B-systems[1]

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Abstract

B-systems are algebras (models) of an essentially algebraic theory that is expected to be constructively equivalent to the essentially algebraic theory of C-systems which is, in turn, constructively equivalent to the theory of contextual categories. The theory of B-systems is closer in its form to the structures directly modeled by contexts and typing judgements of (dependent) type theories and further away from categories than contextual categories and C-systems.

1 Introduction

In [**?**, Def. 2.2] we introduced the concept of a C-system. The type of the C-systems is constructively equivalent to the type of contextual categories defined by Cartmell in [**?**] and [**?**] but the definition of a C-system is slightly different from the Cartmell's foundational definition.

The concept of a B-system is introduced in this paper. It provides an abstract formulation of a structure formed by contexts and "typing judgements" of a type theory relative to the operations of context extensions, weakening and substitutions.

The important difference between B-systems and C-systems is that in B-systems there are no sorts for morphisms between contexts. There are only sorts for contexts of each lengths and for typing judgements, i.e., judgements whose meaning is that a given object has a given type in a given context. This gives us two infinite families of sorts B_n, for contexts of length n, and $\widetilde{B}_{n+1}$, for judgements of the form $\Gamma \vdash o : T$ where $l(\Gamma) = n$.

The operations on these sorts correspond to the empty context (pt), truncation of contexts (ft), taking extended context of a typing judgement (∂), weakening on contexts (T), weakening on typing judgements ($\widetilde{T}$), substitution on contexts (S), substitution on typing judgements ($\widetilde{S}$) and units, also known as projections, (δ).

Of these operations pt, ft, ∂ and δ are everywhere defined while $T, \widetilde{T}, S$ and $\widetilde{S}$ are partially defined with the domains of definition being given by equations that involve only everywhere defined operations ft and ∂.

We may say that operations pt, ft, ∂ and δ are of depth 0 while operations $T, \widetilde{T}, S$ and $\widetilde{S}$ are of depth 1.

We call the structures formed by these sorts and operations with no relations imposed on them pre-B-systems. We distinguish between unital and non-unital pre-B-systems depending on whether operations δ are considered or not. Pre-B-systems are models of an essentially algebraic theory of depth 1 with two infinite families of sorts. The importance of this concept is that while it

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is sufficiently easy to define, categorically it provides a lot of information since homomorphisms between models of essentially algebraic theories depend only on the operations of these theories but not on relations between them.

We next show (Theorem 2.3) that the constructions of [?] define for every C-system a unital pre-B-system and that the first main theorem of [?] can be restated by saying that this theorem establishes a bijection between the C-subsystems of a C-system and sub-pre-B-systems of the corresponding unital pre-B-system.

Theorem 2.4 is saying that the construction of a pre-B-system from a C-system extends to a functor and that this functor is a full embedding. The sketch of a proof of this theorem that we give occupies the rest of Section **??**.

First we define the concept of a B0-system (Definition 2.5) that adds to the concept of a pre-B-system the axioms that include compositions of operations $T, \widetilde{T}, S, \widetilde{S}$ and δ with the everywhere defined operations ft and ∂. We again distinguish between the unital and non-unital cases.

In **??** we show that the pre-B-system defined by a C-system is a B0-system.

We then construct, for any (unital) B0-system BB and any two objects $X \in B_m$, $Y \in B_n$ a set $Mor(X, Y)$ in such a way that when BB is the B0-system that is defined by a C-system the sets $Mor(X, Y)$ are in natural bijection with the sets of morphisms in the C-system.

The construction of the Mor-sets is obviously functorial with respect to homomorphisms of B0-systems.

On the other hand we prove Proposition **??** which shows that for C-systems CC_1, CC_2 and a pair of functions $F_{Ob} : Ob(CC_1) \to Ob(CC_2)$ and $F_{Mor} : Mor(CC_1) \to Mor(CC_2)$ that commute with the source and target maps ∂_0, ∂_1 the condition that $F = (F_{Ob}, F_{Mor})$ is a functor is equivalent to the condition that F is compatible with a set of ??? operations, which does not include the composition operation.

Finally we show that F as above that arises from a homomorphism of the B0-systems corresponding to CC_1 and CC_2 commutes with operations from this list. This completes the proof of Theorem 2.4.

??? Remind that we are using the diagramatic ordering for compositions of morphisms and of maps between sets.

We next start looking for the set of axioms on a pre-B-system that will characterize the image of this functor. We introduce the candidate set of axioms in several layers.

These operations are subject to a number of axioms. We conjecture that the type of B-systems is constructively equivalent to the type of C-systems. A conjecture formulated in more traditional terms would say that the category of B-systems and their homomorphisms is equivalent to the category of C-systems and their homomorphisms. While these two conjectures are not equivalent the former expresses much of what the latter would be used for in practice.

Proving this conjecture is difficult because the definition of sets of morphisms between elements $X \in B_m$, $Y \in B_n$ of a B-system is based on an induction

We define B-systems in several steps. First we describe pre-B-systems that are models of an essentially algebraic theory with countable families of sorts and operations but no relations.

Already at this stage we start to distinguish between unital and non-unital (pre-)B-systems. This distinction continues throughout the paper. While non-unital B-systems have no direct connection to C-systems and therefore no direct connection to categories they have a definition with interesting symmetries and we believe that they are quite interesting in there own right.

Following the ideas of [**?**] we show how to construct a unital pre-B-system from a C-system. This construction is functorial with respect to homomorphisms of C-systems and unital pre-B-systems and moreover defines a full embedding of the category of C-systems to the category of unital pre-B-systems.

It is more or less clear from the proof of the full embedding theorem that the image of this full embedding consists of unital pre-B-systems whose operations satisfy some algebraic conditions. We suggest a form of these conditions in our definition of a non-unital and then unital B-system (Definitions 3.5 and 3.6).

We conclude the first part of the paper with a problem (essentially a conjecture) that the image of the full embedding from C-systems to unital pre-B-systems is precisely the class of unital B-systems. A constructive solution to this problem would also provide an explicit construction of a C-system from a unital B-system.

In the second part we describe an approach to the definition of non-unital B-systems that can be conveniently formalized in Coq and that provide a possible step towards the definition of higher B-systems that is B-systems whose component types are of higher h-levels.

The work on this paper, especially in the part where the axioms TT, SS, TS and ST of B-systems are introduced was influenced and facilitated by recent discussions with Richard Garner and Egbert Rijke. Many other ideas of this work go back to [**?**].

The subject of this paper is closely related to the subject of recent notes by John Cartmell [**?**]. The most important difference between our exposition and that of Cartmell is that we are using the formalism of *essentially algebraic* theories while Cartmell uses the formalism of *generalized algebraic* theories. While there are important connections between these two kinds of theories there are also important distinctions which we intend to discuss in a future paper.

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2 pre-B-systems

Let $B_n, n \geq 0$ be a sequence of sets. Let $ft_i : B_{i+1} \to B_i$ be a sequence of maps. We will simplify our notations by writing ft instead of ft_i and writing $ft^n : B_{i+n} \to B_i$ for the composition $ft_{i+(n-1)} \circ \ldots \circ ft_i$ including writing ft^0 for the identities of B_i.

Definition 2.1 *[**2014.10.10.def1**] A non-unital pre-B-system a collection of data of the following form:*

1. *for all $n \in \mathbf{N}$ two set B_{n+1} and $\widetilde{B}_{n+1}$,*
2. *for all $n \in \mathbf{N}$ maps of the form:*
   (a) $ft : B_{n+1} \to B_n$,
   (b) $\partial : \widetilde{B}_{n+1} \to B_{n+1}$
3. *for all $m, n \in \mathbf{N}$ such that $m \geq n$ maps of the form:*

(a) $T : (X \in B_{n+1}, Y \in B_{m+1}, ft(X) = ft^{m+1-n}(Y)) \to B_{m+2}$,
(b) $\widetilde{T} : (X \in B_{n+1}, r \in \widetilde{B}_{m+1}, ft(X) = ft^{m+1-n}\partial(r)) \to \widetilde{B}_{m+2}$,
(c) $S : (s \in \widetilde{B}_{n+1}, Y \in B_{m+2}, \partial(s) = ft^{m+1-n}(Y)) \to B_{m+1}$,
(d) $\widetilde{S} : (s \in \widetilde{B}_{n+1}, r \in \widetilde{B}_{m+2}, \partial(s) = ft^{m+1-n}\partial(r)) \to \widetilde{B}_{m+1}$,

Definition 2.2 *[***2014.10.20.def1***] A unital pre-B-system is a non-unital pre-B-system together with, for every $n \geq 0$ of an operation*

$$\delta : B_{n+1} \to \widetilde{B}_{n+2}$$

For convenience we will sometimes use the notation B_0 for a one point set $\{pt\}$ and $ft : B_1 \to B_0$ for the unique function to B_0.

