}$,

such that:

1. for any $f \in M_X(Y)$ one has $f^*(Y, 0) = X$,
2. for any $f \in M_X(Y)$, $i, j \ge 0$, $Y' \in B_{n+i}$ such that $Y' \ge Y$ and $Y'' \in B_{n+i+j}$ such that $Y'' \ge Y'$ one has $f^*(Y'', i+j) \ge f^*(Y', i)$.

The construction will proceed by induction on n. For $n = 0$ we set $M_X(Y) = \{p\}$ and for $Y' \in B_i$:

$$p^*(Y') = \begin{cases} X & \text{if } i = 0 \\ T_m(X, Y') & \text{if } i > 0 \end{cases}$$

The second condition follows from Lemma 5.18.

Suppose now that $n > 0$. Then we set

$$M_X(Y) = \sum_{f \in M_X(ft(Y))} \widetilde{B}(f^*(Y,1))$$

For $Y' \in B_{n+i}$, $Y' \geq Y$ and $g = (f, r) \in M_X(Y)$ we define

$$f^*(Y', i) := S(r, f^*(Y, i+1))$$

The conditions are easily verified from the axioms of a B0-system.

Consider the unital B0-system $uB(CC)$ of a C-system CC.

Let $f : X \to Y$ be a morphism such that $X \in B_n$ and $Y \in B_m$. Define a sequence $(s_1(f), \ldots, s_m(f))$ of elements of $\widetilde{B}_{n+1}$ inductively by the rule

$$(s_1(f), \ldots, s_m(f)) = (s_1(ft(f)), \ldots, s_{m-1}(ft(f)), s_f) = (s_{ft^{m-1}(f)}, \ldots, s_{ft(f)}, s_f)$$

where $ft(f) = p_Y f$ and s_f is the s-operation of [**?**, Def. 2.2]. For $m = 0$ we start with the empty sequence. This construction can be illustrated by the following diagram for $f : X \to Y$ where $Y \in B_4$:

$$\begin{array}{ccccccccc}
X & \xrightarrow{s_4(f)} & Z_{4,3} & \longrightarrow & Z_{4,2} & \longrightarrow & Z_{4,1} & \longrightarrow & T_n(X,Y) & \longrightarrow & Y \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & X & \xrightarrow{s_3(f)} & Z_{3,2} & \longrightarrow & Z_{3,1} & \longrightarrow & T_n(X, ft(Y)) & \longrightarrow & ft(Y) \\
& & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & & X & \xrightarrow{s_2(f)} & Z_{2,1} & \longrightarrow & T_n(X, ft^2(Y)) & \longrightarrow & ft^2(Y) \\
& & & & & & \downarrow & & \downarrow & & \downarrow \\
& & & & & & X & \xrightarrow{s_1(f)} & T_n(X, ft^3(Y)) & \longrightarrow & ft^3(Y) \\
& & & & & & & & \downarrow & & \downarrow \\
& & & & & & & & X & \longrightarrow & pt
\end{array} \tag{19}$$

which is completely determined by the condition that the squares are the canonical ones and the composition of morphisms in the i-th arrow from the top is $ft^i(f)$. For the objects Z_i^j we have:

$$\begin{array}{lll}
Z_{4,1} = S(s_1(f), T_n(X,Y)) & Z_{4,2} = S(s_2(f), Z_{4,1}) & Z_{4,3} = S(s_3(f), Z_{4,2}) \\
Z_{3,1} = S(s_1(f), T_n(X, ft(Y))) & Z_{3,2} = S(s_2(f), Z_{3,1}) & \\
Z_{2,1} = S(s_1(f), T_n(X, ft^2(Y))) & &
\end{array} \tag{20}$$

A simple inductive argument similar to the one in the proof of [**?**, Lemma 4.1] show that if $f, f' : X \to Y$ are two morphisms such that $Y \in B_m$ and $s_i(f) = s_i(f')$ for $i = 1, \ldots, m$ then $f = f'$. Therefore, we may consider the set $Mor(CC)$ of morphisms of CC as a subset in $\amalg_{n,m \geq 0} B_n \times B_m \times \widetilde{B}_{n+1}^m$.

Let us show how to describe this subset in terms of the operations introduced above.

Lemma 5.20 *[**2009.11.07.l1**] An element $(X, Y, s_1, \ldots, s_m)$ of $B_n \times B_m \times \widetilde{B}_{n+1}^m$ corresponds to a morphism if and only if the element $(X, ft(Y), s_1, \ldots, s_{m-1})$ corresponds to a morphism and $\partial(s_m) = Z_{m,m-1}$ where $Z_{m,i}$ is defined inductively by the rule:*

$$Z_{m,0} = T_n(X, Y) \qquad Z_{m,i+1} = S(s_{i+1}, Z_{m,i})$$

Proof: Straightforward from the example considered above.

Let us show now how to identify the canonical morphisms $p_{X,i} : X \to ft^i(X)$ and in particular the identity morphisms.

