A note on the Archimedean property

To Cathrine Lelay from Vladimir Voevodsky, March 15, 2016

Hi Cath,

here is something that I suggest to implement that may be nicer than the approach you have used. It only concerns the part of the theory that goes into Algebra.

The relation $>$ below is everywhere supposed to be a transitive binary operation relation.

Definition 1 *[**2016.03.10.def1**] Let M be an abelian (additive) monoid. A relation $>$ on M is said to be Archimedean if for all $x, y_1, y_2 \in M$ such that $y_1 > y_2$ one has:*

1. *there exists $n \in \mathbf{N}$ such that $ny_1 + x > ny_2$,*
2. *there exists $n' \in \mathbf{N}$ such that $n'y_1 > x + n'y_2$.*

Lemma 2 *[**2016.03.14.l1**] Let M be an abelian group and $>$ be a transitive binary operation relation on M. Then $>$ is Archimedean if and only if for all $x, y_1 > y_2$ in M there exists $n \in nat$ such that $ny_1 + x > ny_2$.*

Proof: The "only if" part is obvious since the condition of the lemma is one of the two conditions of Definition 1. To show the 'if" implication we need to prove that if the condition of the lemma holds then the second condition of Definition 1 holds, i.e., that for $x, y_1 > y_2$ in M there exists n' such that $n'y_1 > x + n'y_2$. Let $n' \in \mathbf{N}$ be such that the condition of the lemma is satisfied for $-x, y_1, y_2$, i.e., such that $n'y_1 + (-x) > n'y_2$. Since $>$ is a binary operation relation this inequality implies that $n'y_1 > n'y_2 + x$ or, equivalently using commutativity, that $n'y_1 > x + n'y_2$.

Remark 3 [**2016.03.14.rem1**] The analog of Lemma 2 with the condition $ny_1 + x > ny_2$ replaced with the condition $n'y_1 > n'y_2 + x$ holds as well.

Theorem 4 *[**2016.03.10.th1**] Let R be a rig with a transitive relation $>$ that is a binary operation relation for the addition and satisfies $1 > 0$. Then its additive monoid if Archimedean if and only if the following three conditions hold:*

1. *for all $y_1 > y_2$ there exists $m \in \mathbf{N}$ such that $my_1 > 1 + my_2$,*
2. *for all x there exists $n \in \mathbf{N}$ such that $n > x$,*
3. *for all x there exists $n \in \mathbf{N}$ such that $n + x > 0$.*

Proof: Suppose that the additive monoid is Archimedean. Then one has:

1. let $y_1 > y_2$, applying the second condition of Definition 1 to $1, y_1, y_2$ we find m such that $my_1 > 1 + my_2$,
2. let $x \in M$, applying the second condition of Definition 1 to $x, 1, 0$ we find n such that $n > x$,
3. let $x \in M$, applying the first condition of Definition 1 to $x, 1, 0$ we find n such that $n + x > 0$.

Suppose that the three conditions of the theorem hold. Let $x, y_1, y_2 \in M$ and $y_1 > y_2$. Let m be such that $my_1 > 1 + my_2$. Then one has:

1. let n be such that $n + x > 0$ then $nmy_1 + x > n + nmy_2 + x > nmy_2$,

2. let n be such that $n > x$ then $nmy_1 > n + nmy_2 > x + nmy_2$.

Remark 5 The first condition of Theorem 4 can be interpreted as expressing the property that R does not have infinitesimally close to each other pairs of elements, the second as expressing the fact that R does not have infinitely large elements and the third as expressing the fact that R does not have infinitely small elements.

Corollary 6 *[**2016.03.14.cor1**] Let R be a ring and $>$ be a transitive relation on R that is a binary operation relation relative to the additive structure of R. Then the additive group of R is Archimedean if and only if the following two conditions hold:*

1. *for all $y > 0$ there exists $m \in \mathbf{N}$ such that $my > 1$,*
2. *for all x there exists $n \in \mathbf{N}$ such that $n > x$.*

Proof: Condition (1) of the corollary implies Condition (1) of the theorem by taking $y = y_1 - y_2$ and conversely Condition (1) of the corollary implies Condition (1) of the theorem by taking $y_1 = y$, $y_2 = 0$.