Homomorphisms of non-unital and unital pre-B-systems are defined in the obvious way giving us the corresponding categories. Also in the obvious way one defines the concepts of sub-pre-B-systems.

Let CC be a C-system as defined in [**?**, Def. 2.2]. Recall the following notations. For Y such that $l(Y) \geq i$ and $f : X \to ft^i(Y)$ denote by by $f^*(Y, i)$ the objects and by $q(f, Y, i) : f^*(Y, i) \to Y$ the morphisms defined inductively by the rule

$$f^*(Y, 0) = X \qquad q(f, Y, 0) = f,$$

$$f^*(Y, i+1) = q(f, ft(Y), i)^*(Y) \qquad q(f, Y, i+1) = q(q(f, ft(Y), i), Y).$$

If $l(Y) < i$, then $q(f, Y, i)$ is undefined since $q(-, Y)$ is undefined for $Y = pt$ and again, as in the case of $p_{Y,i}$, *all of the considerations involving $q(f, Y, i)$ are modulo the qualification that $l(Y) \geq i$.*

For $i \geq 1$, $(s : ft(Y) \to Y) \in \widetilde{Ob}$ such that $l(Y) \geq i$, and $f : X \to ft^i(Y)$ let

$$f^*(s, i) : f^*(ft(Y), i-1) \to f^(ft(Y), i)$$

be the pull-back of the section $ft(Y) \to Y$ along the morphism $q(f, ft(Y), i-1)$. We again use the agreement that always when $f^*(s, i)$ is used the condition $l(Y) \geq i$ is part of the assumptions.

One constructs a unital pre-B-system from CC as follows. The B-sets of CC are:

$$B_n(CC) = Ob_n(CC) = \{X \in Ob(CC) \,|\, l(X) = n\}$$

$$\widetilde{B}_{n+1}(CC) = \widetilde{Ob}_n(CC) = \{(X, r) \in \widetilde{Ob}(CC) \,|\, l(X) = n+1\}$$

The definition of ft and ∂ is obvious. The operations T, $\widetilde{T}$, S, $\widetilde{S}$, δ on the B-sets of a C-system are as follows:

1. T sends (X, Y) such that $ft(X) = ft^{m+1-n}(Y)$ to $p_X^*(Y, m+1-n)$,
2. $\widetilde{T}$ sends (X, r) such that $ft(X) = ft^{m+1-n}\partial(r)$ to $p_X^*(r, m+1-n)$,
3. S sends (r, X) such that $\partial(r) = ft^{m+1-n}(X)$ to $r^*(X, m+1-n)$,
4. $\widetilde{S}$ sends (r, s) such that $\partial(r) = ft^{m+1-n}\partial(s)$ to $r^*(s, m+1-n)$.
5. δ sends X to the diagonal section of the projection $p_X^* X \to X$.

The element $pt \in B_0$ is the distinguished object of CC of length 0.

When we need to distinguish between the unital pre-B-system defined by CC and its non-unital analog we will write $uB(CC)$ for the unital version and $nuB(CC)$ for the non-unital one.

One of the main results of [**?**], Proposition 4.3 can be reformulated as follows:

Theorem 2.3 *[**2014.06.26.th1**] There is a natural bijection between C-subsystems of a C-system CC and unital sub-pre-B-systems of $uB(CC)$.*

Another way to construct a pre-B-system is from a pair (R, LM) where R is a monad on sets and LM a left module over R with values in sets as in [**?**] . For the pre-B-system $B(R, LM)$ we have

$$B_n(R, LM) = LM(\emptyset) \times \ldots \times LM(\{1, \ldots, n-1\})$$

$$\widetilde{B}_{n+1}(R, LM) = B_{n+1}(R, LM) \times R(\{1, \ldots, n\})$$

The operations ft and ∂ are the obvious projections. The rest of the operations are defined as follows. For $E \in LM(\{1, \ldots, m\})$ or $E \in R(\{1, \ldots, m\})$ and $n \geq 1$ we set:

$$t_n(E) = E[n+1/n, n+2/n+1, \ldots, m+1/m]$$

$$s_n(E) = E[n/n+1, n+1/n+2, \ldots, m-1/m]$$

1. Operations T:
$$T((E_1, \ldots, E_n, F), (E_1, \ldots, E_n, E_{n+1}, \ldots, E_{m+1})) =$$
$$(E_1, \ldots, E_n, F, t_{n+1}E_{n+1}, \ldots, t_{n+1}E_{m+1})$$

2. Operations $\widetilde{T}$:
$$\widetilde{T}((E_1, \ldots, E_n, F), (E_1, \ldots, E_n, E_{n+1}, \ldots, E_{m+1}, r)) =$$
$$(E_1, \ldots, E_n, F, t_{n+1}E_{n+1}, \ldots, t_{n+1}E_{m+1}, t_{n+1}r)$$

3. Operations S:
$$S((E_1, \ldots, E_n, F, s), (E_1, \ldots, E_n, F, E_{n+1}, \ldots, E_{m+1})) =$$
$$(E_1, \ldots, E_n, s_n(E_{n+1}[s/n]), \ldots, s_n(E_{m+1}[s/n]))$$

4. Operation $\widetilde{S}$:
$$S((E_1, \ldots, E_n, F, s), (E_1, \ldots, E_n, F, E_{n+1}, \ldots, E_{m+1}, r)) =$$
$$(E_1, \ldots, E_n, s_n(E_{n+1}[s/n]), \ldots, s_n(E_{m+1}[s/n]), s_n(r[s/n]))$$

5. Operations δ:
$$\delta(E_1, \ldots, E_n, E_{n+1}) = (E_1, \ldots, E_n, E_{n+1}, \eta_R(n+1))$$
where η_R is the unit of the monad R.

Note that the unit of R also participates in the definition of operations S and $\widetilde{S}$ since the explicit form of the substitution $E \mapsto E[s/n]$ involves η_R.

We can form non-unital pre-B-systems using this construction by considering non-unital sub-pre-B-systems in $uB(R, LM)$ (cf. Example 3.7 below).

For this pre-B-system as well as for its subsystems and regular quotients we can use notations such as $\Gamma \vdash o : T$ directly since in this case $\Gamma \in B_n$, $T \in LM(\{1, \ldots, n\})$ and $o \in R(\{1, \ldots, n\})$ are elements of types or sets that do not depend on elements of other types or sets and the substitution is defined on the level of these sets.

If $CC(R, LM)$ is the C-system corresponding to (R, LM) then there is a constructive isomorphism

$$B(CC(R, LM)) \cong B(R, LM)$$

The construction $CC \mapsto B(CC)$ is clearly compatible with homomorphisms and defines a functor from the category of C-systems to the category of unital pre-B-systems.

Theorem 2.4 *[**2014.10.10.th1**] The functor $CC \mapsto uB(CC)$ is a full embedding.*

The proof of the theorem is completed at the end of this section. We start preparing for the proof by introducing intermediate concepts of B0-systems.