Lemma 5.21 *[**2009.11.10.l1**] Let $X \in B_m$ and $0 \leq i \leq m$. Let $p_{X,i} : X \to ft^i(X)$ be the canonical morphism. Then one has:*

$$s_j(p_{X,i}) = \widetilde{T}_{m-j}(X, \delta_{ft^{m-j}(X)}) \qquad j = 1, \ldots, m-i$$

Proof: Let us proceed by induction on $m-i$. For $i = m$ the assertion is trivial. Assume the lemma proved for $i+1$. Since $ft(p_{X,i}) = p_{X,i+1}$ we have $s_j(p_{X,i}) = s_j(p_{X,i+1})$ for $j = 1, \ldots, m-i-1$. It remains to show that

$$[\mathbf{2009.11.10.eq1}]\, s_{m-i}(p_{X,i}) = \widetilde{T}_i(X, \delta_{ft^i(X)}) \qquad (21)$$

By definition $s_{m-i}(p_{X,i}) = s_{p_{X,i}}$ and (21) follows from the commutative diagram:

$$\begin{array}{ccccc}
X & \longrightarrow & ft^i(X) & & \\
{\scriptstyle s_p}\downarrow & & \downarrow{\scriptstyle \delta_{ft^i(X)}} & & \\
p^*_{X,i+1}(ft^i(X)) & \longrightarrow & p^*_{ft^i(X)}(ft^i(X)) & \longrightarrow & ft^i(X) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{ft^i(X)}} \\
X & \longrightarrow & ft^i(X) & \longrightarrow & ft^{i+1}(X)
\end{array}$$

where $p = p_{X,i}$.

Lemma 5.22 *[**2009.11.10.l2**] Let $(X, s) \in \widetilde{B}_{m+1}$, $X \in B_n$ and $f : X \to ft(Y)$. Define inductively $(f, i)^*(s) \in \widetilde{B}_{n+m+1-i}$ by the rule*

$$(f, 0)^*(s) = \widetilde{T}_n(X, s)$$

$$(f, i+1)^*(s) = \widetilde{S}(s_{i+1}(f), (f, i)^*(s))$$

Then $f^*(s) = (f, m)^*(s)$.

Proof: It follows from the diagram:

$$\begin{array}{ccccccccccc}
X & \xrightarrow{s_m(f)} & * & \longrightarrow & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(Y) \\
{\scriptstyle f^*(s)}\downarrow & & \downarrow{\scriptstyle (f,m-1)^*(s)} & & & & \downarrow{\scriptstyle (f,1)^*(s)} & & \downarrow{\scriptstyle (f,0)^*(s)} & & \downarrow{\scriptstyle s} \\
* & \longrightarrow & * & \longrightarrow & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & Y \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{s_m(f)} & * & \longrightarrow & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(Y) \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
 & & X & \xrightarrow{s_{m-1}(f)} & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft^2(Y) \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & \dots & & \dots & & \dots \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & X & \xrightarrow{s_1(f)} & * & \longrightarrow & ft^{m-1}(Y) \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & X & \longrightarrow & pt
\end{array}$$

Lemma 5.23 *Let $g : Z \to X$, $f : X \to Y$ and $Y \in B_m$. Then $s_i(fg) = g^* s_i(f)$.*

Proof: It follows immediately from the equations $s_{fg} = g^* s_f$ and $ft(fg) = ft(f)g$.

Lemma 5.24 *[***2009.11.10.l4a***] Let $f : X \to ft(Y)$ be a morphism, $X \in B_n$ and $Y \in B_{m+1}$. Define $(f,i)^*(Y)$ inductively by the rule:*

$$(f,0)^*(Y) = T_n(X,Y)$$

$$(f,i+1)^*(Y) = S(s_{i+1}(f),(f,i)^*(Y))$$

Then $f^*(Y) = (f,m)^*(Y)$.

Proof: Similar to the proof of Lemma 5.22.

Lemma 5.25 *[***2009.11.10.l4b***] Let $f : X \to ft(Y)$ be a morphism, $X \in B_n$ and $Y \in B_{m+1}$. Then*

$$s_i(q(f,Y)) = \begin{cases} \widetilde{T}(f^*Y, s_i(f)) & \text{if } i \leq m \\ \widetilde{T}(f^*Y, \delta_Y) & \text{if } i = m+1 \end{cases}$$

Proof: We have $s_i(q(f,Y)) = s_{ft^{m+1-i}(q(f,Y))}$. For $i \leq m$ we have

$$ft^{m+1-i}(q(f,Y)) = ft^{m-i}(f)p_{f^*Y}$$

Therefore,

$$s_{ft^{m+1-i}(q(f,Y))} = s_{ft^{m-i}(f)p_{f^*Y}} = p^*_{f^*Y} s_{ft^{m-i}(f)} = \widetilde{T}(f^*Y, s_i(f))$$

and for $i = m+1$ we have

$$s_i(q(f,Y)) = s_{q(f,Y)} = p^*_{f^*Y}(\delta_Y) = \widetilde{T}(f^*Y, \delta_Y).$$

The lemmas proved above show that a C-system can be reconstructed from the sets $B_n, \widetilde{B}_{n+1}$ and operations ft, ∂, T, $\widetilde{T}$, S, $\widetilde{S}$ and δ. This completes our proof of Theorem **??**.

6 B-systems

The next question that we want to address is the description of the image of the functor $CC \mapsto uB(CC)$. To make this question more precise we introduce below the concepts of non-unital and unital B-systems and formulate a problem whose solution would imply that the functor $CC \mapsto uB(CC)$ defines an equivalence between the category of C-systems and the full subcategory of the category of unital pre-B-systems that consists of unital B-systems.

For $X \in B_i$ let $B(X)_j$ denote the subset of B_{i+j} that consists of Y such that $ft^j(Y) = X$. In particular $B(X)_0$ is the one point subset $\{X\}$. Let also $\widetilde{B(X)}_j$ denote the subset of $\widetilde{B}_{i+j}$ that consists of r such that $ft^j(\partial(r)) = X$.

