Notes on categorical probability

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Contents

1 Introduction

Let us look at the standard approach to mathematical modeling of a deterministic process. One starts with a set X and a family of maps $\phi_{t1,t2} : X \to X$ where $t1, t2$ are two numbers which are

whose points correspond to the possible states of the system in question. A change in the state of the system is modeled as a map from this set to itself. A "process" is usually a family of such maps – one for each interval $[t_0, t_1]$ of the line representing time, which satisfy the obvious composition condition for intervals of the form $[t_0, t_1]$, $[t_1, t_2]$ and $[t_0, t_2]$. For example, any (deterministic) computer program which takes t_0, t_1, and the state of the system at time t_0 as an input and produces the state of the system at time t_1 as an output defines a "process" in the sense specified above.

If the program we use is not deterministic but uses a random number generator to compute new values of the variables from the old ones it does not define such a process.

Consider now the case when we have a process whose computer model is based on a randomized algorithm to produce the new values of the variables from the old ones. As an example we may look at a simple population dynamics model where the the state of the system is determined by the number of organisms currently alive, time is discrete and to produce the state at the next moment of time our algorithm uses a random number generator to determine whether a given organism survives (with probability p) or dies (with probability $1-p$).

Note that all the notions used in the mathematical description of a deterministic process naturally belong to the language of the category theory: we have a set X and a family of morphisms (maps) $f_{[t_0,t_1]} : X \to X$ satisfying the composition condition.

The stochastic category described below allows one to repeat the same description in a randomized case simply by replacing the category of sets with the stochastic category.

For related material see also [3], [4], [10], [2], [7].

2 Stochastic categories

2.1 The category of measurable spaces

Let us first recall the following definition.

Definition 2.1.1 *A σ-algebra $\mathfrak{R}$ on a set X is a collection of subsets of X satisfying the following conditions.*

1. *The empty subset is in $\mathfrak{R}$.*
2. *For a countable family U_i of elements of $\mathfrak{R}$ one has $\cup_i U_i \in \mathfrak{R}$.*
3. *For U in $\mathfrak{R}$ the complement $X \backslash U$ to U in X is in $\mathfrak{R}$.*

For a collection $\mathfrak{R}$ of subsets of X we let $cl_\sigma(\mathfrak{R})$ denote the smallest σ-algebra which contains $\mathfrak{R}$. For a set of σ-algebras $\mathfrak{R}_\alpha$ on X the collection of subsets $\bigcap_\alpha \mathfrak{R}_\alpha$ is the largest σ-algebra contained in all $\mathfrak{R}_\alpha$ and we will write

$$\sum_\alpha \mathfrak{R}_\alpha = cl_\sigma(\cup_\alpha \mathfrak{R}_\alpha)$$

for the smallest σ-algebra which contains all of the $\mathfrak{R}_\alpha$.

Let $f : X \to Y$ be a map of sets.

1. For a collection $\mathfrak{R}$ of subsets of X we let $f(\mathfrak{R})$ denote the collection of subsets of Y of the form $f(U)$ where $U \in \mathfrak{R}$,
2. For a collection $\mathfrak{R}$ of subsets of X we let $f_\#(\mathfrak{R})$ denote the collection of subsets U of Y such that $f^{-1}(U) \in \mathfrak{R}$.
3. For a collection $\mathfrak{S}$ of subsets of Y we let $f^{-1}(\mathfrak{S})$ denote the collection of subsets of X of the form $f^{-1}(U)$ where $U \in \mathfrak{S}$.

It is easily seen that if $\mathfrak{R}$ (resp. $\mathfrak{S}$) is a σ-algebra then $f_\#(\mathfrak{R})$ (resp. $f^{-1}(\mathfrak{S})$) is a σ-algebra. The collection of subsets $f(\mathfrak{R})$ is usually not a σ-algebra.

Definition 2.1.2 *The category $\mathcal{MS}$ of measurable space is defined as follows:*

Objects of $\mathcal{MS}$ are measurable spaces i.e. pairs of the form $(X, \mathfrak{R})$ where X is a set and $\mathfrak{R}$ is a σ-algebra of subsets of X.

Morphisms from $(X, \mathfrak{R})$ to $(Y, \mathfrak{S})$ are maps of sets $f : X \to Y$ such that for each $V \in \mathfrak{S}$ one has $f^{-1}(V) \in \mathfrak{R}$.

Compositions of morphisms and the identity morphisms correspond to the compositions of maps of sets and to the identity maps of sets.

The associativity of the composition and the defining property of the identity maps are obvious and therefore $\mathcal{MS}$ is indeed a category.

Sending $(X, \mathfrak{R})$ to X we get a functor from $\mathcal{MS}$ to the category *Sets* of sets. This functor has two adjoints. The right adjoint sends X to $(X, \{\emptyset, X\})$ and the left adjoint to $(X, 2^X)$ where 2^X is the set of all subsets of X. We will say that a morphism in $\mathcal{MS}$ is surjective, injective or bijective if the morphism of the underlying sets has the corresponding property.

The measurable spaces $(\emptyset, \{\emptyset\})$ and $(pt, 2^{pt})$ give us an initial object and a final object of $\mathcal{MS}$. To simplify the notation we will write $\emptyset$ instead of $(\emptyset, 2^{\emptyset})$ and pt instead of $(pt, 2^{pt})$.

For there are three natural ways to form a new measurable space starting with a family of measurable spaces $(X_\alpha, \mathfrak{R}_\alpha)$:

$$\coprod_\alpha (X_\alpha, \mathfrak{R}_\alpha) = (\coprod X_\alpha, \cap_\alpha i_{\alpha,\#}(R_\alpha))$$

$$\prod_\alpha (X_\alpha, \mathfrak{R}_\alpha) = (\prod X_\alpha, cl_\sigma(\cup_\alpha pr_\alpha^{-1}(R_\alpha)))$$

$$\prod_\alpha^{\mathcal{K}} (X_\alpha, \mathfrak{R}_\alpha) = (\coprod X_\alpha, \sum_\alpha i_{\alpha,\#}(R_\alpha))$$

where i_α and pr_α are the canonical embeddings and projections respectively.

Lemma 2.1.3 *[**prcopr**] The space $\coprod_\alpha (X_\alpha, \mathfrak{R}_\alpha)$ is the coproduct of the family $(X_\alpha, \mathfrak{R}_\alpha)$ in $\mathcal{MS}$ and the space $\prod_\alpha (X_\alpha, \mathfrak{R}_\alpha)$ is the product of the family $(X_\alpha, \mathfrak{R}_\alpha)$ in $\mathcal{MS}$.*

Proof: ???

A categorical meaning for the space $\prod_\alpha^{\mathcal{K}} (X_\alpha, \mathfrak{R}_\alpha)$ will be given in Lemma 2.2.13 below.

Theorem 2.1.4 *[**mscomplete**] The category $\mathcal{MS}$ is a complete category i.e. any small diagram in $\mathcal{MS}$ has a limit.*

Proof: By [9, Theorem 1, p.113] it is sufficient to show that products and equalizers exist in $\mathcal{MS}$. By Lemma 2.1.3 we know that products exist.

Let $f, g : (X, \mathfrak{R}) \to (Y, \mathfrak{S})$ be a pair of morphisms in $\mathcal{MS}$. Consider the equalizer diagram in Sets corresponding to f and g

$$Z \xrightarrow{i} X \rightrightarrows Y$$

and define the equalizer of f and g in $\mathcal{MS}$ by the formula

$$[\mathbf{eqdef}] eq(f, g) = (Z, i^{-1}(\mathfrak{R})) \tag{1}$$

as in the case of the product one verifies easily that together with the obvious morphism $eq(f, g) \to X$ this measurable space is indeed the equalizer of the morphisms f and g in $\mathcal{MS}$.

Remark 2.1.5 [**powerspace**] Let X be a set and $(Y, \mathfrak{S})$ a measure space. The product of as many copies of $(Y, \mathfrak{S})$ as there are elements in X can also be described in a slightly different way. Consider the set Y^X of all maps of sets from X to Y. For any V in $\mathfrak{S}$ and any x in X let $A(x, V)$ be the set of all $g : X \to Y$ such that $g(x) \in V$. Let $\mathfrak{S}^X$ be the σ-algebra on Y^X generated by the subsets $A(x, V)$. Then our product is given by $(Y, \mathfrak{S})^X = (Y^X, \mathfrak{S}^X)$.

Theorem 2.1.6 *[***mscocomplete***] The category $\mathcal{MS}$ is co-complete i.e. any small diagram in $\mathcal{MS}$ has a colimit.*

Proof: By [9, Theorem 1, p.113] applied to the category $\mathcal{MS}^{op}$ it is sufficient to show that $\mathcal{MS}$ has coproducts and coequalizers. By Lemma 2.1.3 we know that coproducts exist.