Definition 2.5 *[**2014.10.16.def1.fromold**] [**2014.10.16.def1**] A non-unital pre-B-system is called a non-unital B0-system if the following conditions hold:*

1. *for $X \in B_{n+1}$, $Y \in B_{m+1}$ such that $ft(X) = ft^{m+1-n}(Y)$ and $m \geq n \geq 0$ one has:*

$$[\mathbf{oldeq1}] ft(T(X, Y)) = \begin{cases} T(X, ft(Y)) & \text{if } m > n \\ X & \text{if } m = n \end{cases} \quad (1)$$

2. *for $X \in B_{n+1}$, $r \in \widetilde{B}_{m+1}$ such that $ft(X) = ft^{m+1-n}\partial(r)$ and $m \geq n \geq 0$ one has:*

$$\partial(\widetilde{T}(X, r)) = T(X, \partial(r)) \quad (2)$$

3. *for $s \in \widetilde{B}_{n+1}$, $X \in B_{m+2}$ such that $\partial(s) = ft^{m+1-n}(X)$ and $m \geq n \geq 0$ one has:*

$$ft(S(s, X)) = \begin{cases} S(s, ft(X)) & \text{if } m > n \\ ft(\partial(s)) & \text{if } m = n \end{cases} \quad (3)$$

4. *for $s \in \widetilde{B}_{n+1}$, $r \in \widetilde{B}_{m+2}$ such that $\partial(s) = ft^{m+1-n}\partial(r)$ and $m \geq n \geq 0$ one has:*

$$\partial(\widetilde{S}(s, r)) = S(s, \partial(r)) \quad (4)$$

Definition 2.6 *[**2014.10.20.def2**] A unital pre-B-system is called a unital B0-system if the underlying non-unital pre-B-system is a non-unital B0-system and for all $i \geq 0$, $X \in B_{n+1}$ one has*

$$[\mathbf{2009.12.27.eq1}] \partial(\delta(X)) = T(X, X) \quad (5)$$

Lemma 2.7 *[**2014.12.17.l1**] Let BB be a non-unital pre-B-system. Then one has:*

1. *for* $X \in B_{n+1}$, $Y \in B_{m+1}$ *such that* $ft(X) = ft^{m+1-n}(Y)$ *and* $m \geq n \geq 0$ *one has:*

$$ft^k(T(X,Y)) = \begin{cases} T(X, ft^k(Y)) & \text{if } m-n \geq k \\ ft^{(k-1)-(m-n)}X & \text{if } m-n < k \end{cases} \tag{6}$$

2. *for* $s \in \widetilde{B}_{n+1}$, $X \in B_{m+2}$ *such that* $\partial(s) = ft^{m+1-n}(X)$ *and* $m \geq n \geq 0$ *one has:*

$$ft^k(S(s,X)) = \begin{cases} S(s, ft^k(X)) & \text{if } m-n \geq k \\ ft^{k-(m-n)}(\partial(s)) & \text{if } m-n < k \end{cases} \tag{7}$$

Proof: See T_l1 and S_l1 in [**?**].

In a B0-system let us denote by

$$T_j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}} (B_{m+1}) \to B_{m+1+j}$$

$$\widetilde{T}_j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \to \widetilde{B}_{m+1+j}$$

the maps which are defined inductively by

$$T_j(X,Y) = \begin{cases} Y & \text{if } j=0 \\ T(X, T_{j-1}(ft(X), Y)) & \text{if } j>0 \end{cases} \tag{8}$$

$$\widetilde{T}_j(X,s) = \begin{cases} s & \text{if } j=0 \\ \widetilde{T}(X, \widetilde{T}_{j-1}(ft(X), s)) & \text{if } j>0 \end{cases} \tag{9}$$

Note that for any $i = 0, \ldots, j$ we have

$$T_j(X,Y) = T_i(X, T_{j-i}(ft^i(X), Y))$$

and

$$\widetilde{T}_j(X,s) = \widetilde{T}_i(X, \widetilde{T}_{j-i}(ft^i(X), s))$$

Lemma 2.8 *[**Tnft**] One has*

$$T_j(X, ft(Y)) = ft(T_j(X,Y))$$

Proof: For $j = 0$ the statement is obvious. For $j > 0$ we have by induction on j

$$T_j(X, ft(Y)) = T(X, T_{j-1}(ft(X), ft(Y))) = T(X, ft(T_{j-1}(ft(X), Y))) =$$

$$= ft(T(X, T_{j-1}(ft(X), Y))) = ft(T_j(X,Y)).$$

Lemma 2.9 *[**l2014.10.10.l1**] Let B be a unital pre-B-system of the form $uB(CC)$. Then B is a unital B0-system.*

Proof: Straightforward.

Given a sequence of sets and maps such as (B_i, ft_i) and two elements $X \in B_m$, $Y \in B_n$ let us write $X \leq Y$ if $m \leq n$ and $X = ft^{n-m}Y$. This defines a reflexive transitive relation on $\amalg_{n\geq 0} B_n$.

Let us also denote by $\widetilde{B}(Y)$ the subset in $\widetilde{B}$ of elements r such that $\partial(r) = Y$.

Given a B0-system and two elements $X \in B_m$, $Y \in B_n$ let us define a set that we will denote later $Mor(X, Y)$. For the purpose of this construction we fix X and proceed to construct for each n and each $Y \in B_n$ a pair

1. a set $M_X(Y)$,
2. for any $f \in M_X(Y)$, $i \geq 0$ and $Y' \in B_{n+i}$ such that $Y' \geq Y$, an element $f^*(Y', i) \in B_{m+i}$,

such that:

1. for any $f \in M_X(Y)$ one has $f^*(Y, 0) = X$,
2. for any $f \in M_X(Y)$, $i, j \geq 0$, $Y' \in B_{n+i}$ such that $Y' \geq Y$ and $Y'' \in B_{n+i+j}$ such that $Y'' \geq Y'$ one has $f^*(Y'', i+j) \geq f^*(Y', i)$.

The construction will proceed by induction on n. For $n = 0$ we set $M_X(Y) = \{p\}$ and for $Y' \in B_i$:

$$p^*(Y') = \begin{cases} X & \text{if } i = 0 \\ T_m(X, Y') & \text{if } i > 0 \end{cases}$$

The second condition follows from Lemma 2.8.

Suppose now that $n > 0$. Then we set

$$M_X(Y) = \sum_{f \in M_X(ft(Y))} \widetilde{B}(f^*(Y, 1))$$

For $Y' \in B_{n+i}$, $Y' \geq Y$ and $g = (f, r) \in M_X(Y)$ we define

$$f^*(Y', i) := S(r, f^*(Y, i+1))$$

The conditions are easily verified from the axioms of a B0-system.

Consider the unital B0-system $uB(CC)$ of a C-system CC.

Let $f : X \to Y$ be a morphism such that $X \in B_n$ and $Y \in B_m$. Define a sequence $(s_1(f), \ldots, s_m(f))$ of elements of $\widetilde{B}_{n+1}$ inductively by the rule

$$(s_1(f), \ldots, s_m(f)) = (s_1(ft(f)), \ldots, s_{m-1}(ft(f)), s_f) = (s_{ft^{m-1}(f)}, \ldots, s_{ft(f)}, s_f)$$

where $ft(f) = p_Y f$ and s_f is the s-operation of [**?**, Def. 2.2]. For $m = 0$ we start with the empty sequence. This construction can be illustrated by the following diagram for $f : X \to Y$ where

$Y \in B_4$:

$$
\begin{array}{cccccccccccc}
X & \xrightarrow{s_4(f)} & Z_{4,3} & \longrightarrow & Z_{4,2} & \longrightarrow & Z_{4,1} & \longrightarrow & T_n(X,Y) & \longrightarrow & Y \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & X & \xrightarrow{s_3(f)} & Z_{3,2} & \longrightarrow & Z_{3,1} & \longrightarrow & T_n(X, ft(Y)) & \longrightarrow & ft(Y) \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & X & \xrightarrow{s_2(f)} & Z_{2,1} & \longrightarrow & T_n(X, ft^2(Y)) & \longrightarrow & ft^2(Y) \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & X & \xrightarrow{s_1(f)} & T_n(X, ft^3(Y)) & \longrightarrow & ft^3(Y) \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & X & \longrightarrow & pt
\end{array} \tag{10}
$$

which is completely determined by the condition that the squares are the canonical ones and the composition of morphisms in the i-th arrow from the top is $ft^i(f)$. For the objects Z_i^j we have:

$$
\begin{array}{lll}
Z_{4,1} = S(s_1(f), T_n(X,Y)) & Z_{4,2} = S(s_2(f), Z_{4,1}) & Z_{4,3} = S(s_3(f), Z_{4,2}) \\
Z_{3,1} = S(s_1(f), T_n(X, ft(Y))) & Z_{3,2} = S(s_2(f), Z_{3,1}) & \\
Z_{2,1} = S(s_1(f), T_n(X, ft^2(Y))) & &
\end{array} \tag{11}
$$

A simple inductive argument similar to the one in the proof of [**?**, Lemma 4.1] show that if $f, f' : X \to Y$ are two morphisms such that $Y \in B_m$ and $s_i(f) = s_i(f')$ for $i = 1, \ldots, m$ then $f = f'$. Therefore, we may consider the set $Mor(CC)$ of morphisms of CC as a subset in $\amalg_{n,m \geq 0} B_n \times B_m \times \widetilde{B}_{n+1}^m$.