Then the operations T, $\widetilde{T}$, S and $\widetilde{S}$ can be seen as follows:

$$T(X, -) : B(ft(X))_* \to B(X)_*$$

$$\widetilde{T}(X, -) : \widetilde{B}(ft(X))_* \to \widetilde{B}(X)_*$$

$$S(s, -) : B(\partial(s))_* \to B(ft(\partial(s)))_*$$

$$\widetilde{S}(s, -) : \widetilde{B}(\partial(s))_* \to \widetilde{B}(ft(\partial(s)))_*$$

Definition 6.1 *[***2014.10.16.def2***] [***was.2014.06.18.eq2.to.eq11***] Let B be a non-unital B0-system. Define the following conditions on B:*

1. *The TT-condition. For all $GT \in B_{i+1}$, $GDT' \in B(ft(GT))_{j+1}$ one has*
   (a) *for all $R \in B(ft(GDT'))_{*+1}$*
   $$T(T(GT, GDT'), T(GT, R)) = T(GT, T(GDT', R))$$
   (b) *for all $r \in \widetilde{B}(ft(GDT'))_{*+1}$*
   $$\widetilde{T}(T(GT, GDT'), \widetilde{T}(GT, r)) = \widetilde{T}(GT, \widetilde{T}(GDT', r))$$
2. *The SS-condition. For all $s \in \widetilde{B}_{i+1}$, $s' \in \widetilde{B(\partial(s))}_{j+1}$ one has*
   (a) *for all $R \in B(\partial(s'))_*$*
   $$S(\widetilde{S}(s, s'), S(s, R)) = S(s, S(s', R))$$

(b) for all $r \in \widetilde{B}(\partial(s'))_*$

$$\widetilde{S}(\widetilde{S}(s,s'),\widetilde{S}(s,r)) = \widetilde{S}(s,\widetilde{S}(s',r))$$

3. The TS-condition. For any $s \in \widetilde{B}_{i+1}$ and $GTDT' \in \widetilde{B(\partial(s))}_{j+1}$ one has

   (a) for all $R \in B(ft(GTDT'))_*$

   $$T(S(s,GTDT'),S(s,R)) = S(s,T(GTDT',R))$$

   (b) for all $r \in \widetilde{B}(ft(GTDT'))_*$

   $$\widetilde{T}(S(s,GTDT'),\widetilde{S}(s,r)) = \widetilde{S}(s,\widetilde{T}(GTDT',r))$$

4. The ST-condition. For any $GT \in B_{i+1}$ and $s' \in \widetilde{B(ft(GT))}_{j+1}$ one has

   (a) for all $R \in B(\partial(s'))_*$

   $$S(\widetilde{T}(GT,s'),T(GT,R)) = T(GT,S(s',R))$$

   (b) for all $r \in \widetilde{B}(\partial(s'))_*$

   $$\widetilde{S}(\widetilde{T}(GT,s'),\widetilde{T}(GT,r)) = \widetilde{T}(GT,\widetilde{S}(s',r))$$

5. The STid-condition. For any $s \in \widetilde{B}_{i+1}$ one has

   (a) for all $R \in B(ft(\partial(s)))_*$

   $$S(s,T(\partial(s),R)) = R$$

   (b) for all $r \in \widetilde{B}(ft(\partial(s)))_*$

   $$\widetilde{S}(s,\widetilde{T}(\partial(s),r)) = r$$

Definition 6.2 *[***2014.10.20.def3***] Let B be a unital B0-system. Define the following conditions on B:*

1. The δT-condition. For any $GT \in B_{i+1}$ and $GDT' \in B(ft(GT))_{j+1}$ one has

   $$\widetilde{T}(GT,\delta(GDT')) = \delta(T(GT,GDT'))$$

2. The δS-condition. For any $s \in \widetilde{B}_{i+1}$ and $GTDT' \in B(\partial(s))_{j+1}$ one has

   $$\widetilde{S}(s,\delta(GTDT')) = \delta(S(s,GTDT'))$$

3. The δSid-condition. For any $s \in \widetilde{B}_{i+1}$ one has

   $$\widetilde{S}(s,\delta(\partial(s))) = s$$

4. The $S\delta T$-condition. For any $GT \in B_{i+1}$ one has

   (a) for $R \in B(GT)_{*+1}$ one has:

   $$S(\delta(GT),T(GT,R)) = R$$

(b) *for* $r \in \widetilde{B(GT)}_{*+1}$ *one has*

$$\widetilde{S}(\delta(GT), \widetilde{T}(GT, r)) = r$$

Remark 6.3 *[***2014.06.14.rem2***] The conditions defined above can be shown as follows:*

1. *The TT-condition:*

$$\dfrac{\Gamma, T \rhd \quad \Gamma, \Delta, T' \rhd \quad \Gamma, \Delta \vdash \mathcal{J}}{\dfrac{\Gamma, T \rhd \quad \Gamma, \Delta, T' \vdash \mathcal{J}}{\Gamma, T, \Delta, T' \vdash \mathcal{J}} \quad \dfrac{\Gamma, T, \Delta, T' \rhd \quad \Gamma, T, \Delta \vdash \mathcal{J}}{\Gamma, T, \Delta, T' \vdash \mathcal{J}}}$$