Let $f, g : (X, \mathfrak{R}) \to (Y, \mathfrak{S})$ be a pair of morphisms in $\mathcal{MS}$. Consider the coequalizer diagram in Sets corresponding to f and g

$$X \rightrightarrows Y \stackrel{p}{\to} Z$$

and define the coequalizer of f and g in $\mathcal{MS}$ by the formula

$$[\mathbf{coeqdef}] coeq(f, g) = (Z, p_{\#}(\mathfrak{S})). \tag{2}$$

As in the case of the coproduct one verifies easily that together with the obvious morphism $(Y, \mathfrak{S}) \to coeq(f, g)$ this measurable space is indeed the coequalizer of the morphisms f and g in $\mathcal{MS}$.

Lemma 2.1.7 *[***epimono1***] A morphism $f : (X, \mathfrak{R}) \to (Y, \mathfrak{S})$ in $\mathcal{MS}$ is an epimorphism (resp. a monomorphism) if and only if it is surjective (resp. injective).*

Proof: The 'if' part is obvious both for epimorphisms and for monomorphisms. Let us prove the 'only if' parts. Assume that f is a monomorphism. Then it is injective since otherwise there would be two different morphisms from the point *pt* to X whose compositions with f coincide. Assume that f is an epimorphism. Then it is surjective since otherwise there would be two different morphisms from Y to $(\{0, 1\}, 2^{\{0,1\}})$ whose compositions with f coincide.

Recall that a morphism $X \to Y$ is called an effective epimorphism if $X \times_Y X \rightrightarrows X \stackrel{f}{\to} Y$ is a coequalizer diagram and an effective monomorphism if it is an effective epimorphism in the opposite category.

Lemma 2.1.8 *[***epimono2***] A morphism $f : (X, \mathfrak{R}) \to (Y, \mathfrak{S})$ in $\mathcal{MS}$ is an effective epimorphism iff it is an epimorphism and $\mathfrak{S} = f_{\#}(\mathfrak{R})$. It is an effective monomorphism iff it is a monomorphism and $\mathfrak{R} = f^{-1}(\mathfrak{S})$.*

Proof: The statement for the epimorphisms follows from (2) and the statement for the monomorphisms from (1).

Example 2.1.9 [**bijective**] Let X be a set and $\mathfrak{R}_2 \subset \mathfrak{R}_1$ be two σ-algebras on X. Then the identity of X defines a bijective morphism $(X, \mathfrak{R}_1) \to (X, \mathfrak{R}_2)$. This morphism is an epimorphism and a monomorphism but unless $\mathfrak{R}_2 = \mathfrak{R}_1$ it is not an isomorphism.

Proposition 2.1.10 *[***epimono3***] For any morphism $f : (X, \mathfrak{R}) \to (Y, \mathfrak{S})$ there exists a unique decomposition of the form $f = i \circ b \circ p$ where i is an effective monomorphism, b is a bijection and p is an effective epimorphism.*

Proof: Let $X \stackrel{p}{\to} Z \stackrel{i}{\to} Y$ be the decomposition of f into a surjection and an injection in the category of sets. It defines a decomposition of f in the category $\mathcal{MS}$ of the form

$$(X, \mathfrak{R}) \stackrel{p}{\to} (Z, p_{\#}(\mathfrak{R})) \stackrel{b}{\to} (Z, i^{-1}(\mathfrak{S})) \stackrel{i}{\to} (Y, \mathfrak{S})$$

which satisfies the conditions of the proposition by Lemmas 2.1.7 and 2.1.8. The uniqueness easily follows from the same two lemmas.

2.2 Category of kernels

We define the *category of kernels* $\mathcal{K}$ as follows. Objects of $\mathcal{K}$ are pairs $(X, \mathfrak{R})$ where X is a set and $\mathfrak{R}$ is a σ-algebra of subsets of X i.e. objects are measurable spaces. Morphisms in $\mathcal{K}$ are called *kernels*.

Definition 2.2.1 *[***d1***] A kernel $f = f(x, U)$ from $(X, \mathfrak{R})$ to $(Y, \mathfrak{S})$ is a function*

$$f(-,-) : X \times \mathfrak{S} \to [0, \infty]$$

such that for any $x \in X$ the function

$$f(x,-) : U \mapsto f(x, U)$$

is a measure on $(Y, \mathfrak{S})$ and for any $U \in \mathfrak{S}$ the function

$$f(-, U) : x \mapsto f(x, U)$$

is a measurable function on $(X, \mathfrak{R})$.

For a measure μ on $(X, \mathfrak{R})$, a measurable function f on the same space and a measurable subset Y of X we let

$$\int_Y f d\mu$$

denote the integral of f restricted to Y with respect to μ.

Lemma 2.2.2 *[***comp1***] Let f be a kernel $(X, \mathfrak{R}) \to (Y, \mathfrak{S})$ and $g : Y \to [0, \infty]$ be a non-negative measurable function on Y. Then the function*

$$f^*(g) : x \mapsto \int_Y g df(x,-)$$

is a measurable function on $(X, \mathfrak{R})$.

Proof: Consider the class C of all g such that $f^*(g)$ is measurable. By definition of a kernel this class contains defining functions I_U of subsets U in $\mathfrak{S}$. Hence it contains all non-negative simple functions on $(Y, \mathfrak{S})$. The continuity property of the integral (e.g. [1, Th.15.1(iii),p.204]) implies that if $0 \leq g_n \uparrow g$ where g_n are in C then g is in C. By [1, Th.13.5, p.185] the smallest class satisfying these two properties contains all measurable functions.

Now let $f : (X, \mathfrak{R}) \to (Y, \mathfrak{S})$, $g : (Y, \mathfrak{S}) \to (Z, \mathfrak{T})$ be two kernels. Consider the function on $X \times \mathfrak{T}$ of the form

$$[\textbf{comp2}](x, W) \mapsto \int_Y g(-, W) df(x,-) \tag{3}$$

This function is well defined since $g(-,W)$ is measurable. For each W it is a measurable function on $(X,\mathfrak{R})$ by Lemma 2.2.2. On the other hand for any x the function

$$W \mapsto \int_Y g(-,W)df(x,-)$$

is a measure on $(Z,\mathfrak{T})$ by the standard properties of the integral. Therefore, (3) defines a kernel from $(X,\mathfrak{R})$ to $(Z,\mathfrak{T})$ which we denote by $g \circ f$ and call the composition of f and g.

For every $(X,\mathfrak{R})$ the kernel Id which takes x to the measure δ_x concentrated in x is the identity morphism. The following three lemmas imply that our composition is associative and therefore measure spaces, kernels and compositions (3) define a category. We denote this category by $\mathcal{K}$ and call the category of kernels.

Lemma 2.2.3 *[**funcmes**] Let μ be a measure on $(X,\mathfrak{R})$ and $f : (X,\mathfrak{R}) \to (Y,\mathfrak{S})$ a kernel. Then the function $f_*(\mu)$ on $\mathfrak{S}$ of the form*

$$U \mapsto \int_X f(-,U)d\mu$$

is a measure on $(Y,\mathfrak{S})$.

Proof: Obvious.

Lemma 2.2.4 *[**tudysyudy**] Let $f : (X,\mathfrak{R}) \to (Y,\mathfrak{S})$ be a kernel, μ a measure on $(X,\mathfrak{R})$ and g a measurable non-negative function on $(Y,\mathfrak{S})$. Then one has*

$$\int f^*(g)d\mu = \int gdf_*(\mu)$$

Proof: If g is the simple function corresponding to a subset $U \in \mathfrak{S}$ then our equality holds by definitions. For a general g the result follows by the same continuity argument as in the proof of Lemma 2.2.2.

Lemma 2.2.5 *[**assos**] The composition of kernels defined by (3) is associative.*

Proof: It follows immediately from definitions and Lemma 2.2.4.

For a topological space X we will write simply X instead of the usual $(X,\mathcal{B})$ for the measure space with the underlying set X and the underlying σ-algebra the Borel σ-algebra on X. We will further consider sets as topological spaces with the discrete topology (all subsets are open). Combining these two conventions we will write X for the measure space with the underlying set X and the underlying σ-algebra of all subsets of X.

Example 2.2.6 *[**ex0**]*For any $(X,\mathfrak{R})$ there is a unique kernel from $\emptyset$ to $(X,\mathfrak{R})$. Therefore $\emptyset$ is the initial object of the category of kernels. Since there is a unique measure on $\emptyset$ there is also a unique kernel from any $(X,\mathfrak{R})$ to the empty set i.e. $\emptyset$ is also the final object.

Example 2.2.7 *[**ex1**]*We will denote the object of $\mathcal{K}$ corresponding to the one element set by **1**. A morphism from **1** to $(X,\mathfrak{R})$ is the same as a measure on $(X,\mathfrak{R})$. A morphism from $(X,\mathfrak{R})$ to **1** is a non-negative measurable function or an (unbounded) random variable on $(X,\mathfrak{R})$. In particular

$$[\mathbf{h11}]Hom(\mathbf{1},\mathbf{1}) = \mathbf{R}_{\geq 0} \cup \{\infty\} \qquad (4)$$

and for any $(X,\mathfrak{R})$ the composition pairing

$$Hom(\mathbf{1},(X,\mathfrak{R}))\times Hom((X,\mathfrak{R}),\mathbf{1})\rightarrow Hom(\mathbf{1},\mathbf{1})$$

takes (μ,f) to $\int f\mu$. Note that the composition on (4) is of the form $(a,b)\mapsto ab$ where $0\infty=\infty 0=0$ as is usually assumed in measure theory.