Let us show how to describe this subset in terms of the operations introduced above.

Lemma 2.10 *[***2009.11.07.l1***] An element $(X, Y, s_1, \ldots, s_m)$ of $B_n \times B_m \times \widetilde{B}_{n+1}^m$ corresponds to a morphism if and only if the element $(X, ft(Y), s_1, \ldots, s_{m-1})$ corresponds to a morphism and $\partial(s_m) = Z_{m,m-1}$ where $Z_{m,i}$ is defined inductively by the rule:*

$$Z_{m,0} = T_n(X,Y) \qquad Z_{m,i+1} = S(s_{i+1}, Z_{m,i})$$

Proof: Straightforward from the example considered above.

Let us show now how to identify the canonical morphisms $p_{X,i} : X \to ft^i(X)$ and in particular the identity morphisms.

Lemma 2.11 *[***2009.11.10.l1***] Let $X \in B_m$ and $0 \leq i \leq m$. Let $p_{X,i} : X \to ft^i(X)$ be the canonical morphism. Then one has:*

$$s_j(p_{X,i}) = \widetilde{T}_{m-j}(X, \delta_{ft^{m-j}(X)}) \qquad j = 1, \ldots, m-i$$

Proof: Let us proceed by induction on $m-i$. For $i = m$ the assertion is trivial. Assume the lemma proved for $i+1$. Since $ft(p_{X,i}) = p_{X,i+1}$ we have $s_j(p_{X,i}) = s_j(p_{X,i+1})$ for $j = 1, \ldots, m-i-1$. It remains to show that

$$[\textbf{2009.11.10.eq1}] s_{m-i}(p_{X,i}) = \widetilde{T}_i(X, \delta_{ft^i(X)}) \tag{12}$$

By definition $s_{m-i}(p_{X,i}) = s_{p_{X,i}}$ and (12) follows from the commutative diagram:

$$\begin{array}{ccccc}
X & \longrightarrow & ft^i(X) & & \\
{\scriptstyle s_p}\downarrow & & \downarrow{\scriptstyle \delta_{ft^i(X)}} & & \\
p^*_{X,i+1}(ft^i(X)) & \longrightarrow & p^*_{ft^i(X)}(ft^i(X)) & \longrightarrow & ft^i(X) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{ft^i(X)}} \\
X & \longrightarrow & ft^i(X) & \longrightarrow & ft^{i+1}(X)
\end{array}$$

where $p = p_{X,i}$.

Lemma 2.12 *[**2009.11.10.l2**] Let $(X, s) \in \widetilde{B}_{m+1}$, $X \in B_n$ and $f : X \to ft(Y)$. Define inductively $(f, i)^*(s) \in \widetilde{B}_{n+m+1-i}$ by the rule*

$$(f, 0)^*(s) = \widetilde{T}_n(X, s)$$

$$(f, i+1)^*(s) = \widetilde{S}(s_{i+1}(f), (f, i)^*(s))$$

Then $f^*(s) = (f, m)^*(s)$.

Proof: It follows from the diagram:

$$\begin{array}{ccccccccccc}
X & \xrightarrow{s_m(f)} & * & \longrightarrow & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(Y) \\
{\scriptstyle f^*(s)}\downarrow & & \downarrow{\scriptstyle (f,m-1)^*(s)} & & & & \downarrow{\scriptstyle (f,1)^*(s)} & & \downarrow{\scriptstyle (f,0)^*(s)} & & \downarrow{\scriptstyle s} \\
* & \longrightarrow & * & \longrightarrow & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & Y \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{s_m(f)} & * & \longrightarrow & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(Y) \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
 & & X & \xrightarrow{s_{m-1}(f)} & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft^2(Y) \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & \ldots & & \ldots & & \ldots \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & X & \xrightarrow{s_1(f)} & * & \longrightarrow & ft^{m-1}(Y) \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & X & \longrightarrow & pt
\end{array}$$

Lemma 2.13 *Let $g : Z \to X$, $f : X \to Y$ and $Y \in B_m$. Then $s_i(fg) = g^* s_i(f)$.*

Proof: It follows immediately from the equations $s_{fg} = g^* s_f$ and $ft(fg) = ft(f)g$.

Lemma 2.14 *[**2009.11.10.l4a**] Let $f : X \to ft(Y)$ be a morphism, $X \in B_n$ and $Y \in B_{m+1}$. Define $(f,i)^*(Y)$ inductively by the rule:*

$$(f,0)^*(Y) = T_n(X,Y)$$

$$(f,i+1)^*(Y) = S(s_{i+1}(f),(f,i)^*(Y))$$

Then $f^*(Y) = (f,m)^*(Y)$.

Proof: Similar to the proof of Lemma 2.12.

Lemma 2.15 *[**2009.11.10.l4b**] Let $f : X \to ft(Y)$ be a morphism, $X \in B_n$ and $Y \in B_{m+1}$. Then*

$$s_i(q(f,Y)) = \begin{cases} \widetilde{T}(f^*Y, s_i(f)) & \text{if } i \le m \\ \\ \widetilde{T}(f^*Y, \delta_Y) & \text{if } i = m+1 \end{cases}$$

Proof: We have $s_i(q(f,Y)) = s_{ft^{m+1-i}(q(f,Y))}$. For $i \le m$ we have

$$ft^{m+1-i}(q(f,Y)) = ft^{m-i}(f)p_{f^*Y}$$

Therefore,

$$s_{ft^{m+1-i}(q(f,Y))} = s_{ft^{m-i}(f)p_{f^*Y}} = p^*_{f^*Y} s_{ft^{m-i}(f)} = \widetilde{T}(f^*Y, s_i(f))$$

and for $i = m+1$ we have

$$s_i(q(f,Y)) = s_{q(f,Y)} = p^*_{f^*Y}(\delta_Y) = \widetilde{T}(f^*Y, \delta_Y).$$

The lemmas proved above show that a C-system can be reconstructed from the sets $B_n, \widetilde{B}_{n+1}$ and operations ft, ∂, T, $\widetilde{T}$, S, $\widetilde{S}$ and δ. This completes our proof of Theorem 2.4.

3 B-systems

The next question that we want to address is the description of the image of the functor $CC \mapsto uB(CC)$. To make this question more precise we introduce below the concepts of non-unital and unital B-systems and formulate a problem whose solution would imply that the functor $CC \mapsto uB(CC)$ defines an equivalence between the category of C-systems and the full subcategory of the category of unital pre-B-systems that consists of unital B-systems.

For $X \in B_i$ let $B(X)_j$ denote the subset of B_{i+j} that consists of Y such that $ft^j(Y) = X$. In particular $B(X)_0$ is the one point subset $\{X\}$. Let also $\widetilde{B(X)}_j$ denote the subset of $\widetilde{B}_{i+j}$ that consists of r such that $ft^j(\partial(r)) = X$.