2. *The SS-condition:*

$$\dfrac{\Gamma \vdash s : T \quad \Gamma, T, \Delta \vdash s' : T' \quad \Gamma, T, \Delta, T' \vdash \mathcal{J}}{\dfrac{\Gamma \vdash s:T \quad \Gamma, T, \Delta \vdash \mathcal{J}[s]}{\Gamma, \Delta[s] \vdash \mathcal{J}[s'][s]} \quad \dfrac{\Gamma, \Delta[s] \vdash s'[s] : T'[s] \quad \Gamma, \Delta[s], T'[s] \vdash \mathcal{J}[s]}{\Gamma, \Delta[s] \vdash \mathcal{J}[s][s']}}$$

3. *The TS-condition:*

$$\dfrac{\Gamma, T \rhd \quad \Gamma, \Delta \vdash s' : T' \quad \Gamma, \Delta, T' \vdash \mathcal{J}}{\dfrac{\Gamma, T \rhd \quad \Gamma, \Delta \vdash \mathcal{J}[s']}{\Gamma, T, \Delta \vdash \mathcal{J}[s']} \quad \dfrac{\Gamma, T, \Delta \vdash s' : T' \quad \Gamma, T, \Delta, T' \vdash \mathcal{J}}{\Gamma, T, \Delta \vdash \mathcal{J}[s']}}$$

4. *The ST-condition:*

$$\dfrac{\Gamma \vdash s : T \quad \Gamma, T, \Delta, T' \rhd \quad \Gamma, T, \Delta \vdash \mathcal{J}}{\dfrac{\Gamma \vdash s:T \quad \Gamma, T, \Delta, T' \vdash \mathcal{J}[s]}{\Gamma, \Delta[s], T'[s] \vdash \mathcal{J}[s]} \quad \dfrac{\Gamma, \Delta[s], T'[s] \rhd \quad \Gamma, \Delta[s] \vdash \mathcal{J}[s]}{\Gamma, \Delta[s], T'[s] \vdash \mathcal{J}[s]}}$$

5. *The STid-condition:*

$$\dfrac{\Gamma \vdash s : T \quad \Gamma, T \rhd \quad \Gamma \vdash \mathcal{J}}{\dfrac{\Gamma \vdash s:T \quad \Gamma, T \vdash \mathcal{J}}{\Gamma \vdash \mathcal{J}[s]}}$$

6. *The δT-condition:*

$$\dfrac{\Gamma, T \rhd \quad \Gamma, \Delta, x : T' \rhd}{\dfrac{\Gamma, T \rhd \quad \Gamma, \Delta, x:T' \vdash x:T'}{\Gamma, T, \Delta, x:T' \vdash x:T'} \; \dfrac{\Gamma, T, \Delta, x:T' \rhd}{\Gamma, T, \Delta, x:T' \vdash x:T'}}$$

7. *The δS-condition:*

$$\dfrac{\Gamma \vdash s : T \quad \Gamma, T, \Delta, x : T' \rhd}{\dfrac{\Gamma \vdash s:T \quad \Gamma, T, \Delta, x:T' \vdash x:T'}{\Gamma, \Delta[s], x:T'[s] \vdash x:T'[s]} \; \dfrac{\Gamma, \Delta[s], x:T[s]' \rhd}{\Gamma, \Delta[s], x:T'[s] \vdash x:T'[s]}}$$

8. *The δSid-condition:*

$$\dfrac{\Gamma \vdash s : T \quad \Gamma, x : T \rhd}{\dfrac{\Gamma \vdash s:T \quad \Gamma, x:T \vdash x:T}{\Gamma \vdash s:T}}$$

9. *The $S\delta T$-condition:*

$$\dfrac{\Gamma, y : X, \Delta \vdash \mathcal{J}}{\dfrac{\Gamma, y_1:X, y:X, \Delta \vdash \mathcal{J} \quad \Gamma, y_1:X \vdash y_1:X}{\Gamma, y_1:X, \Delta[y_1/y] \vdash \mathcal{J}[y_1/y]}}$$

Lemma 6.4 *[***2014.10.20.l1***] [***2014.10.16.l1***] Let B be a unital B0-system and let δ_1, δ_2 be two families of operations as in Definition 3.4. Suppose that both δ_1 and δ_2 satisfy the δT, δSid and $S\delta T$ conditions. Then $\delta_1 = \delta_2$.*

Proof: We have:

$$\delta_1(GT) = \widetilde{S}(\delta_2(GT), \widetilde{T}(GT, \delta_1(GT))) = \widetilde{S}(\delta_2(GT), \delta_1(T(GT, GT))) = \delta_2(GT)$$

where the first equality is the $S\delta T$-condition for δ_2, the second equality is the δT-condition for δ_1 and the third equality is the δSid-condition for δ_1.

Definition 6.5 *[**2014.10.10.def2a**] [**2014.10.20.def4**] A non-unital B-system is a non-unital B0-system that satisfy the conditions TT, SS, TS, ST and $STid$ of Definition 6.1.*

Definition 6.6 *[**2014.10.10.def2b**] [**2014.10.20.def5**] A unital B-system is a unital B0-system that satisfy the conditions TT, SS, TS, ST, $STid$ of Definition 6.1 and the conditions δT, δS, δSid and $S\delta T$ of Definition 6.2.*

Equivalently, a unital B-system is non-unital B-system such that there exists a family of operations δ satisfying the conditions δT, δS, δSid and $S\delta T$ of Definition 6.2.

Example 6.7 [**2014.10.20.eX**] While being unital is a property of non-unital B-systems not any homomorphism of non-unital B-systems preserves units. Here is a sketch of an example of a homomorphism that does not preserve units.

Consider the following pairs of a monad and a left module over it. In both cases pt is the constant functor corresponding to the one point set $\{T\}$ that has a unique left module structure over any monad.