Example 2.2.8 [**matrixex**] Let $\mathbf{n}$ be the measure space with the underlying set $\{1,\ldots,n\}$ and the σ-algebra of all subsets. Then $Hom(\mathbf{n},\mathbf{n})$ is the set of $n\times n$ matrices with entries from $[0,\infty]$. The composition is given by the product of matrices.

Example 2.2.9 [**ex0new**] Let $(X,\mathfrak{R})$ be a measurable space and f a non-negative measurable function on it. Then the mapping which sends a point x of X to the measure $f(x)\delta_x$ is a kernel which we denote I_f. If $\mu:\mathbf{1}\rightarrow(X,\mathfrak{R})$ is a measure on $(X,\mathfrak{R})$ the the composition $I_f\circ\mu$ is the 'product measure' which sends $U\in\mathfrak{R}$ to $\int_U f d\mu$. We will denote this measure by $f*\mu$.

Let $(X,\mathfrak{R})$, $(Y,\mathfrak{S})$ be measurable spaces and let $f:X\rightarrow Y$ be a measurable map. Sending $x\in X$ to the measure $\delta_{f(x)}$ on Y concentrated in $f(y)$ defines a morphism from $(X,\mathfrak{R})$ to $(Y,\mathfrak{S})$ in $\mathcal{K}$. To verify the integrability condition note that for a subset U in Y the function $x\mapsto\delta_{f(x)}(U)$ is the characteristic function of the subset $f^{-1}(U)$. Hence the second condition of Definition 2.2.1 is equivalent to the condition that f is measurable. This construction defines a functor from the category of measurable spaces and measurable maps to the category of kernels. To distinguish morphisms in $\mathcal{K}$ which correspond to maps of measure spaces from the general morphisms we will call the former *deterministic* morphisms.

Example 2.2.10 *[***ex5***]*Let $\mu:\mathbf{1}\rightarrow(X,\mathfrak{R})$ be a measure on $(X,\mathfrak{R})$ and $f:(X,\mathfrak{R})\rightarrow(Y,\mathfrak{S})$ a measurable map considered as a kernel. Then $f\circ\mu=f_*(\mu)$ is the "direct image" of μ with respect to f.

Example 2.2.11 *[***retract***]*Let $(X,\mathfrak{R})$ be a measure set and $(U,\mathfrak{R}_U)$ be a measurable subset of X considered with the induced σ-algebra. Then the embedding $(U,\mathfrak{R}_U)\rightarrow(X,\mathfrak{R})$ can be split by a projection p where $p(x,-)$ is zero for $x\in X-U$ and is the measure concentrated in x for $x\in U$. Hence any measurable subset (including the empty one) of a measure space is canonically a retract of this space in $\mathcal{K}$.

The functor from the category of measurable spaces to $\mathcal{K}$ does not reflect isomorphisms i.e. some morphisms of measurable spaces may become isomorphisms when considered in $\mathcal{K}$. Let $(Y,\mathfrak{S})$ be a measurable space and $f:X\rightarrow Y$ a be any surjection of sets. Then measures on $(X,f^{-1}(\mathfrak{S}))$ are in one-to-one correspondence with measures on $(Y,\mathfrak{S})$. In particular for each point $y\in Y$ we have a measure f_y on $(X,f^{-1}(\mathfrak{S}))$ corresponding to the delta measure δ_y on $(Y,\mathfrak{S})$. Sending y to f_y gives us a kernel $(Y,\mathfrak{S})\rightarrow(X,f^{-1}(\mathfrak{S}))$ and one verifies easily that it is inverse to the obvious kernel $(X,f^{-1}(\mathfrak{S}))\rightarrow(Y,\mathfrak{S})$. Hence, from the point of view of the category of kernels, the measurable spaces $(Y,\mathfrak{S})$ and $(X,f^{-1}(\mathfrak{S}))$ are indistinguishable.

Lemma 2.2.12 *[***copr***] Let $(X_\alpha,\mathfrak{R}_\alpha)$ be a family of measure spaces. The measure space $\coprod(X_\alpha,\mathfrak{R}_\alpha)$ is the coproduct of the family $(X_\alpha,\mathfrak{R}_\alpha)$ in $\mathcal{K}$.*

Proof: ???

Lemma 2.2.13 *[***pr***] Let $(X_\alpha,\mathfrak{R}_\alpha)$ be a family of measure spaces. The measure space $\prod^{\mathcal{K}}(X_\alpha,\mathfrak{R}_\alpha)$ is the product of the family $(X_\alpha,\mathfrak{R}_\alpha)$ in $\mathcal{K}$.*

Proof: ???

Lemmas 2.2.12 and 2.2.13 together with Example 2.2.6 show that $\mathcal{K}$ has both finite products and finite coproducts which coincide. The set of morphisms between any two objectsis an abelian semi-group and moreover a "module" over $\mathbf{R}_{\geq 0} \cup \{\infty\}$. However (since we do not allow negative measures) morphisms can not be subtracted and therefore $\mathcal{K}$ is not an additive category.

Lemmas 2.2.12 and 2.2.13 also imply that the countable products and coproducts in $\mathcal{K}$ coincide.

Example 2.2.14 [**prcopr2**]The set of natural numbers $\mathbf{N}$ considered with the σ-algebra of all subsets is both the product and the coproduct of a countable number of copies of $\mathbf{1}$. The sets $Hom_{\mathcal{K}}(\mathbf{N}, \mathbf{1})$ and $Hom_{\mathcal{K}}(\mathbf{1}, \mathbf{N})$ can both be identified with the set $[0, \infty]^{\mathbf{N}}$ of infinite sequences of (extended) non-negative real numbers.

Lemma 2.2.15 [**l1**] *Let G be a finite group of measurable automorphisms of a measure space $(X, \mathfrak{R})$. Then the measure space $(X/G, \mathfrak{R}^G)$ is the categorical quotient of $(X, \mathfrak{R})$ in $\mathcal{K}$ with respect to the action of G.*

Proof: ???

2.3 Category of bounded kernels

A kernel $f : (X, \mathfrak{R}) \to (Y, \mathfrak{S})$ is called bounded if the function

$$\beta_f : x \mapsto f(x, Y)$$

is a bounded function on X. Note that this condition means in particular that β_f takes only finite values i.e. that for any x the measure $f(x, -)$ on $(Y, \mathfrak{S})$ is finite. The composition of bounded kernels is bounded and therefore measure spaces and bounded kernels form a subcategory $\mathcal{K}^b$ in $\mathcal{K}$ called the category of bounded kernels.

Lemma 2.3.1 [**whenk**] *Let $(X, \mathfrak{R})$, $(Y, \mathfrak{S})$ be two measurable spaces and $f : X \times \mathfrak{S} \to R_{\geq 0}$ a mapping such that for any $x \in X$ the map $f(x, -)$ is a measure on $(Y, \mathfrak{S})$. Let further S be a collection of subsets of Y which is closed under finite unions and contains $\emptyset$ (resp. is closed under finite intersections and contains Y) such that $cl_\sigma(S) = \mathfrak{S}$. Then if the map $x \mapsto f(x, U)$ is measurable for any $U \in S$ then f is a kernel.*

Proof: ???

For $(X, \mathfrak{R})$, $(X', \mathfrak{R}')$ consider the measure space $(X \times X', \mathfrak{R} \times \mathfrak{R}')$ where $\mathfrak{R} \times \mathfrak{R}'$ is the σ-algebra generated by $U \times V$ with $U \in \mathfrak{R}$ and $V \in \mathfrak{R}'$. If $f : (X, \mathfrak{R}) \to (Y, \mathfrak{S})$ and $f' : (X', \mathfrak{R}') \to (Y', \mathfrak{S}')$ are bounded kernels define $f \times f'$ as the family which takes (x, x') to the product measure $f(x, -) \times f'(x', -)$ on $Y \times Y'$. Standard results about products of finite measures imply that $f \times f'$ is a bounded kernel. One can easily see that this construction defines a symmetric monoidal structure on $\mathcal{K}^b$ which we will denote by $\otimes$ instead of $\times$ to avoid confusion with the categorical product. The one element set is the unit of this monoidal structure which is why we denote it by $\mathbf{1}$.

Example 2.3.2 [**net1**] The standard example of a problem which one encounters if one tries to define the product of two measures one of which is not necessarily finite can be found in [12, p.78]. The source of the problem seems to lie in the fact that while all measures are continuous with respect to countable filtered colimits (cf. [12, Lemma 1.10(a)]) only finite measures are continuous with respect to countable filtered limits ([12, Lemma 1.10(b)]). Since limits are required to produce measurable subsets of the product of two measure spaces (e.g. the diagonal), a pair of measures on the factors can not be canonically extended to a measure on the product.