Then the operations T, $\widetilde{T}$, S and $\widetilde{S}$ can be seen as follows:

$$T(X,-) : B(ft(X))_* \to B(X)_*$$
$$\widetilde{T}(X,-) : \widetilde{B}(ft(X))_* \to \widetilde{B}(X)_*$$
$$S(s,-) : B(\partial(s))_* \to B(ft(\partial(s)))_*$$
$$\widetilde{S}(s,-) : \widetilde{B}(\partial(s))_* \to \widetilde{B}(ft(\partial(s)))_*$$

Definition 3.1 *[**2014.10.16.def2**] [**was.2014.06.18.eq2.to.eq11**] Let B be a non-unital B0-system. Define the following conditions on B:*

1. *The TT-condition. For all $GT \in B_{i+1}$, $GDT' \in B(ft(GT))_{j+1}$ one has*
   (a) *for all $R \in B(ft(GDT'))_{*+1}$*
   $$T(T(GT, GDT'), T(GT, R)) = T(GT, T(GDT', R))$$
   (b) *for all $r \in \widetilde{B}(ft(GDT'))_{*+1}$*
   $$\widetilde{T}(T(GT, GDT'), \widetilde{T}(GT, r)) = \widetilde{T}(GT, \widetilde{T}(GDT', r))$$
2. *The SS-condition. For all $s \in \widetilde{B}_{i+1}$, $s' \in \widetilde{B(\partial(s))}_{j+1}$ one has*
   (a) *for all $R \in B(\partial(s'))_*$*
   $$S(\widetilde{S}(s, s'), S(s, R)) = S(s, S(s', R))$$
   (b) *for all $r \in \widetilde{B}(\partial(s'))_*$*
   $$\widetilde{S}(\widetilde{S}(s, s'), \widetilde{S}(s, r)) = \widetilde{S}(s, \widetilde{S}(s', r))$$
3. *The TS-condition. For any $s \in \widetilde{B}_{i+1}$ and $GTDT' \in \widetilde{B(\partial(s))}_{j+1}$ one has*
   (a) *for all $R \in B(ft(GTDT'))_*$*
   $$T(S(s, GTDT'), S(s, R)) = S(s, T(GTDT', R))$$
   (b) *for all $r \in \widetilde{B}(ft(GTDT'))_*$*
   $$\widetilde{T}(S(s, GTDT'), \widetilde{S}(s, r)) = \widetilde{S}(s, \widetilde{T}(GTDT', r))$$
4. *The ST-condition. For any $GT \in B_{i+1}$ and $s' \in B(\widetilde{ft(GT)})_{j+1}$ one has*
   (a) *for all $R \in B(\partial(s'))_*$*
   $$S(\widetilde{T}(GT, s'), T(GT, R)) = T(GT, S(s', R))$$
   (b) *for all $r \in \widetilde{B}(\partial(s'))_*$*
   $$\widetilde{S}(\widetilde{T}(GT, s'), \widetilde{T}(GT, r)) = \widetilde{T}(GT, \widetilde{S}(s', r))$$
5. *The STid-condition. For any $s \in \widetilde{B}_{i+1}$ one has*

(a) *for all* $R \in B(ft(\partial(s)))_*$

$$S(s, T(\partial(s), R)) = R$$

(b) *for all* $r \in \widetilde{B}(ft(\partial(s)))_*$

$$\widetilde{S}(s, \widetilde{T}(\partial(s), r)) = r$$

Definition 3.2 *[***2014.10.20.def3***] Let* B *be a unital B0-system. Define the following conditions on* B*:*

1. *The* δT*-condition. For any* $GT \in B_{i+1}$ *and* $GDT' \in B(ft(GT))_{j+1}$ *one has*

$$\widetilde{T}(GT, \delta(GDT')) = \delta(T(GT, GDT'))$$

2. *The* δS*-condition. For any* $s \in \widetilde{B}_{i+1}$ *and* $GTDT' \in B(\partial(s))_{j+1}$ *one has*

$$\widetilde{S}(s, \delta(GTDT')) = \delta(S(s, GTDT'))$$

3. *The* δSid*-condition. For any* $s \in \widetilde{B}_{i+1}$ *one has*

$$\widetilde{S}(s, \delta(\partial(s))) = s$$

4. *The* $S\delta T$*-condition. For any* $GT \in B_{i+1}$ *one has*

   (a) *for* $R \in B(GT)_{*+1}$ *one has:*

$$S(\delta(GT), T(GT, R)) = R$$

   (b) *for* $r \in \widetilde{B(GT)}_{*+1}$ *one has*

$$\widetilde{S}(\delta(GT), \widetilde{T}(GT, r)) = r$$

Remark 3.3 *[***2014.06.14.rem2***] The conditions defined above can be shown as follows:*

1. *The* TT*-condition:*

$$\frac{\Gamma, T \rhd \quad \Gamma, \Delta, T' \rhd \quad \Gamma, \Delta \vdash \mathcal{J}}{\frac{\Gamma, T\rhd \quad \Gamma, \Delta, T' \vdash \mathcal{J}}{\Gamma, T, \Delta, T' \vdash \mathcal{J}} \quad \frac{\Gamma, T, \Delta, T' \rhd \quad \Gamma, T, \Delta \vdash \mathcal{J}}{\Gamma, T, \Delta, T' \vdash \mathcal{J}}}$$

2. *The* SS*-condition:*

$$\frac{\Gamma \vdash s : T \quad \Gamma, T, \Delta \vdash s' : T' \quad \Gamma, T, \Delta, T' \vdash \mathcal{J}}{\frac{\Gamma \vdash s:T \quad \Gamma, T, \Delta \vdash \mathcal{J}[s]}{\Gamma, \Delta[s] \vdash \mathcal{J}[s'][s]} \quad \frac{\Gamma, \Delta[s] \vdash s'[s] : T'[s] \quad \Gamma, \Delta[s], T'[s] \vdash \mathcal{J}[s]}{\Gamma, \Delta[s] \vdash \mathcal{J}[s][s']}}$$

3. *The* TS*-condition:*

$$\frac{\Gamma, T \rhd \quad \Gamma, \Delta \vdash s' : T' \quad \Gamma, \Delta, T' \vdash \mathcal{J}}{\frac{\Gamma, T\rhd \quad \Gamma, \Delta \vdash \mathcal{J}[s']}{\Gamma, T, \Delta \vdash \mathcal{J}[s']} \quad \frac{\Gamma, T, \Delta \vdash s' : T' \quad \Gamma, T, \Delta, T' \vdash \mathcal{J}}{\Gamma, T, \Delta \vdash \mathcal{J}[s']}}$$

4. *The* ST*-condition:*

$$\frac{\Gamma \vdash s : T \quad \Gamma, T, \Delta, T' \rhd \quad \Gamma, T, \Delta \vdash \mathcal{J}}{\frac{\Gamma \vdash s:T \quad \Gamma, T, \Delta, T' \vdash \mathcal{J}[s]}{\Gamma, \Delta[s], T'[s] \vdash \mathcal{J}[s]} \quad \frac{\Gamma, \Delta[s], T'[s] \rhd \quad \Gamma, \Delta[s] \vdash \mathcal{J}[s]}{\Gamma, \Delta[s], T'[s] \vdash \mathcal{J}[s]}}$$

5. *The STid-condition:*

$$\frac{\Gamma \vdash s : T \quad \Gamma, T \rhd \quad \Gamma \vdash \mathcal{J}}{\frac{\Gamma \vdash s:T \quad \Gamma, T \vdash \mathcal{J}}{\Gamma \vdash \mathcal{J}[s]}}$$

6. *The δT-condition:*

$$\frac{\Gamma, T \rhd \quad \Gamma, \Delta, x : T' \rhd}{\frac{\Gamma, T\rhd \quad \Gamma,\Delta,x:T'\vdash x:T'}{\Gamma,T,\Delta,x:T'\vdash x:T'} \frac{\Gamma,T,\Delta,x:T'\rhd}{\Gamma,T,\Delta,x:T'\vdash x:T'}}$$

7. *The δS-condition:*

$$\frac{\Gamma \vdash s : T \quad \Gamma, T, \Delta, x : T' \rhd}{\frac{\Gamma\vdash s:T \quad \Gamma,T,\Delta,x:T'\vdash x:T'}{\Gamma,\Delta[s],x:T'[s]\vdash x:T'[s]} \frac{\Gamma,\Delta[s],x:T[s]'\rhd}{\Gamma,\Delta[s],x:T'[s]\vdash x:T'[s]}}$$