Condition (2) of the corollary coincides with the Condition (2) of the theorem.

Condition (2) of the corollary implies Condition (3) of the theorem when applied to $-x$.

Definition 7 *[**2016.03.06.def1**] For an abelian (additive) monoid M with a relation $>$ define a new relation $>_c$ by the rule that $x_1 >_c x_2$ if there exists $c \in$ such that $x_1 + c >_c x_2 + c$.*

Lemma 8 *[**2016.03.10.l1**] In the context of Definition 7, if $>$ is a transitive binary operation relation then so is $>_c$.*

Proof: Omitted.

Lemma 9 *[**2016.03.07.l1**] Let M, $>$ be as above. Suppose that M is a group. Then $>$ equals $>_c$.*

Proof: If $x > y$ then $x >_c y$ for $c = 0$. If $x + c > y + c$ then, since $>$ is a binary operation relation, $x + c + (-c) > y + c + (-c)$ which is equivalent to $x > y$.

Definition 10 *[**2016.03.07.def1**] For M and $>$ as above we say that $>$ is differentially Archimedean if $>_c$ is Archimedean.*

Let M be an abelian monoid. Denote by M^+ the abelian group of differences of M. For a transitive binary operation relation $>$ on M define a relation $>$ on M^+ setting that $(x_1, x_2) > (y_1, y_2)$ if and only if $x_1 + y_2 >_c y_1 + x_2$.

Lemma 11 *[**2016.03.10.l2**] Let M, $>$ be as above. Then the relation $>$ on M^+ is a transitive binary operation relation.*

Proof: Omitted.

Theorem 12 *[**2016.03.10.th2**] Let M, $>$ be as above. Then $>$ on M^+ is Archimedean if and only if $>$ on M is differentially Archimedean.*

Proof: Suppose that $(M^+, >)$ is Archimedean. Let $x, y_1, y_2 \in M$ be such that $y_1 >_c y_2$. Then $(y_1, 0) > (y_2, 0)$ in M^+ and one has:

1. There exists n such that $n(y_1, 0) + (x, 0) > n(y_2, 0)$ in M^+. By definition of $>$ on M^+ this is equivalent to $ny_1 + x >_c ny_2$ in M.
2. There exists n' such that $n'(y_1, 0) > (x, 0) + n'(y_2, 0)$ in M^+. By definition of $>$ on M^+ this is equivalent to $n'y_1 >_c x + n'y_2$ in M.

This improves one implication.

To prove the second implication suppose that $(M, >)$ is differentially Archimedean. Let

$$(x, x'), (y_1, y_1') > (y_2, y_2') \in M^+.$$

By definition this means that $y_1 + y_2' >_c y_2 + y_1'$. In view of Lemma 2 it is sufficient to find N such that

$$N(y_1, y_1') + (x, x') > N(y_2, y_2')$$

that is, such that

$$[\mathbf{2016.03.14.eq1}] Ny_1 + x + Ny_2' >_c Ny_2 + Ny_1' + x' \qquad (1)$$

Let n be such that

$$ny_1 + ny_2' + x > ny_2 + ny_1'$$

and n' be such that

$$n'y_1 + n'y_2' > n'y_2 + n'y_1' + x'$$

Then

$$(n + n')y_1 + (n + n')y_2' + x > (n + n')y_2 + (n + n')y_1' + x'$$

that is, $N = n + n'$ satisfies the inequality (1). This completes the proof of Theorem 12.

Corollary 13 *[**2016.03.14.cor2**] Let R be a rig with a transitive, additive binary operation relation $>$ such that $1 > 0$. Then the ring of differences R^+ of R is Archimedean if and only if $>$ is differentially Archimedean on the additive monoid of R. In particular, R^+ is Archimedean if $>$ on R satisfies the three conditions of Theorem 4.*

Proof: The first assertion is straightforward. The second assertion follows from a simple proof that if $>$ satisfies the three conditions of Theorem 4 then so does $>_c$ and therefore the first assertion of the corollary applies.