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Lemma 2.3.3 *[**kol**] Let T be a set, $(Y,\mathfrak{S})$ a measurable space and P_t a collection of probability measures on $(Y,\mathfrak{S})$ one for each $t \in T$. Then there exists a unique probability measure P on $(Y,\mathfrak{S})^T$ such that for any finite set of pairwise distinct elements $t_1, \ldots, t_n$ of T and any finite set $V_1, \ldots, V_n$ of elements of $\mathfrak{S}$ one has*

$$P(\cap_{i=1}^n A(t_i, V_i)) = \prod_{i=1}^n P_{t_i}(V_i)$$

where $A(t,V)$ is the set of all $f : T \to Y$ such that $f(t) \in V$.

Proof: See e.g. [11] or [8].

Example 2.3.4 [**paths1**] Let T be an interval of the real line. Then Y^T is the space of paths in Y. An elementary measurable subset $A(t,V)$ in $(Y,\mathfrak{S})^T$ is the subset of all paths γ such that $\gamma(t) \in V$. More generally $\cap_{i=1}^n A(t_i, V_i)$ in Y^T is the subset of all paths which pass through V_i at time t_i. Lemma 2.3.3 asserts that any non-deterministic path $\phi : T \to (Y,\mathfrak{S})$ defines a measure on $(Y,\mathfrak{S})^T$ such that the "size" of $\cap_{i=1}^n A(t_i, V_i)$ relative to this measure is the product of the probabilities (determined by ϕ) that t_i lands in V_i.

Let $ev : (Y,\mathfrak{S})^X \otimes X \to (Y,\mathfrak{S})$ be the evaluation morphism $(g,x) \mapsto g(x)$. Our choice of the σ-algebra on Y^X implies immediately that it is a measurable map. Consider μ_f as a morphism $\mathbf{1} \to (Y,\mathfrak{S})^X$. Then the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_f \otimes Id} & (Y,\mathfrak{S})^X \otimes X \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle ev} \\ X & \xrightarrow{f} & (Y,\mathfrak{S}) \end{array}$$

commutes and provides a canonical implementation of the morphism f. The obvious extension of this construction to bounded kernels $(X,\mathfrak{R}) \to (Y,\mathfrak{S})$ implies the following result.

Lemma 2.3.5 *[**hasanimpl**] For any bounded kernel $f : (X,\mathfrak{R}) \to (Y,\mathfrak{S})$ the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\mu_f \otimes Id} & (Y,\mathfrak{S})^X \otimes X \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle ev} \\ (X,\mathfrak{R}) & \xrightarrow{f} & (Y,\mathfrak{S}) \end{array}$$

where μ_f is the measure of Lemma 2.3.3, is an implementation of f.

Remark 2.3.6 For each $(X,\mathfrak{R})$ the diagonal $(X,\mathfrak{R}) \to (X,\mathfrak{R}) \otimes (X,\mathfrak{R})$ and the projection $(X,\mathfrak{R}) \to \mathbf{1}$ make $(X,\mathfrak{R})$ into a (commutative) comonoid in $\mathcal{K}^b$ with respect to the product $\otimes$. Note however that this structure is not natural with respect to morphisms in $\mathcal{K}$.

Remark 2.3.7 Let $f_\alpha : (X_\alpha, \mathfrak{R}_\alpha) \to (Y, \mathfrak{S})$ be a countable family of morphisms in $\mathcal{K}^b$. Our definitions imply that $\coprod f_\alpha$ is a bounded kernel if and only if the functions β_{f_α} are uniformly bounded. This observation shows in particular that the coproduct of our family in $\mathcal{K}$ is not its coproduct in $\mathcal{K}^b$.

Similarly for $f_\alpha : (X, \mathfrak{R}) \to (Y_\alpha, \mathfrak{S}_\alpha)$, the family which sends x to the measure $\sum f_\alpha(x, -)$ is not a bounded kernel unless this measure is finite i.e. unless

$$\sum \beta_{f_\alpha} < \infty$$

everywhere on X, which shows that the product of our family in $\mathcal{K}$ is not its product in $\mathcal{K}^b$.

One can also see (cf. 2.5.3 below) that sending a family $(X_\alpha, \mathfrak{R}_\alpha)$ to the coproduct space $\coprod(X_\alpha, \mathfrak{R}_\alpha)$ is not even a functor from the category of families of objects in $\mathcal{K}^b$ to $\mathcal{K}^b$. These properties make the category of bounded kernels to be of limited use. Instead one uses the stochastic category considered in the following section.

Let us also include in this section some very elementary facts about bounded measures on intervals and their distribution functions. For a measure μ on an interval $[u, v]$ of the real line the distribution function of μ is given by

$$Distr(\mu)(x) = \mu([u, x])$$

For any μ the function $Distr(\mu)$ is monotone non-decreasing, right continuous and has the property that $Distr(\mu)(u) = 0$. Conversely, for any function F with these properties there exists a unique measure μ (called Lebesgue-Stieltjes measure of F) such that $Distr(\mu) = F$ (see e.g. [12, p.33-34]).

For any bounded measure μ on $[u, v]$ define a function

$$X^+_\mu : [0, \mu([u, v])] \to [u, v]$$

by the rule

$$X^+_\mu = sup\{x \in [u, v] \,|\, Distr(x) \leq y\}$$

Then

$$\mu = (X^+_\mu)_*(dy)$$

where dy is the Lebesgue measure on $[0, \mu([u, v])]$ (see [12, p.34]). This is called Skorokhod representation of μ.

A measure μ is called non-atomic if $\mu(\{x\}) = 0$ for any point x. A measure is non-atomic if and only if its distribution function is continuous.

Lemma 2.3.8 *[**skor1**] Let μ be a non-atomic measure and $G = Distr(\mu)$. Then one has:*

1. *X^+_μ is an order preserving embedding whose image is the complement to the disjoint union of a countable number of intervals of the form $[x, x')$,*
2. *$G \circ X^+ = Id$,*
3. *$G_*(\mu) = dy$*

Proof: Since μ is non-atomic the function G is continuous. A continuous monotone non-decreasing function is strictly increasing on the complement of a countable number of intervals of the form $[x, x')$ and therefore defines an order preserving bijection between this complement and $[0, G(v)]$. The mapping X^+ is the composition of the inverse to this bijection with the inclusion of its image into $[u, v]$ which proves the first two assertions.

To prove the third assertion note that $G^{-1}([0, y]) = [u, x]$ where $G(x) = y$.

2.4 The stochastic category

A kernel $f : (X, \mathfrak{R}) \to (Y, \mathfrak{S})$ is called stochastic if for any x one has $f(x, Y) = 1$ i.e. if the corresponding measures are probability measures (in probability theory such kernels are also known as Markov kernels). Composition of stochastic kernels is stochastic. The subcategory generated by stochastic kernels is called the *stochastic category*. We denote it by $\mathcal{S}$. One may also consider the category of sub-probability kernels whose morphism are kernels such that $f(x, Y) \leq 1$.

Example 2.4.1 [**exsc1**]One obtains an important class of stochastic kernels as follows. Consider an (idealized) randomized computer algorithm A which takes as an input a sequence of real numbers $r_1, \ldots, r_m$ and produces as an output a sequence of real numbers $s_1, \ldots, s_n$. Let us assume that our computer has access only to the usual (i.e. equally distributed) random numbers on the interval $I = [0, 1]$. Then such an algorithm defines a map

$$\tilde{a} : \mathbf{R}^m \times I^\infty \to \mathbf{R}^n$$

where $\tilde{a}(s_1, \ldots, s_m; \rho_1, \ldots)$ is the result our algorithm will produce for the input $r_1, \ldots, r_m$ if its i-th request for a random number gives ρ_i. Consider the usual Lebesgue measure λ on I^∞. Then sending every $(r_1, \ldots, r_m)$ to the push-out of λ with respect to

$$\tilde{a}_{|(r_1,\ldots,r_m)\times I^\infty} : I^\infty \to \mathbf{R}^n$$

we get a stochastic kernel $a : \mathbf{R}^m \to \mathbf{R}^n$ which we call the kernel corresponding to A. This kernel takes (r, U) where $r \in \mathbf{R}^m$ and $U \subset \mathbf{R}^n$ to the probability that our algorithm will produce a result lying in U when given $r = (r_1, \ldots, r_m)$ as an input.

If A and B are two randomized algorithms such that the output of A can be used as an input for B we map consider the composed algorithm $B \circ A$. It is easy to see that the stochastic kernel corresponding to $B \circ A$ is the composition $b \circ a$ of the stochastic kernels corresponding to A and B. It is also easy to see that the stochastic kernel corresponding to an algorithm is a deterministic morphism if and only if our algorithm is essentially deterministic i.e. while it may request random numbers at some point the output does not depend on which random number it gets.