8. *The δSid-condition:*

$$\frac{\Gamma \vdash s : T \quad \Gamma, x : T \rhd}{\frac{\Gamma\vdash s:T \quad \Gamma,x:T\vdash x:T}{\Gamma\vdash s:T}}$$

9. *The $S\delta T$-condition:*

$$\frac{\Gamma, y : X, \Delta \vdash \mathcal{J}}{\frac{\Gamma,y_1:X,y:X,\Delta\vdash\mathcal{J} \quad \Gamma,y_1:X\vdash y_1:X}{\Gamma,y_1:X,\Delta[y_1/y]\vdash\mathcal{J}[y_1/y]}}$$

Lemma 3.4 *[**2014.10.20.l1**] [**2014.10.16.l1**] Let B be a unital B0-system and let δ_1, δ_2 be two families of operations as in Definition 2.2. Suppose that both δ_1 and δ_2 satisfy the δT, δSid and $S\delta T$ conditions. Then $\delta_1 = \delta_2$.*

Proof: We have:

$$\delta_1(GT) = \widetilde{S}(\delta_2(GT), \widetilde{T}(GT, \delta_1(GT))) = \widetilde{S}(\delta_2(GT), \delta_1(T(GT, GT))) = \delta_2(GT)$$

where the first equality is the $S\delta T$-condition for δ_2, the second equality is the δT-condition for δ_1 and the third equality is the δSid-condition for δ_1.

Definition 3.5 *[**2014.10.10.def2a**] [**2014.10.20.def4**] A non-unital B-system is a non-unital B0-system that satisfy the conditions TT, SS, TS, ST and STid of Definition 3.1.*

Definition 3.6 *[**2014.10.10.def2b**] [**2014.10.20.def5**] A unital B-system is a unital B0-system that satisfy the conditions TT, SS, TS, ST, STid of Definition 3.1 and the conditions δT, δS, δSid and $S\delta T$ of Definition 3.2.*

Equivalently, a unital B-system is non-unital B-system such that there exists a family of operations δ satisfying the conditions δT, δS, δSid and $S\delta T$ of Definition 3.2.

Example 3.7 [**2014.10.20.eX**] While being unital is a property of non-unital B-systems not any homomorphism of non-unital B-systems preserves units. Here is a sketch of an example of a homomorphism that does not preserve units.

Consider the following pairs of a monad and a left module over it. In both cases *pt* is the constant functor corresponding to the one point set $\{T\}$ that has a unique left module structure over any monad.

1. (R_1, pt) where R_1 is the monad corresponding to one unary operation $s_1(x)$ and the relation

$$s_1(s_1(x)) = s_1(x)$$

2. (R_2, pt) where R_2 is the monad corresponding to two unary operations $s_1(x)$ and $s_2(x)$ and relations:

$$s_1(s_1(x)) = s_1(x) \quad s_1(s_2(x)) = s_1(x) \quad s_2(s_1(x)) = s_1(x) \quad s_2(s_2(x)) = s_2(x)$$

Consider the unital B-systems $uB(R_1, pt)$ and $uB(R_2, pt)$. In $uB(R_1, pt)$ consider the non-unital sub-B-system nuB_1 generated by $(T \vdash s_1(1) : T)$. In $uB(R_2, pt)$ consider the non-unital sub-B-system nuB_2 generated by $(T \vdash s_1(1) : T)$ and $(T \vdash s_2(1) : T)$.

Observe that both nuB_1 and nuB_2 are in fact unital with the unit in the first one given by $(T, \ldots, T \vdash s_1(n) : T)$ and unit in the second one is given by $(T, \ldots, T \vdash s_2(n) : T)$ where n is the number of T's before the turnstile $\vdash$ symbol.

We also have an obvious (unital) homomorphism from $uB(R_1, pt)$ to $uB(R_2, pt)$ that defines a homomorphism $nuB_1 \to nuB_2$ and that latter homomorphism is not unital.

Remark 3.8 For a unital B-systems operations S and T can be expressed as follows.

$$[\mathbf{2014.10.14.eq1}]T(X,Y) = \begin{cases} X & \text{if } l(Y) = l(X) - 1 \\ ft(\partial(\widetilde{T}(X, \delta(Y)))) & \text{if } l(Y) \geq l(X) \end{cases} \tag{13}$$

$$[\mathbf{2014.10.14.eq2}]S(s,X) = \begin{cases} ft(\partial(s)) & \text{if } l(X) = l(\partial(s)) \\ ft(\partial(\widetilde{S}(s, \delta(X)))) & \text{if } l(X) > l(\partial(s)) \end{cases} \tag{14}$$

I would like to end this section with the formulation of the following problem. I am reasonably sure that it has a straightforward solution.

Problem 3.9 *[**2014.10.10.prob2**] To show that a unital B0-system is isomorphic to a unital B0-system of the form $uB(CC)$ if and only if it is a unital B-system.*

4 B-systems in Coq

While our main interest is in pre-B-systems and B-systems in sets we would like to be able to formalize their definitions in Coq without assuming that B_n and $\widetilde{B}_{n+1}$ are of h-level 2.

This suggests the following reformulation of our definitions. In what follows we give a presentation of non-unital B-systems in "functional terms". The presentation of the axioms related to the δ-operations is more complex as can be see already in the case of the δT-axiom and we leave it for the future.

Let us define a tower as a sequence of functions $T := (\ldots \to T_{i+1} \stackrel{p_i}{\to} T_i \to \ldots \to T_0)$.

For a tower T and $i, j \geq 0$ define $ft_i^j : T_{i+j} \to T_i$ as the composition of the functions p_k for $k = i, \ldots, i+j-1$. When no ambiguity can arise we will write ft^j instead of ft_i^j and we will write ft instead of ft^1.

For a tower T, $i \geq 0$ and $G \in T_i$ define a new tower $T(G)$ setting:

$$T(G)_j = \{GD \in T_{i+j} | ft_i^j(x) = G\}$$

and defining the functions $T(G)_{j+1} \to T(G)_j$ in the obvious way. More categorically this can expressed by saying that $T(G)_j$ is defined by the standard (homotopy) pull-back square

$$\begin{array}{ccc} T(G)_j & \longrightarrow & T_{i+j} \\ \downarrow & & \downarrow {\scriptstyle ft_i^j} \\ pt & \xrightarrow{G} & T_i \end{array}$$

For $G \in T_{i+j}$ we let $\phi_j(G) \in T(ft^{i+j}(G))_j$ denote the obvious element.

For towers T and T' define a function or morphism of towers $F : T \to T'$ as a sequence of morphisms $F_i : T_i \to T'_i$ which commute in the obvious sense with the functions p_i and p'_i.

The identity function of towers id_T and the composition of functions of towers are defined in the obvious way.

For T, $i, j, k \geq 0$, $G \in T_i$ and $GD \in T_j(G)$ we have the digrams:

$$\begin{array}{ccccc} T(G)(GD)_k & \longrightarrow & T(G)_{j+k} & \longrightarrow & T_{i+(j+k)} \\ \downarrow & & \downarrow {\scriptstyle ft^k_{T(G),j}} & & \downarrow \\ pt & \xrightarrow{GD} & T(G)_j & \xrightarrow{u_{G,j}} & T_{i+j} \\ & & \downarrow & & \downarrow {\scriptstyle ft^j_{T,i}} \\ & & pt & \xrightarrow{G} & T_i \end{array} \qquad \begin{array}{ccc} T(u_{G,i}(GD))_k & \longrightarrow & T_{(i+j)+k} \\ \downarrow & & \downarrow \\ pt & \xrightarrow[u_{G,j}(GD)]{} & T_{i+j} \end{array}$$

which shows that we have natural equivalences (isomorphisms)

$$[\mathbf{2014.06.12}] T(G)(GD)_k \cong T(u_{G,j}(GD))_k \tag{15}$$

The equivalences (15) commute with the functions $p(G)(GD)_i$ and $p(u_{G,j}(GD))$ in the obvious sense and define an equivalence of towers

$$[\mathbf{2014.06.14.eq2}] T(G)(GD) \cong T(u_{G,j}(GD)) \tag{16}$$

Remark 4.1 In the case when standard pull-backs are pull-backs in a category, the functions $u_{G,j}$ from $T_j(G)$ to T_{i+j} are pull-backs of (split) monomorphisms and therefore are monomorphisms. In this case $T_k(G)(GD)$ is a sub-object of $T_{i+(j+k)}$ and $T_k(u_{G,j}(GD))$ is a sub-object of $T_{(i+j)+k}$ which are canonically equal. Then we can say that

$$[\mathbf{2014.06.14.eq1}] T(G)(GD)_k = T(u_{G,j}(GD))_k \tag{17}$$

where the equality is the equality of sub-objects of $T_{(i+j)+k}$.