1. (R_1, pt) where R_1 is the monad corresponding to one unary operation $s_1(x)$ and the relation

$$s_1(s_1(x)) = s_1(x)$$

2. (R_2, pt) where R_2 is the monad corresponding to two unary operations $s_1(x)$ and $s_2(x)$ and relations:

$$s_1(s_1(x)) = s_1(x) \quad s_1(s_2(x)) = s_1(x) \quad s_2(s_1(x)) = s_1(x) \quad s_2(s_2(x)) = s_2(x)$$

Consider the unital B-systems $uB(R_1, pt)$ and $uB(R_2, pt)$. In $uB(R_1, pt)$ consider the non-unital sub-B-system nuB_1 generated by $(T \vdash s_1(1) : T)$. In $uB(R_2, pt)$ consider the non-unital sub-B-system nuB_2 generated by $(T \vdash s_1(1) : T)$ and $(T \vdash s_2(1) : T)$.

Observe that both nuB_1 and nuB_2 are in fact unital with the unit in the first one given by $(T, \ldots, T \vdash s_1(n) : T)$ and unit in the second one is given by $(T, \ldots, T \vdash s_2(n) : T)$ where n is the number of T's before the turnstile $\vdash$ symbol.

We also have an obvious (unital) homomorphism from $uB(R_1, pt)$ to $uB(R_2, pt)$ that defines a homomorphism $nuB_1 \to nuB_2$ and that latter homomorphism is not unital.

Remark 6.8 For a unital B-systems operations S and T can be expressed as follows.

$$[\mathbf{2014.10.14.eq1}]T(X, Y) = \begin{cases} X & \text{if } l(Y) = l(X) - 1 \\ ft(\partial(\widetilde{T}(X, \delta(Y)))) & \text{if } l(Y) \geq l(X) \end{cases} \tag{22}$$

$$[\mathbf{2014.10.14.eq2}]S(s, X) = \begin{cases} ft(\partial(s)) & \text{if } l(X) = l(\partial(s)) \\ ft(\partial(\widetilde{S}(s, \delta(X)))) & \text{if } l(X) > l(\partial(s)) \end{cases} \tag{23}$$

I would like to end this section with the formulation of the following problem. I am reasonably sure that it has a straightforward solution.

Problem 6.9 *[***2014.10.10.prob2***] To show that a unital B0-system is isomorphic to a unital B0-system of the form $uB(CC)$ if and only if it is a unital B-system.*

7 B-systems in Coq

While our main interest is in pre-B-systems and B-systems in sets we would like to be able to formalize their definitions in Coq without assuming that B_n and $\widetilde{B}_{n+1}$ are of h-level 2.

This suggests the following reformulation of our definitions. In what follows we give a presentation of non-unital B-systems in "functional terms". The presentation of the axioms related to the δ-operations is more complex as can be see already in the case of the δT-axiom and we leave it for the future.

Let us define a tower as a sequence of functions $T := (\ldots \to T_{i+1} \stackrel{p_i}{\to} T_i \to \ldots \to T_0)$.

For a tower T and $i, j \geq 0$ define $ft_i^j : T_{i+j} \to T_i$ as the composition of the functions p_k for $k = i, \ldots, i+j-1$. When no ambiguity can arise we will write ft^j instead of ft_i^j and we will write ft instead of ft^1.

For a tower T, $i \geq 0$ and $G \in T_i$ define a new tower $T(G)$ setting:

$$T(G)_j = \{GD \in T_{i+j} | ft_i^j(x) = G\}$$

and defining the functions $T(G)_{j+1} \to T(G)_j$ in the obvious way. More categorically this can expressed by saying that $T(G)_j$ is defined by the standard (homotopy) pull-back square

$$\begin{array}{ccc} T(G)_j & \longrightarrow & T_{i+j} \\ \downarrow & & \downarrow{\scriptstyle ft_i^j} \\ pt & \stackrel{G}{\longrightarrow} & T_i \end{array}$$

For $G \in T_{i+j}$ we let $\phi_j(G) \in T(ft^{i+j}(G))_j$ denote the obvious element.

For towers T and T' define a function or morphism of towers $F : T \to T'$ as a sequence of morphisms $F_i : T_i \to T'_i$ which commute in the obvious sense with the functions p_i and p'_i.

The identity function of towers id_T and the composition of functions of towers are defined in the obvious way.

For T, $i, j, k \geq 0$, $G \in T_i$ and $GD \in T^j(G)$ we have the digrams:

$$\begin{array}{ccccc} T(G)(GD)_k & \longrightarrow & T(G)_{j+k} & \longrightarrow & T_{i+(j+k)} \\ \downarrow & & \downarrow{\scriptstyle ft^k_{T(G),j}} & & \downarrow \\ pt & \stackrel{GD}{\longrightarrow} & T(G)_j & \stackrel{u_{G,j}}{\longrightarrow} & T_{i+j} \\ & & \downarrow & & \downarrow{\scriptstyle ft^j_{T,i}} \\ & & pt & \stackrel{G}{\longrightarrow} & T_i \end{array} \qquad \begin{array}{ccc} T(u_{G,i}(GD))_k & \longrightarrow & T_{(i+j)+k} \\ \downarrow & & \downarrow \\ pt & \underset{u_{G,j}(GD)}{\longrightarrow} & T_{i+j} \end{array}$$

which shows that we have natural equivalences (isomorphisms)