Note that for a non-empty $(X, \mathfrak{R})$ there are no stochastic kernels from $(X, \mathfrak{R})$ to $\emptyset$. Therefore, while $\emptyset$ is an initial object of the stochastic category it is not a finial object. On the other hand for any $(X, \mathfrak{R})$ there is exactly one stochastic kernel from $(X, \mathfrak{R})$ to $\mathbf{1}$. Therefore, $\mathbf{1}$ is the final object of the stochastic category but not of the category of kernels.

For $(X, \mathfrak{R})$ and $(X', \mathfrak{R}')$ the coproduct $(X, \mathfrak{R}) \coprod (X', \mathfrak{R}')$ in $\mathcal{K}$ is easily seen too be the coproduct of $(X, \mathfrak{R})$ and $(X', \mathfrak{R}')$ in the stochastic category. However it is not the product of $(X, \mathfrak{R})$ and $(X', \mathfrak{R}')$ in the stochastic category since the sum of two probability measures is not a probability measure.

For any measurable map of measure spaces $(X, \mathfrak{R}) \to (Y, \mathfrak{S})$ the corresponding morphism in $\mathcal{K}$ is stochastic. Therefore the functor from measurable spaces to the category of kernels factors through the stochastic category.

Our description of morphisms from infinite coproducts given above implies the following result.

Lemma 2.4.2 [**l3**] *Let $(X_\alpha, \mathfrak{R}_\alpha)$ be a family of measure spaces. Then $\coprod(X_\alpha, \mathfrak{R}_\alpha)$ of this family in $\mathcal{K}$ is also a coproduct in the stochastic category.*

Proof: ???

Note also that the finite group quotients of Lemma 2.2.15 remain quotients in the stochastic category.

The tensor product of two stochastic kernels is a stochastic kernel and therefore the symmetric monoidal structure defined above for the category of bounded kernels gives a similar structure on $\mathcal{S}$.

Example 2.4.3 *[***markov2***]*Let G be a set which is finite or countable. We consider G as a measure space with respect to the σ-algebra which contains all subsets of G. Then $Hom_{\mathcal{K}^b}(G, G)$ is the set of matrices $(p_{ij})_{i,j \in G}$ such that $p_{ij} \geq 0$, for any i the sum $p_i = \sum_j p_{ij}$ is finite and the set of numbers p_i is bounded. The set $Hom_{\mathcal{S}}(G, G)$ is the set of stochastic matrices with rows and columns numbered by elements of G. The composition of kernels corresponds in this description to multiplication of matrices. If P is an element of this set and $f : G \to \mathbf{1}$ a morphism in $\mathcal{K}$ (corresponding to a random variable by 2.2.7) then the sequence of random variables $f_n = f \circ G^n$ is called the Markov chain generated by the stochastic matrix P.

2.5 Branching morphisms and branching category

For a measure space $(X, \mathfrak{R})$ let $S^n(X, \mathfrak{R}) = (X, \mathfrak{R})^n / \Sigma_n$ be the n-th symmetric power of $(X, \mathfrak{R})$. For $n = 0$ we set $S^0(X, \mathfrak{R}) := \mathbf{1}$ for all $(X, \mathfrak{R})$ including the empty set. We further set

$$S^\bullet(X, \mathfrak{R}) = \coprod_{n \geq 0} S^n(X, \mathfrak{R})$$

Example 2.5.1 *[***ex6***]*We obviously have:

$$S^\bullet(\emptyset) = \mathbf{1}$$

and

$$S^\bullet(\mathbf{1}) = \mathbf{N}$$

Lemma 2.2.15 shows that for each n, $S^n(-)$ is a functor from the category of bounded kernels to itself. Since $S^\bullet(X, \mathfrak{R})$ is the coproduct of $S^n(X, \mathfrak{R})$ in $\mathcal{K}$ we conclude that $S^\bullet(-)$ is a functor from the category of bounded kernels to the category of all kernels. Finally, since coproduct of stochastic kernels is stochastic we conclude that both the individual symmetric powers $S^n(X, \mathfrak{R})$ and the total symmetric power $S^\bullet(X, \mathfrak{R})$ are functors from the stochastic category to itself.

Remark 2.5.2 For a sufficiently nice space $(X, \mathfrak{R})$ the space $S^\bullet(X, \mathfrak{R})$ is isomorphic to the space of integer-valued measures $M((X, \mathfrak{R}), \mathbf{Z}_+)$ on $(X, \mathfrak{R})$. This interpretation of the total symmetric power appears in some probabilistic texts on branching processes (e.g. [5]). The theory of measure valued branching processes studies the analogs of branching processes with the integer-valued measures replaced by more general measures.

Remark 2.5.3 *[***ex7***]*One can easily see that the total symmetric power $S^\bullet$ is not a functor from $\mathcal{K}^b$ to $\mathcal{K}^b$. Indeed consider a kernel $a : \mathbf{1} \to \mathbf{1}$ where $a > 1$ (see (4)). Then $S^n(a) = a^n$ and $S^\bullet(a)$ is not bounded since the volumes of corresponding measures on $\mathbf{N}$ are $a, a^2, \ldots$ which is an unbounded function on $\mathbf{N}$.

Definition 2.5.4 *[***d2***] A branching morphism ϕ from $(X, \mathfrak{R})$ to $(Y, \mathfrak{S})$ is a morphism in $\mathcal{S}$ of the form $(X, \mathfrak{R}) \to S^\bullet(Y, \mathfrak{S})$.*

The functor $S^\bullet(-)$ is an extension to $\mathcal{S}$ of a functor with the same notation and meaning on the category of measure spaces and measurable maps to itself. In particular the obvious monad structure

$$S^\bullet \circ S^\bullet \to S^\bullet$$

$$Id \to S^\bullet$$

of the total symmetric power functor on sets defines a monad structure on $S^\bullet$ on $\mathcal{S}$. We define the *branching category* $\mathcal{B}$ as the category of free algebras over $S^\bullet$. The objects of $\mathcal{B}$ are again measure spaces $(X, \mathfrak{R})$ and morphisms from $(X, \mathfrak{R})$ to $(Y, \mathfrak{S})$ are the branching morphisms of Definition 2.5.4.

Remark 2.5.5 [**notfree**] In view of Lemma 2.4.2 algebras over $S^\bullet$ are exactly commutative monoids in $\mathcal{S}$ with respect to $\otimes$.

We will write $\phi : [X, \mathfrak{R}] \to [Y, \mathfrak{S}]$ for branching morphisms to distinguish them from morphisms in $\mathcal{K}$ and $\mathcal{S}$. Let us describe the composition of branching morphisms more explicitly. Observe first that there is a measurable map of measure spaces

$$m : S^\bullet(Y, \mathfrak{S}) \times S^\bullet(Y, \mathfrak{S}) \to S^\bullet(Y, \mathfrak{S})$$

which makes $S^\bullet(Y, \mathfrak{S})$ into a commutative monoid. In view of Lemma 2.2.15 and the definition of the symmetric product it shows that any kernel ϕ from $(X, \mathfrak{R})$ to $S^\bullet(Y, \mathfrak{S})$ in $\mathcal{K}^b$ defines a family of kernels of the form

$$\phi_n : S^n(X, \mathfrak{R}) \to S^\bullet(Y, \mathfrak{S})$$

(where we set ϕ_0 to be identically 1). If the original kernel is stochastic so are the kernels ϕ_n and therefore by Lemma 2.4.2 they define a kernel

$$\phi_* = \coprod \phi_n : S^\bullet(X, \mathfrak{R}) \to S^\bullet(Y, \mathfrak{S})$$

We can now define the composition of two branching morphisms by the rule:

$$\psi \circ_{\mathcal{B}} \phi := \psi \circ \phi_*$$

Forgetting the $S^\bullet$ algebra structure defines a functor

$$F : \mathcal{B} \to \mathcal{S}$$

which takes $(X, \mathfrak{R})$ to $S^\bullet(X, \mathfrak{R})$ and ϕ to the kernel ϕ_* defined above.

Example 2.5.6 *[***ex8***]*Consider morphisms in the branching category of the form $\phi : [\mathbf{1}] \to [\mathbf{1}]$. Since $S^\bullet(\mathbf{1}) = \mathbf{N}$ we may identify this set with the set of probability measures on $\mathbf{N}$. For any ϕ let $p_\phi = \sum p_i t^i$ be the generating function of this measure. This construction identifies $Hom_{\mathcal{B}}([\mathbf{1}], [\mathbf{1}])$ with formal power series $\sum p_i t^i$ satisfying $p_i \geq 0$ and $\sum p_i = 1$. If ϕ, ψ two endomorphisms of $[\mathbf{1}]$ in $\mathcal{B}$ then one has

$$[\textbf{compseries}] p_{\phi \circ \psi} = p_\psi(p_\phi(t)) \tag{5}$$

i.e. in this description the composition of morphisms corresponds to the composition of power series in the reverse order.