More generally, if T_i are objects of h-level 2, the functions $u_{G,j}$ are of h-level 1 (monic inclusions) and we again can say that the equality (17) holds as the unique equality of monic sub-objects of $T_{(i+j)+k}$.

For a function $F : T \to T'$ and $G \in T_i$ we obtain a function $F(G) : T(G) \to T'(G)$ using functoriality of standard pull-backs.

Define a B-system carrier or a B-carrier as a pair $\mathbf{B} = (B, \widetilde{B})$ where B is a tower such that $B_0 = \{pt\}$ and $\widetilde{B}$ is a family $\widetilde{B}_{i+1}$, $i \geq 0$ together with functions $\partial_i : \widetilde{B}_{i+1} \to B_{i+1}$. The B-system carriers in sets are the same as the "type-and-term structures" of [**?**].

We will denote the standard fiber of ∂_i over $GT \in B_{i+1}$ by $\widetilde{B}_{GT}$.

For a B-carrier $\mathbf{B}$, $i \geq 0$ and $G \in B_i$, define a B-carrier $\mathbf{B}(G)$ as the pair $(B(G), \widetilde{B(G)})$ where

$$\widetilde{B(G)}_{j+1} = \{s \in \widetilde{B}_{i+j+1} | \partial(s) \in B(G)_{j+1}\}$$

or, categorically, $\widetilde{B(G)}_{j+1}$ is defined by the standard pull-back square

$$\begin{array}{ccc} \widetilde{B(G)}_{j+1} & \xrightarrow{\widetilde{u}_{G,j+1}} & \widetilde{B}_{i+(j+1)} \\ \partial(G)\downarrow & & \downarrow\partial \\ B(G)_{j+1} & \xrightarrow{u_{G,j+1}} & B_{i+(j+1)} \end{array}$$

For a B-carrier $\mathbf{B}$, $i, j \geq 0$, $G \in B_i$ and $GD \in B_{i+j}$ the equivalence (16) clearly extends to an equivalence

$$[\mathbf{2014.06.14.eq3}]\mathbf{B}(G)(GD) \cong \mathbf{B}(u_G(GD)) \quad (18)$$

For B-carriers $\mathbf{B}$ and $\mathbf{B}'$ define a function of B-carriers $\mathbf{F} : \mathbf{B} \to \mathbf{B}'$ as a pair $\mathbf{F} = (F, \widetilde{F})$ where $F : B \to B'$ is a function of towers and for every $i \geq 0$, $\widetilde{F}_{i+1}$ is a function $\widetilde{B}_{i+1} \to \widetilde{B}'_{i+1}$ which commutes in the obvious sense with the functions ∂', F_{i+1} and ∂.

The identity function of B-carriers $id_{\mathbf{B}}$ and the composition of functions of B-carriers are defined in the obvious way.

For a function of B-carriers $\mathbf{F} : \mathbf{B} \to \mathbf{B}'$ and $G \in B_i$ we obtain a function of B-carriers $\mathbf{F}(G) : \mathbf{B}(G) \to \mathbf{B}'(F(G))$ using functoriality of standard pull-backs.

Definition 4.2 *[***Bdata***] Non-unital B-system data is given by the following:*

1. *a B-system carrier* $\mathbf{B}$ *,*
2. *for every* $m \geq 0$*,* $X \in B_{n+1}$ *a B-carrier function* $\mathbf{T}_X : \mathbf{B}(p_n(X)) \to \mathbf{B}(X)$*,*
3. *for every* $m \geq 0$*,* $s \in \widetilde{B}_{n+1}$*, a B-carrier function* $\mathbf{S}_s : \mathbf{B}(\partial(s)) \to \mathbf{B}(p_n(\partial(s)))$*,*

Problem 4.3 *[***2014.10.10.prob1***] Construct an equivalence between the type of non-unital B0-systems and the type of non-unital B-system data such that the types* B_* *and* $\widetilde{B}$ *are sets.*

Construction 4.4 [**2014.10.10.constr1**] A non-unital B-system carrier is the same as two families of sets B_n, $\widetilde{B}_{n+1}$ together with maps $p_n : B_{n+1} \to B_n$ and $\partial : \widetilde{B_{n+1}} \to B_{n+1}$.

For a given $X \in B_{n+1}$ a B-carrier function $\mathbf{T}_X : \mathbf{B}(ft(X)) \to \mathbf{B}(X)$ is the same as:

1. for all $i \geq 0$, $Y \in B_{n+i}$ such that $ft^i(Y) = ft(X)$, an element $T(X,Y) \in B_{n+i+1}$ such that $ft^i(T(X,Y)) = X$,
2. for all $i \geq 0$, $r \in \widetilde{B}_{n+i+1}$ such that $ft^{i+1}(\partial(r)) = ft(X)$, an element $\widetilde{T}(X,r)$ such that $ft^{i+1}(\partial(r)) = ft(X)$.

For $i = 0$, the operation T is uniquely determined by the condition $ft^i(T(X,Y)) = X$ which leaves us with the operations T and $\widetilde{T}$ as in Definition 2.1 satisfying the conditions of Lemma 2.9.

The same reasoning applies to S, $\widetilde{S}$.

From this point on everything is assumed to be non-unital. Let $\mathbf{BD} = (\mathbf{B}, \mathbf{T}, \mathbf{S}, \delta)$ be B-data and $G \in B_i$. Define B-data $\mathbf{BD}(G)$ over G as follows. The B-carrier of $\mathbf{BD}(G)$ is $\mathbf{B}(G)$.

For $GDT \in B(G)_{i+1}$ we need to define a B-carrier function

$$\mathbf{T}(G)_{GDT} : \mathbf{B}(G)(p_i(GDT)) \to \mathbf{B}(G)(GDT)$$

We define it through the condition of commutativity of the pentagon:

[**2014.06.14.eq4**]

$$\begin{array}{ccc} \mathbf{B}(G)(p_i(GDT)) & \xrightarrow{\mathbf{T}(G)_{GDT}} & \mathbf{B}(G)(GDT) \\ \cong\downarrow & & \downarrow\cong \\ \mathbf{B}(u_G(p_i(GDT))) \cong \mathbf{B}(p_i(u_G(GDT))) & \xrightarrow{\mathbf{T}_{-}} & \mathbf{B}(u_G(GDT)) \end{array} \tag{19}$$

where the vertical equivalences are from (18).

Similarly for $s \in \widetilde{B(G)}_{j+1}$ we define a B-carrier function

$$\mathbf{S}(G)_s : \mathbf{B}(G)(\partial(s)) \to \mathbf{B}(G)(p_j(\partial(s)))$$

by the diagram:

[**2014.06.14.eq5**]

$$\begin{array}{ccc} \mathbf{B}(G)(\partial(s)) & \xrightarrow{\mathbf{S}(G)_s} & \mathbf{B}(G)(p_j(\partial(s))) \\ \downarrow & & \downarrow \\ \mathbf{B}(u_G(\partial(s))) \cong \mathbf{B}(\partial(\widetilde{u}_G(s))) & \xrightarrow{\mathbf{S}(\widetilde{u}_G(s))} & \mathbf{B}(p_j(\partial(\widetilde{u}_G(s)))) \cong \mathbf{B}(u_G(p_j(\partial(s)))) \end{array} \tag{20}$$

We can now give formulations for the conditions TT, SS, TS, ST and STid.