$$[\mathbf{2014.06.12}]T(G)(GD)_k \cong T(u_{G,j}(GD))_k \tag{24}$$

The equivalences (24) commute with the functions $p(G)(GD)_i$ and $p(u_{G,j}(GD))$ in the obvious sense and define an equivalence of towers

$$[\mathbf{2014.06.14.eq2}]T(G)(GD) \cong T(u_{G,j}(GD)) \tag{25}$$

Remark 7.1 In the case when standard pull-backs are pull-backs in a category, the functions $u_{G,j}$ from $T^j(G)$ to T_{i+j} are pull-backs of (split) monomorphisms and therefore are monomorphisms. In this case $T_k(G)(GD)$ is a sub-object of $T_{i+(j+k)}$ and $T_k(u_{G,j}(GD))$ is a sub-object of $T_{(i+j)+k}$ which are canonically equal. Then we can say that

$$[\mathbf{2014.06.14.eq1}]T(G)(GD)_k = T(u_{G,j}(GD))_k \tag{26}$$

where the equality is the equality of sub-objects of $T_{(i+j)+k}$.

More generally, if T_i are objects of h-level 2, the functions $u_{G,j}$ are of h-level 1 (monic inclusions) and we again can say that the equality (26) holds as the unique equality of monic sub-objects of $T_{(i+j)+k}$.

For a function $F : T \to T'$ and $G \in T_i$ we obtain a function $F(G) : T(G) \to T'(G)$ using functoriality of standard pull-backs.

Define a B-system carrier or a B-carrier as a pair $\mathbf{B} = (B, \widetilde{B})$ where B is a tower such that $B_0 = \{pt\}$ and $\widetilde{B}$ is a family $\widetilde{B}_{i+1}$, $i \geq 0$ together with functions $\partial_i : \widetilde{B}_{i+1} \to B_{i+1}$. The B-system carriers in sets are the same as the "type-and-term structures" of [**?**].

We will denote the standard fiber of ∂_i over $GT \in B_{i+1}$ by $\widetilde{B}_{GT}$.

For a B-carrier $\mathbf{B}$, $i \geq 0$ and $G \in B_i$, define a B-carrier $\mathbf{B}(G)$ as the pair $(B(G), \widetilde{B(G)})$ where

$$\widetilde{B(G)}_{j+1} = \{s \in \widetilde{B}_{i+j+1} | \partial(s) \in B(G)_{j+1}\}$$

or, categorically, $\widetilde{B(G)}_{j+1}$ is defined by the standard pull-back square

$$\begin{array}{ccc} \widetilde{B(G)}_{j+1} & \xrightarrow{\widetilde{u}_{G,j+1}} & \widetilde{B}_{i+(j+1)} \\ {\scriptstyle \partial(G)}\downarrow & & \downarrow{\scriptstyle \partial} \\ B(G)_{j+1} & \xrightarrow{u_{G,j+1}} & B_{i+(j+1)} \end{array}$$

For a B-carrier $\mathbf{B}$, $i, j \geq 0$, $G \in B_i$ and $GD \in B_{i+j}$ the equivalence (25) clearly extends to an equivalence

$$[\mathbf{2014.06.14.eq3}]\mathbf{B}(G)(GD) \cong \mathbf{B}(u_G(GD)) \tag{27}$$

For B-carriers $\mathbf{B}$ and $\mathbf{B}'$ define a function of B-carriers $\mathbf{F} : \mathbf{B} \to \mathbf{B}'$ as a pair $\mathbf{F} = (F, \widetilde{F})$ where $F : B \to B'$ is a function of towers and for every $i \geq 0$, $\widetilde{F}_{i+1}$ is a function $\widetilde{B}_{i+1} \to \widetilde{B}'_{i+1}$ which commutes in the obvious sense with the functions ∂', F_{i+1} and ∂.

The identity function of B-carriers $id_{\mathbf{B}}$ and the composition of functions of B-carriers are defined in the obvious way.

For a function of B-carriers $\mathbf{F} : \mathbf{B} \to \mathbf{B}'$ and $G \in B_i$ we obtain a function of B-carriers $\mathbf{F}(G) : \mathbf{B}(G) \to \mathbf{B}'(F(G))$ using functoriality of standard pull-backs.

Definition 7.2 *[**Bdata**] Non-unital B-system data is given by the following:*

1. *a B-system carrier* $\mathbf{B}$,
2. *for every* $m \geq 0$*,* $X \in B_{n+1}$ *a B-carrier function* $\mathbf{T}_X : \mathbf{B}(p_n(X)) \to \mathbf{B}(X)$*,*
3. *for every* $m \geq 0$*,* $s \in \widetilde{B}_{n+1}$*, a B-carrier function* $\mathbf{S}_s : \mathbf{B}(\partial(s)) \to \mathbf{B}(p_n(\partial(s)))$*,*

Problem 7.3 *[**2014.10.10.prob1**] Construct an equivalence between the type of non-unital B0-systems and the type of non-unital B-system data such that the types* B_* *and* $\widetilde{B}$ *are sets.*

Construction 7.4 [**2014.10.10.constr1**] A non-unital B-system carrier is the same as two families of sets B_n, $\widetilde{B}_{n+1}$ together with maps $p_n : B_{n+1} \to B_n$ and $\partial : \widetilde{B_{n+1}} \to B_{n+1}$.