Example 2.5.7 *[ex10]*The previous example has an immediate generalization to branching morphisms of the form $\phi : [\mathbf{n}] \rightarrow [\mathbf{n}]$ where $\mathbf{n} := \coprod_{i=1}^{n} \mathbf{1}$ is the set of n elements considered as a measure space with respect to the maximal σ-algebra. Such morphism is a collection of n probability measures on $\mathbf{N}^n$. If we describe these measures through their generating functions we may identify $Hom_{\mathcal{B}}([\mathbf{n}], [\mathbf{n}])$ with the set of n-tuples $(f_1, \dots, f_n)$ where each f_i is a formal power series in n-variables with non-negative coefficients satisfying the condition $f_i(1, \dots, 1) = 1$. The composition of morphisms corresponds to the substitution composition for such n-tuples.

For any $(X, \mathfrak{R})$ let

$$[\mathbf{tr1}] tr_n = \sum_{i=1}^{n} pr_i : (X, \mathfrak{R})^{\otimes n} \rightarrow (X, \mathfrak{R}) \tag{6}$$

be the kernel which sends a point $(x_1, \dots, x_n)$ to the measure $\sum_{i=0}^{n} \delta_{x_i}$. For $n = 0$ we take tr_0 to be the zero kernel.

The kernel (6) is clearly invariant under the action of the symmetric group and by Lemma 2.4.2 it defines a bounded kernel

$$tr_n : S^n(X, \mathfrak{R}) \rightarrow (X, \mathfrak{R})$$

which sends the point $x_1, \dots, x_n$ to the sum of δ-measures $\delta_{x_1} + \dots + \delta_{x_n}$ (for $n = 0$ our kernel is 0) and which we continue to denote by tr_n. The coproduct of tr_n's is a kernel $tr_* : S^{\bullet}(X, \mathfrak{R}) \rightarrow (X, \mathfrak{R})$. For a stochastic kernel $(X, \mathfrak{R}) \rightarrow S^{\bullet}(Y, \mathfrak{S})$ (i.e. for a branching morphism $\phi : [X, \mathfrak{R}] \rightarrow [Y, \mathfrak{S}]$) define a kernel

$$tr(\phi) : (X, \mathfrak{R}) \rightarrow (Y, \mathfrak{S})$$

as the composition $tr_* \circ \phi$.

Lemma 2.5.8 *[comm] For any stochastic kernel $f : (X, \mathfrak{R}) \rightarrow (Y, \mathfrak{S})$ and any $n \geq 0$ the diagram*

$$\begin{array}{ccc} (X, \mathfrak{R})^{\otimes n} & \xrightarrow{f^{\otimes n}} & (Y, \mathfrak{S})^{\otimes n} \\ {\scriptstyle tr_n}\downarrow & & \downarrow{\scriptstyle tr_n} \\ (X, \mathfrak{R}) & \xrightarrow{f} & (Y, \mathfrak{S}) \end{array}$$

commutes.

Proof: In view of the definition of tr_n it is sufficient to verify that $pr_i \circ f^{\otimes n} = f \circ pr_i$ for all i. More generally it is sufficient to see that for a kernel $f : X \rightarrow Y$ and a stochastic kernel $f' : X' \rightarrow Y'$ one has $pr_Y \circ (f \otimes f') = f \circ pr_X$ i.e. that the square

$$\begin{array}{ccc} X \otimes X' & \xrightarrow{f \otimes f'} & Y \otimes Y' \\ {\scriptstyle pr_X}\downarrow & & \downarrow{\scriptstyle pr_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Let e be the canonical stochastic kernel from an object to the point. We have

$$pr_Y \circ (f \otimes f') = (Id_Y \otimes e) \circ (f \otimes f') = f \otimes (e \circ f') = f \otimes e = f \circ pr_X$$

where the third equality holds since $e \circ f' = e$ eactly means that f' is stochastic.

Proposition 2.5.9 *[comm2] For any ϕ as above the diagram*

$$\begin{array}{ccc} S^\bullet(X,\mathfrak{R}) & \xrightarrow{\phi_*} & S^\bullet(Y,\mathfrak{S}) \\ {\scriptstyle tr_*}\downarrow & & {\scriptstyle tr_*}\downarrow \\ (X,\mathfrak{R}) & \xrightarrow{tr(\phi)} & (Y,\mathfrak{S}) \end{array}$$

commutes.

Proof: By definition of ϕ_* it is sufficient to verify that for any n the diagram

$$\begin{array}{ccccc} (X,\mathfrak{R})^{\otimes n} & \xrightarrow{\phi^{\otimes n}} & S^\bullet(Y,\mathfrak{S})^{\otimes n} & \xrightarrow{m} & S^\bullet(Y,\mathfrak{S}) \\ {\scriptstyle tr_n}\downarrow & & {\scriptstyle tr_n}\downarrow & & \downarrow{\scriptstyle tr_*} \\ (X,\mathfrak{R}) & \xrightarrow{\phi} & S^\bullet(Y,\mathfrak{S}) & \xrightarrow{tr_*} & (Y,\mathfrak{S}) \end{array}$$

commutes. The right hand side square consists of kernels which take a point to the sum of finitely many points and it is easy to verify its commutativity explicitly. The left hand side square commutes by Lemma 2.5.8.

Corollary 2.5.10 *[main1] For a pair of branching morphisms $\phi : [X,\mathfrak{R}] \to [Y,\mathfrak{S}]$, $\psi : [Y,\mathfrak{S}] \to [Z,\mathfrak{T}]$ one has*

$$tr(\psi \circ \phi) = tr(\psi) \circ tr(\phi)$$

Proof: This follows immediately from the explicit description of the composition of branching morphisms given above and Proposition 2.5.9.

Example 2.5.11 [**ex11**]Consider a branching morphism $\phi : [\mathbf{1}] \to [\mathbf{1}]$ which we describe through the corresponding probability generating function $p_\phi = \sum p_i t^i$ as in Example 2.5.6. Then $tr(\phi)$ is a kernel $\mathbf{1} \to \mathbf{1}$ i.e. a non-negative number. One can easily see that

$$tr(\phi) = \sum i p_i = p'_\phi(1)$$

where p'_ϕ is the formal derivative of p_ϕ with respect to t. In other words, $tr(\phi)$ is in this case the expectation value of ϕ. For two morphisms ϕ, ψ of this form Corollary 2.5.10 asserts that

$$tr(\psi \circ \phi) = tr(\psi) tr(\phi).$$

In view of (5) this follows from the equation

$$(p_\phi \circ p_\psi)'(1) = p'_\psi(1) p'_\phi(p_\psi(1)) = p'_\psi(1) p'_\phi(1)$$

where the last equation holds since the $p_\psi(1) = 1$ because ψ is a stochastic kernel.

Example 2.5.12 [**ex12**]Consider now branching morphisms $[\mathbf{n}] \to [\mathbf{n}]$ as in Example 2.5.7. For a morphism ϕ of this form $tr(\phi)$ is a kernel $\mathbf{n} \to \mathbf{n}$ i.e. an $n \times n$-matrix (a_{ij}) with entries from $[0,\infty]$. If we represent ϕ a sequence of power series $(f_1, \ldots, f_n)$ in variables $t_1, \ldots, t_n$ then one gets

$$a_{ij} = \frac{\partial f_i}{\partial t_j}(1)$$

If $\psi = (g_1, \ldots, g_n)$ is another such morphism then the statement of Corollary 2.5.10 is again equivalent to the formula for the differential of a composition combined with the fact that $g_i(1) = 1$ since ψ is stochastic.

3 Standard notions of probability

3.1 Stochastic processes

Definition 3.1.1 *[***sproc***] Let T be a subset of* $\mathbf{R}$. *A stochastic process with time window T and values in a measurable space $(Y, \mathfrak{S})$ is the following collection of data:*

1. *a measurable space $(\Omega, \mathfrak{F})$,*
2. *a probability measure $P : \mathbf{1} \to (\Omega, \mathfrak{F})$,*
3. *a measurable map $X : (\Omega, \mathfrak{F}) \to (Y, \mathfrak{S})^T$.*

Two stochastic processes $((\Omega, \mathfrak{F}), P, X)$ and $((\Omega', \mathfrak{F}'), P', X')$ are said to be equivalent in the wide sense if the corresponding measures $X \circ P$, $X' \circ P'$ on $(Y, \mathfrak{S})^T$ coincide.

Since $(Y, \mathfrak{S})^T$ is the product of T copies of $(Y, \mathfrak{S})$ in $\mathcal{MS}$, specification of a map X is equivalent to the specification of measurable maps $X_t : (\Omega, \mathcal{F}) \to (Y, \mathfrak{S})$, one for each $t \in T$.

The projections $P_{t_1,\ldots,t_n}$ of $X \circ P$ to the products $(Y, \mathfrak{S})^n$ corresponding to finite subsets $\{t_1, \ldots, t_n\}$ of T are called finite dimensional distributions (or marginal distributions) of the process. Since the product σ-algebra on the infinite product is generated in the strong sense by the pull-backs of the product σ-algebras on the corresponding finite products, two stochastic processes are equivalent in the wide sense if and only if their "marginal distributions" coincide.

The main result towards the existence of a stochastic process with given family of finite-dimensional distribution is the following theorem.