Definition 4.5 *[**2014.10.16.def3.fromold**] Let us define the following conditions on a B-system data* $(\mathbf{B}, \mathbf{T}, \mathbf{S}, \delta)$*:*

1. *The TT-condition. For any* $GT \in B_{i+1}$*,* $GDT' \in B_{j+1}(p_i(GT))$ *the pentagon of B-carrier functions*

$$\begin{array}{ccc} \mathbf{B}(p_i(GT))(p_j(GDT')) & \xrightarrow{\mathbf{T}(p_i(GT))_{GDT'}} & \mathbf{B}(p_i(GT))(GDT') \\ \mathbf{T}_{GT}(p_j(GDT'))\downarrow & & \\ \mathbf{B}(GT)(T_{GT}(p_j(GDT'))) & & \downarrow\mathbf{T}_{GT}(GDT') \\ \cong\downarrow & & \\ \mathbf{B}(GT)(p_j(T_{GT}(GDT'))) & \xrightarrow{\mathbf{T}_{T_{GT}(GDT')}(GT)} & \mathbf{B}(GT)(\mathbf{T}_{GT}(GDT')) \end{array} \tag{21}$$

commutes.

2. *The SS-condition. For any* $s \in \widetilde{B}_{i+1}$, $s' \in \widetilde{B}_{j+1}(\partial(s))$ *the diagram of B-carrier functions*

$$\begin{array}{ccc}
\mathbf{B}(\partial(s))(\partial(s')) & \xrightarrow{\mathbf{S}(\partial(s))_{s'}} & \mathbf{B}(\partial(s))(p_j(\partial(s'))) \\
\mathbf{S}_s(\partial(s'))\downarrow & & \downarrow \mathbf{S}_s(p_j(\partial(s'))) \\
\mathbf{B}(p_i(\partial(s)))(S_s(\partial(s'))) & & \mathbf{B}(p_i(\partial(s))(S_s(p_j(\partial(s'))))) \\
\cong\downarrow & & \downarrow\cong \\
\mathbf{B}(p_i(\partial(s)))(\partial(\widetilde{S}_s(s'))) & \xrightarrow{\mathbf{S}(p_i(\partial(s)))_{\widetilde{S}_s(s')}} & \mathbf{B}(p_i(\partial(s)))(p_j(\partial(\widetilde{S}_s(s'))))
\end{array} \tag{22}$$

commutes.

3. *The TS-condition. For any* $GT \in B_{i+1}$, $s' \in \widetilde{B}_{j+1}(p_i(GT))$ *the diagram of B-carrier functions*

$$\begin{array}{ccc}
\mathbf{B}(p_i(GT))(\partial(s')) & \xrightarrow{\mathbf{S}(p_i(GT))_{s'}} & \mathbf{B}(p_i(GT))(p_j(\partial(s'))) \\
\mathbf{T}_{GT}(\partial(s'))\downarrow & & \downarrow \mathbf{T}_{GT}(p_j(\partial(s'))) \\
\mathbf{B}(GT)(T_{GT}(\partial(s'))) & & \mathbf{B}(GT)(T_{GT}(p_j(\partial(s')))) \\
\cong\downarrow & & \downarrow\cong \\
\mathbf{B}(GT)(\partial(\widetilde{T}_{TG}(s'))) & \xrightarrow{\mathbf{S}(GT)_{\widetilde{T}_{GT}(s')}} & \mathbf{B}(GT)(p_j(\partial(\widetilde{T}_{GT}(s'))))
\end{array} \tag{23}$$

4. *The ST-condition. For any* $s \in \widetilde{B}_{i+1}$, $GTDT' \in B_{j+1}(\partial(s))$ *the diagram of B-carrier functions*

$$\begin{array}{ccc}
\mathbf{B}(\partial(s))(p_j(GTDT')) & \xrightarrow{\mathbf{T}(\partial(s))_{GTDT'}} & \mathbf{B}(\partial(s))(GTDT') \\
\mathbf{S}_s(p_j(GTDT'))\downarrow & & \\
\mathbf{B}(p_i(\partial(s)))(S_s(p_j(GTDT'))) & & \downarrow \mathbf{S}_s(GTDT') \\
\cong\downarrow & & \\
\mathbf{B}(p_i(\partial(s)))(p_j(S_s(GTDT'))) & \xrightarrow{\mathbf{T}(p_i(\partial(s)))_{S_s(GTDT')}} & \mathbf{B}(p_i(\partial(s)))(S_s(GTDT'))
\end{array} \tag{24}$$

5. *The STid-condition. For any* $s \in \widetilde{B}_{i+1}$ *one has*

$$(\mathbf{B}(p_i(\partial(s))) \xrightarrow{T_{\partial(s)}} \mathbf{B}(\partial(s)) \xrightarrow{S_s} \mathbf{B}(p_i(\partial(s)))) = id_{\mathbf{B}(p_i(\partial(s))}$$

Formulation of the remaining four conditions that involve δ is more difficult since their formulation using this approach leads to conditions that depend on the conditions from the first group. We leave their study for the future.

5 An approach to B-systems using the length function.

In formalization of B-systems (as well as C-systems) in Coq one of the main technical difficulties that arises is the need to work with a family of types B_n which are dependent on $n \in \mathbf{N}$. Due to the absence of strong substitutional equality in Coq types such as $B_{n+(m+1)}$ and $B_{(n+m)+1}$ do not have same elements and can only be dealt with as being connected by an equivalence. Eventually we hope that this issue will be resolved but at the moment an alternative approach to formalization where the families of types B_n and $\widetilde{B}_n$ are replaced by their total spaces together with the functions from these total spaces to $\mathbf{N}$ may be useful.

In this approach we will have only two sorts B and $\widetilde{B}$ but the presentation will cease to be essentially algebraic.

Instead we consider the following:

Definition 5.1 *[***2014.10.26.def1***] A non-unital pre-l-B-system (in sets) is the following collection of data:*

1. *two sets B and $\widetilde{B}$,*
2. *a function $l : B \to \mathbf{N}$,*
3. *a function $\partial : \widetilde{B} \to B$ such that for all $s \in \widetilde{B}$, $l(\partial(s)) > 0$,*
4. *a function $ft : B \to B$ such that*
   - *(a) for all b such that $l(b) > 0$ one has $l(ft(b)) = l(b) - 1$,*
   - *(b) for all b such that $l(b) = 0$ one has $l(ft(b)) = 0$,*
5. *for each $i \geq 0$ four operations:*

$$T_i : (X \in B, Y \in B, l(X) > 0, l(Y) > i, ft(X) = ft^{i+1}(Y)) \to B$$
$$\widetilde{T}_i : (X \in B, r \in \widetilde{B}, l(X) > 0, l(\partial(r)) > i, ft(X) = ft^{i+1}(\partial(r))) \to \widetilde{B}$$
$$S_i : (s \in \widetilde{B}, Y \in B, \partial(s) = ft^{i+1}(Y)) \to B$$
$$\widetilde{S}_i : (s \in \widetilde{B}, r \in \widetilde{B}, \partial(s) = ft^{i+1}(\partial(r))) \to \widetilde{B}$$

   such that:
   - *(a) $l(T_i(X, Y)) = l(Y) + 1$,*
   - *(b) $l(\partial(\widetilde{T}_i(X, r))) = l(\partial(r)) + 1$,*
   - *(c) $l(S_i(s, X)) = l(X) - 1$,*
   - *(d) $l(\partial(\widetilde{S}_i(s, r))) = l(\partial(r)) - 1$.*

Definition 5.2 *[***2014.10.26.def2***] A unital pre-l-B-system is a non-unital pre-l-B-system together with an operation*

$$\delta : (X \in B, l(X) > 0) \to \widetilde{B}$$

such that $l(\partial(\delta(X))) = l(X) + 1$.

Definition 5.3 *[***2014.12.05.def3***] A pre-l-B-system is a pre-l-B-system together with an element $pt \in B$ such that $l(pt) = 0$.*

It is easy now to define non-unital and unital l-B0-systems and l-B-systems.