For a given $X \in B_{n+1}$ a B-carrier function $\mathbf{T}_X : \mathbf{B}(ft(X)) \to \mathbf{B}(X)$ is the same as:

1. for all $i \geq 0$, $Y \in B_{n+i}$ such that $ft^i(Y) = ft(X)$, an element $T(X, Y) \in B_{n+i+1}$ such that $ft^i(T(X, Y)) = X$,
2. for all $i \geq 0$, $r \in \widetilde{B}_{n+i+1}$ such that $ft^{i+1}(\partial(r)) = ft(X)$, an element $\widetilde{T}(X, r)$ such that $ft^{i+1}(\partial(r)) = ft(X)$.

For $i = 0$, the operation T is uniquely determined by the condition $ft^i(T(X, Y)) = X$ which leaves us with the operations T and $\widetilde{T}$ as in Definition 3.3 satisfying the conditions of Lemma 5.19.

The same reasoning applies to S, $\widetilde{S}$.

From this point on everything is assumed to be non-unital. Let $\mathbf{BD} = (\mathbf{B}, \mathbf{T}, \mathbf{S}, \delta)$ be B-data and $G \in B_i$. Define B-data $\mathbf{BD}(G)$ over G as follows. The B-carrier of $\mathbf{BD}(G)$ is $\mathbf{B}(G)$.

For $GDT \in B(G)_{i+1}$ we need to define a B-carrier function

$$\mathbf{T}(G)_{GDT} : \mathbf{B}(G)(p_i(GDT)) \to \mathbf{B}(G)(GDT)$$

We define it through the condition of commutativity of the pentagon:

[**2014.06.14.eq4**]

$$\begin{array}{ccc} \mathbf{B}(G)(p_i(GDT)) & \xrightarrow{\mathbf{T}(G)_{GDT}} & \mathbf{B}(G)(GDT) \\ \cong\downarrow & & \downarrow\cong \\ \mathbf{B}(u_G(p_i(GDT))) \cong \mathbf{B}(p_i(u_G(GDT))) & \xrightarrow{\mathbf{T}_{-}} & \mathbf{B}(u_G(GDT)) \end{array} \quad (28)$$

where the vertical equivalences are from (27).

Similarly for $s \in \widetilde{B(G)}_{j+1}$ we define a B-carrier function

$$\mathbf{S}(G)_s : \mathbf{B}(G)(\partial(s)) \to \mathbf{B}(G)(p_j(\partial(s)))$$

by the diagram:

$$
[\mathbf{2014.06.14.eq5}] \qquad \begin{array}{ccc} \mathbf{B}(G)(\partial(s)) & \xrightarrow{\mathbf{S}(G)_s} & \mathbf{B}(G)(p_j(\partial(s))) \\ \downarrow & & \downarrow \\ \mathbf{B}(u_G(\partial(s))) \cong \mathbf{B}(\partial(\widetilde{u}_G(s))) & \xrightarrow{\mathbf{S}(\widetilde{u}_G(s))} & \mathbf{B}(p_j(\partial(\widetilde{u}_G(s)))) \cong \mathbf{B}(u_G(p_j(\partial(s)))) \end{array} \tag{29}
$$

We can now give formulations for the conditions TT, SS, TS, ST and STid.

Definition 7.5 *[**2014.10.16.def3.fromold**] Let us define the following conditions on a B-system data* $(\mathbf{B}, \mathbf{T}, \mathbf{S}, \delta)$*:*

1. *The TT-condition. For any* $GT \in B_{i+1}$, $GDT' \in B_{j+1}(p_i(GT))$ *the pentagon of B-carrier functions*

$$
\begin{array}{ccc} \mathbf{B}(p_i(GT))(p_j(GDT')) & \xrightarrow{\mathbf{T}(p_i(GT))_{GDT'}} & \mathbf{B}(p_i(GT))(GDT') \\ \mathbf{T}_{GT}(p_j(GDT'))\downarrow & & \\ \mathbf{B}(GT)(T_{GT}(p_j(GDT'))) & & \downarrow \mathbf{T}_{GT}(GDT') \\ \cong\downarrow & & \\ \mathbf{B}(GT)(p_j(T_{GT}(GDT'))) & \xrightarrow{\mathbf{T}_{T_{GT}(GDT')}(GT)} & \mathbf{B}(GT)(\mathbf{T}_{GT}(GDT')) \end{array} \tag{30}
$$

   commutes.

2. *The SS-condition. For any* $s \in \widetilde{B}_{i+1}$, $s' \in \widetilde{B}_{j+1}(\partial(s))$ *the diagram of B-carrier functions*

$$
\begin{array}{ccc} \mathbf{B}(\partial(s))(\partial(s')) & \xrightarrow{\mathbf{S}(\partial(s))_{s'}} & \mathbf{B}(\partial(s))(p_j(\partial(s'))) \\ \mathbf{S}_s(\partial(s'))\downarrow & & \downarrow \mathbf{S}_s(p_j(\partial(s'))) \\ \mathbf{B}(p_i(\partial(s)))(S_s(\partial(s'))) & & \mathbf{B}(p_i(\partial(s))(S_s(p_j(\partial(s'))))) \\ \cong\downarrow & & \downarrow\cong \\ \mathbf{B}(p_i(\partial(s)))(\partial(\widetilde{S}_s(s'))) & \xrightarrow{\mathbf{S}(p_i(\partial(s)))_{\widetilde{S}_s(s')}} & \mathbf{B}(p_i(\partial(s)))(p_j(\partial(\widetilde{S}_s(s')))) \end{array} \tag{31}
$$

   commutes.