Theorem 3.1.2 (Kolmogorov) *[***kol1***] Let Y be a separable complete topological space and $\mathfrak{B}_Y$ be its Borel σ-algebra. Then for any compatible (in the obvious sense) system of probability measures P_A on the spaces $(Y, \mathfrak{S})^A$ where A runs through finite subsets of T, there exists a unique probability measure P on $(Y, \mathfrak{S})^T$ whose partial projections are P_A.*

Corollary 3.1.3 *[***projlim***] Under the assumption of Theorem 3.1.2 the space $(Y, \mathfrak{S})^T$ is the inverse limit of the system of spaces $\{(Y, \mathfrak{S})^A\}_{A \in Fin(T)}$ in $\mathcal{S}$ where $Fin(T)$ is the partially ordered set of finite subsets of T.*

Proof: Follows immediately from the theorem and Lemma 2.3.1.

For the proof of Theorem 3.1.2 as well as for a discussion of its variants and generalizations see [11].

An issue which often arises in probability in connection with stochastic processes on some subset T of the real line is the possibility of finding a process which is equivalent to the given one in the wide sense and has trajectories lying in some subset C of $(X, \mathfrak{R})^T$ i.e. such that $Im(X) \subset C$.

Lemma 3.1.4 *[***smallertr***] Let C be a subset of $(Y, \mathfrak{R})^T$. A process $((\Omega, \mathfrak{F}), P, X)$ on $(Y\mathfrak{R})$ with time window T is equivalent to a process $((\Omega', \mathfrak{F}'), P', X')$ satisfying the condition*

$$Im(X') \subset C$$

if and only if for any $A, B \in \mathfrak{R}^T$ such that $A \cap C = B \cap C$ one has $X_*(P)(A) = X_*(P)(B)$.

Proof: The 'only if' part is obvious. To prove the 'if' part one may take $\Omega' = C$ and $\mathfrak{F}' = i^{-1}(\mathfrak{R}^T)$ where $i : C \to (Y, \mathfrak{R})^T$ is the inclusion.

3.2 Markov processes - classical approach

Let us start with a definition of a Markov process from [5].

Definition 3.2.1 *[**dynkindef**] A Markov process is a collection of data of the form:*

1. *a measurable space $X = (X, \mathfrak{R})$,*
2. *a set Ω,*
3. *a function $\zeta : \Omega \to [0, \infty]$,*
4. *a function $x : U(\Omega, \zeta) \to X$ where*

$$U(\Omega, \zeta) = \{(t, \omega) \in [0, \infty) \times \Omega \,|\, t < \zeta(\omega)\}$$

5. *for each s in $[0, \infty)$ a σ-algebra $\mathfrak{M}^s$ on Ω,*
6. *for each $s \leq t$ in $[0, \infty)$ a σ-algebra $\mathfrak{M}^s_t$ on Ω_t where*

$$\Omega_t = \{\omega \in \Omega \,|\, \zeta(\omega) > t\}$$

7. *for each $s \in [0, \infty)$, $x \in X$ a probability measure $P_{s,x}$ on $(\Omega, \mathfrak{M}^s)$,*

which satisfies the following conditions:

1. *for each $s \leq t$ in $[0, \infty)$ one has $i_t(\mathfrak{M}^s_t) \subset \mathfrak{M}^s$ where $i_t : \Omega_t \to \Omega$ is the inclusion,*
2. *for each $s \leq t$ in $[0, \infty)$ the map $x_t : (\Omega_t, \mathfrak{M}^s_t) \to (X, \mathfrak{R})$ is measurable,*

3.3 Path systems

Let T be a time window i.e. a subset of $\mathbf{R}$. The pairs of elements u, v of T such that $u \leq v$ form a partially ordered set where $(u', v') \leq (u, v)$ if $u' \geq u$ and $v' \leq v$. A path system over T with values in a category $\mathcal{C}$ is a contravariant functor from this partially ordered set to $\mathcal{C}$. In what follows we will work almost exclusively with path systems with values in the category of measurable spaces for which we have the following explicit definition.

Definition 3.3.1 *[**pathsystem**] Let T be a subset in $\mathbf{R}$. A path system X_{**} over T is a mapping which assigns to each $u \leq v$ in T a measurable space X_{uv} and to each $u \leq u' \leq v' \leq v$ in T a measurable map $res^{u,v}_{u',v'} : X_{uv} \to X_{u'v'}$, such that $res^{u,v}_{u,v} = Id$ and for $u \leq u' \leq u'' \leq v'' \leq v' \leq v$ one has*

$$res^{u,v}_{u'',v''} = res^{u',v'}_{u'',v''} \circ res^{u,v}_{u',v'}$$

Isomorphisms between path systems are defined in the obvious way. More general morphisms between path systems can be of different types and will be considered later.

Path systems arise whenever we model some dynamical processes especially under time inhomogeneous conditions. The space X_{uu} is the space of all possible immediate states of the system at time u and the space X_{uv} is the space of possible, in our model, trajectories or paths with time window $[u, v]$. In what follows we will often write X_u instead of X_{uu} and ξ_t instead of $res^{u,v}_{t,t}$ for $u \leq t \leq v$.

Let $[u,v]_T = \{x \in T \,|\, u \leq x \leq v\}$. A family of spaces X_t given for all $t \in T$ defines a path system with

$$X_{uv} = \prod_{t\in[u,v]_T} X_t$$

and $res^{u,v}_{u',v'}$ being the partial projection maps. This path system will be called the canonical path system defined by the family X_t. In an important case when all X_t are the same space X we get a path system where $X_{uv} = X^{[u,v]_T}$ is the space of all maps from $[u,v]_T$ to X.

Many of path systems which one encounters are sub-objects of the canonical path system defined by a single space X. For example, if X is a topological space then one may consider the path systems whose uv-spaces are the spaces of continuous or right continuous maps from $[u,v]_T$ to X with the σ-algebras defined by the inclusion to $X^{[u,v]_T}$.

The constant path systems with $X_{uv} = X$ for all $u \leq v$ is a sub-object of the canonical path system given by diagonals in $X^{[u,v]_T}$.

3.4 A categorical view of Markov processes

A pre-process P on a path system $(X_{uv}, res^{u,v}_{u',v'})$ with time window $T \subset \mathbf{R}$ is a collection of sub-probability kernels $\mu^v_u : X_u \to X_{uv}$ given for all $u \leq v$ in T such that for any $x \in X_u$, $\xi_u \mu^v_u(x)$ is zero on $X_u - \{x\}$.

The kernels

$$\phi_{uv} = \xi_v \circ \mu^v_u : X_u \to X_v$$

are called the transition kernels of the pre-process. The projection $X_u \overset{\phi_{uv}}{X}{}_v \to pt$ is a function on X_u which we denote by υ_{uv}.

A pre-process is called non-degenerate if for all $u \in T$ one has $\phi_{uu} = Id$ or equivalently $\upsilon^u_u \equiv 1$. A pre-process is called a process if it satisfies the following equivalent conditions

1. for all $u \leq v$ in T, $\upsilon^v_u \equiv 1$,
2. for all $u \leq v$ in T, ϕ_{uv} is a probability kernel,
3. for all $u \leq v$ in T, μ^v_u is a probability kernel,
4. for all $u \leq v$ in T, μ^v_u is a section of ξ_u.

For any pre-process and any $u \leq w \leq v$ the composition

$$X_{uw} \overset{Id\times\xi_u}{\longrightarrow} X_{uw} \times X_w \overset{Id\otimes\mu^v_w}{\to} X_{uw} \times X_{wv}$$

is a kernel which sends a point $\omega \in X_{uv}$ to the measure $\delta_\omega \otimes \mu^v_w(\xi_w(\omega))$ and which we denote by $Id \otimes (\mu^v_w \circ \xi_w)$.

Definition 3.4.1 *[***submarkov***] A (pre-)process is said to be a Markov (pre-)process if it satisfies the condition*

M *For any $u \leq v \leq w$ the square*

$$\begin{array}{ccc} X_u & \xrightarrow{\mu^w_u} & X_{uw} \\ {\scriptstyle \mu^v_u}\downarrow & & \downarrow{\scriptstyle Id\otimes(\mu^v_w\circ\xi_w)} \\ X_{uv} & \xrightarrow{res^{u,v}_{u,w}\times res^{u,v}_{w,v}} & X_{uw}\times X_{wv} \end{array} \quad \textbf{[md]} \qquad (7)$$

commutes.

Example 3.4.2 [**2009.04.29.5**] A pre-process on the constant path system X is a collection of measurable functions υ_u^v on X with values in $[0,1]$. It satisfies (M) if and only if for all $u \leq w \leq v$ in T one has $\upsilon_u^w \upsilon_w^v = \upsilon_u^v$.

To compare our definition with other definitions which appear in the literature on probability it will be convenient for us to introduce to weaker versions of condition (M).