3. *The TS-condition. For any* $GT \in B_{i+1}$, $s' \in \widetilde{B}_{j+1}(p_i(GT))$ *the diagram of B-carrier functions*

$$
\begin{array}{ccc} \mathbf{B}(p_i(GT))(\partial(s')) & \xrightarrow{\mathbf{S}(p_i(GT))_{s'}} & \mathbf{B}(p_i(GT))(p_j(\partial(s'))) \\ \mathbf{T}_{GT}(\partial(s'))\downarrow & & \downarrow \mathbf{T}_{GT}(p_j(\partial(s'))) \\ \mathbf{B}(GT)(T_{GT}(\partial(s'))) & & \mathbf{B}(GT)(T_{GT}(p_j(\partial(s')))) \\ \cong\downarrow & & \downarrow\cong \\ \mathbf{B}(GT)(\partial(\widetilde{T}_{TG}(s'))) & \xrightarrow{\mathbf{S}(GT)_{\widetilde{T}_{GT}(s')}} & \mathbf{B}(GT)(p_j(\partial(\widetilde{T}_{GT}(s')))) \end{array} \tag{32}
$$

4. *The ST-condition. For any $s \in \widetilde{B}_{i+1}$, $GTDT' \in B_{j+1}(\partial(s))$ the diagram of B-carrier functions*

$$\begin{array}{ccc}
\mathbf{B}(\partial(s))(p_j(GTDT')) & \xrightarrow{\mathbf{T}(\partial(s))_{GTDT'}} & \mathbf{B}(\partial(s))(GTDT') \\
\mathbf{S}_s(p_j(GTDT'))\downarrow & & \\
\mathbf{B}(p_i(\partial(s)))(S_s(p_j(GTDT'))) & & \downarrow\mathbf{S}_s(GTDT') \\
\cong\downarrow & & \\
\mathbf{B}(p_i(\partial(s)))(p_j(S_s(GTDT'))) & \xrightarrow{\mathbf{T}(p_i(\partial(s)))_{S_s(GTDT')}} & \mathbf{B}(p_i(\partial(s)))(S_s(GTDT'))
\end{array} \quad (33)$$

5. *The STid-condition. For any $s \in \widetilde{B}_{i+1}$ one has*

$$(\mathbf{B}(p_i(\partial(s))) \stackrel{T_{\partial(s)}}{\to} \mathbf{B}(\partial(s)) \stackrel{S_s}{\to} \mathbf{B}(p_i(\partial(s)))) = id_{\mathbf{B}(p_i(\partial(s))}$$

Formulation of the remaining four conditions that involve δ is more difficult since their formulation using this approach leads to conditions that depend on the conditions from the first group. We leave their study for the future.

8 An approach to B-systems using the length function.

In formalization of B-systems (as well as C-systems) in Coq one of the main technical difficulties that arises is the need to work with a family of types B_n which are dependent on $n \in \mathbf{N}$. Due to the absence of strong substitutional equality in Coq types such as $B_{n+(m+1)}$ and $B_{(n+m)+1}$ do not have same elements and can only be dealt with as being connected by an equivalence. Eventually we hope that this issue will be resolved but at the moment an alternative approach to formalization where the families of types B_n and $\widetilde{B}_n$ are replaced by their total spaces together with the functions from these total spaces to $\mathbf{N}$ may be useful.

In this approach we will have only two sorts B and $\widetilde{B}$ but the presentation will cease to be essentially algebraic.

Instead we consider the following:

Definition 8.1 *[**2014.10.26.def1**] A non-unital pre-l-B-system (in sets) is the following collection of data:*

1. *two sets B and $\widetilde{B}$,*
2. *a function $l : B \to \mathbf{N}$,*
3. *a function $\partial : \widetilde{B} \to B$ such that for all $s \in \widetilde{B}$, $l(\partial(s)) > 0$,*
4. *a function $ft : B \to B$ such that*
   (a) *for all b such that $l(b) > 0$ one has $l(ft(b)) = l(b) - 1$,*
   (b) *for all b such that $l(b) = 0$ one has $l(ft(b)) = 0$,*

5. *for each $i \geq 0$ four operations:*

$$T_i : (X \in B, Y \in B, l(X) > 0, l(Y) > i, ft(X) = ft^{i+1}(Y)) \to B$$

$$\widetilde{T}_i : (X \in B, r \in \widetilde{B}, l(X) > 0, l(\partial(r)) > i, ft(X) = ft^{i+1}(\partial(r))) \to \widetilde{B}$$

$$S_i : (s \in \widetilde{B}, Y \in B, \partial(s) = ft^{i+1}(Y)) \to B$$

$$\widetilde{S}_i : (s \in \widetilde{B}, r \in \widetilde{B}, \partial(s) = ft^{i+1}(\partial(r))) \to \widetilde{B}$$

such that:

(a) $l(T_i(X, Y)) = l(Y) + 1$,
(b) $l(\partial(\widetilde{T}_i(X, r))) = l(\partial(r)) + 1$,
(c) $l(S_i(s, X)) = l(X) - 1$,
(d) $l(\partial(\widetilde{S}_i(s, r))) = l(\partial(r)) - 1$.

Definition 8.2 *[***2014.10.26.def2***] A unital pre-l-B-system is a non-unital pre-l-B-system together with an operation*

$$\delta : (X \in B, l(X) > 0) \to \widetilde{B}$$

such that $l(\partial(\delta(X))) = l(X) + 1$.

Definition 8.3 *[***2014.12.05.def3***] A pre-l-B-system is a pre-l-B-system together with an element $pt \in B$ such that $l(pt) = 0$.*

It is easy now to define non-unital and unital l-B0-systems and l-B-systems.