Mf For any $u \leq v \leq w$ the diagram

$$[\mathbf{mfd}]\begin{array}{ccccc} X_u & \xrightarrow{\mu_u^w} & X_{uw} & & \\ \mu_u^v\downarrow & & \downarrow Id\otimes(\mu_w^v\circ\xi_w) & & \\ X_{uv} & \xrightarrow{res_{u,w}^{u,v}\times res_{w,v}^{u,v}} & X_{uw}\times X_{wv} & \xrightarrow{\xi_u\times Id} & X_u\times X_{wv} \end{array} \quad (8)$$

commutes.

Mb For any $u \leq v \leq w$ the diagram

$$[\mathbf{mbd}]\begin{array}{ccccc} X_u & \xrightarrow{\mu_u^w} & X_{uw} & & \\ \mu_u^v\downarrow & & \downarrow Id\otimes(\mu_w^v\circ\xi_w) & & \\ X_{uv} & \xrightarrow{res_{u,w}^{u,v}\times res_{w,v}^{u,v}} & X_{uw}\times X_{wv} & \xrightarrow{Id\times\xi_v} & X_{uw}\times X_v \end{array} \quad (9)$$

commutes.

The first of this conditions is a generalization of the "forward Markov property" and the second one of the "backward Markov property". Our main condition (M) expresses the "two-sided Markov property".

Lemma 3.4.3 *[**mtr**] For any pre-process μ_*^* which satisfies (Mf) (resp. (Mb)) and any $u \leq w \leq v$ one has*

$$[\mathbf{wm}]\phi_{wv}\circ\phi_{uw} = \phi_{uv} \quad (10)$$

Proof: For μ_*^* satisfying (Mf) one gets the equation (10) combining diagram (8) with the commutative square

$$\begin{array}{ccc} X_{uw} & \xrightarrow{\xi_w} & X_w \\ \xi_u\otimes(\mu_w^v\circ\xi_w)\downarrow & & \downarrow\xi_v\circ\mu_w^v \\ X_u\times X_{wv} & \xrightarrow{\xi_v\circ pr} & X_v \end{array}$$

For μ_*^* satisfying (Mb) one gets the equation (10) combining diagram (9) with the commutative square

$$\begin{array}{ccc} X_{uw} & \xrightarrow{\xi_w} & X_w \\ Id\otimes(\xi_v\circ\mu_w^v\circ\xi_w)\downarrow & & \downarrow\xi_v\circ\mu_w^v \\ X_{uw}\times X_w & \xrightarrow{pr} & X_v \end{array}$$

Remark 3.4.4 For a process which satisfies (M) the projection of μ_u^v to X_{uw} coincides with μ_u^w. If T has a maximal element t_{max} and μ_*^* is a process then it is sufficient to verify the condition (M) for $v = t_{max}$.

If μ_*^* satisfies (M) but is not a process then the projection of μ_u^v to X_{uw} does not coincide with μ_u^w. Instead by (M) we get

$$res_{u,w}^{u,v}(\mu_u^v) = (\upsilon_w^v \circ \xi_w) * \mu_u^w$$

where υ_w^v is the function on X_w which takes x to $\mu_w^v(x, X_{wv})$ and which equals 1 if and only if μ_*^* is a process.

Remark 3.4.5 [**ff**] For any pre-process, ϕ_{uu} is the kernel of the form $x \mapsto \upsilon_u^u(x)\delta_u$. For a pre-process satisfying (M) the equation (10) applied to u, u, u implies that for any u one has $(\upsilon_u^u)^2 = \upsilon_u^u$ and therefore this function may take only values 0 and 1. Note also that for a Markov pre-process one has

$$[\mathbf{canuneq1}]\upsilon_u^v = (X_u \stackrel{\phi_{u,v}}{\to} X_v \to pt) \tag{11}$$

and for $u \le w \le v$

$$[\mathbf{canuneq2}]\xi_w(\mu_u^v) = \upsilon_w^v * \phi_{u,w} = (x \mapsto \upsilon_w^v(x)\delta_x) \circ \phi_{u,w} \tag{12}$$

Denote the σ-algebra on X_{uv} by $\mathfrak{S}_u^v$.

Lemma 3.4.6 *[**smf**] A pre-process satisfies condition (Mf) if and only if for any $u \le w \le v$ in T, $x \in X_u$ and any A in $\mathfrak{S}_w^v$ one has $\mu_u^v(x, A) = (\mu_w^v \circ \phi_{uw})(x, A)$.*

Proof: The condition of our lemma is equivalent to the commutativity of the external rectangle of the diagram

$$[\mathbf{ad1}]\quad \begin{array}{ccccc} X_u & \xrightarrow{\mu_u^w} & X_{uw} & \xrightarrow{\xi_w} & X_w \\ \downarrow{\mu_u^v} & & \downarrow{Id\otimes(\mu_w^v\circ\xi_w)} & & \downarrow{\mu_w^v} \\ X_{uv} & \longrightarrow & X_{uw}\times X_{wv} & \xrightarrow{pr} & X_{wv} \end{array} \tag{13}$$

Since the right hand side square of this diagram always commutes the commutativity of (8) implies commutativity of (13). On the other hand since both μ_u^w and μ_u^v are supported on the fibers of ξ_u the commutativity of (13) implies the commutativity of (8).

Lemma 3.4.7 *[**smb**] A pre-process satisfies condition (Mb) if and only if for any $u \le w \le v$ in T, $x \in X_u$, $A \in \mathfrak{S}_u^w$ and B in $\mathfrak{S}_v^v$ one has*

$$[\mathbf{eqgik2}]\mu_u^v(x, A \cap \xi_v^{-1}(B)) = \int_{\omega\in A} \mu_w^v(\xi_w(\omega), \xi_v^{-1}(B))d\mu_u^w(x). \tag{14}$$

Proof: Since the σ-algebra on $X_{uw} \times X_v$ is generated in the strong sense by subsets of the form $A \times B$ where $A \in \mathfrak{S}_u^w$ and $B \in \mathfrak{S}_v^v$ the commutativity of (9) is equivalent to the assertion that for any $x \in X_v$ and any such A, B one has:

$$\mu_u^v(x, res^{-1}(A) \cap \xi_v^{-1}(B)) = ((Id_{X_{uw}} \otimes (\xi_v \circ \mu_w^v \circ \xi_w))) \circ \mu_u^w)(x, A \times B)$$

By definition of kernel composition the right hand side is of the form

$$((Id_{X_{uw}} \otimes (\xi_v \circ \mu_w^v \circ \xi_w))) \circ \mu_u^w)(x, A \times B) = \int (\delta_\omega \otimes (\xi_v \circ \mu_w^v))(\xi_w(\omega), A \times B)d\mu_u^w(x) =$$

$$= \int_{\omega\in A} (\xi_v \circ \mu_w^v)(\xi_w(\omega), B)d\mu_u^w(x) = \int_{\omega\in A} \mu_w^v(\xi_w(\omega), \xi_v^{-1}(B))d\mu_u^w(x).$$

Lemma 3.4.8 *[cr1] A pre-process satisfies condition (M) if and only if for any $u \leq w \leq v$ in T, $x \in X_u$ and any $A \in \mathfrak{S}_u^w$, $B \in \mathfrak{S}_w^v$ one has*

$$\mu_u^v(x, A \cap B) = \int_{\omega \in A} \mu_w^v(\xi_w(\omega), B) d\mu_u^w(x).$$

Proof: The σ-algebra on $X_{uw} \times X_{wv}$ is generated in the strong sense by subsets of the form $A \times B$ where $A \in \mathfrak{S}_u^w$ and $B \in \mathfrak{S}_w^v$. The value of the image of a point $x \in X_u$ on $A \times B$ under the path of our diagram going through the lower left corner is $\mu_u^v(x, A \cap B)$. If we go through the upper right corner we get

$$\int_{X_{uw}} (\delta_\omega \otimes \mu_w^v(\xi_w(\omega)))(A \times B) d\mu_u^w(x) = \int_{\omega \in A} \mu_w^v(\xi_w(\omega), B) d\mu_u^w(x).$$

Lemma 3.4.9 *[canun] Let X_t be a family of measurable spaces given for all $t \in T$ and let P, P' be two pre-processes satisfying condition (M) on the canonical path system defined by this family such that for all $u \leq v$ in T one has $\phi_{uv}(P) = \phi_{uv}(P')$. Then $P = P'$.*

Proof: From (11) we conclude that for all $u \leq v$ in T we have $\upsilon_u^v(P) = \upsilon_u^v(P')$ and from (12) that for all $u \leq w \leq v$ in T we have

$$\xi_w(\mu_u^v(P)) = \xi_w(\mu_u^v(P'))$$

Since the σ-algebra on $X_{uv} = \prod_{w \in [u,v]_T} X_w$ is generated in the strong sense by the pull-backs of σ-algebras on X_t with respect to projections ξ_w the claim of the lemma follows.

In what follows we will omit the product sign and write XY instead of $X \times Y$. We will also write $[u, v]$ instead of $[u, v]_T$. For $t_1, \ldots, t_n \in [u, v]$ we let $pr_{t_1,\ldots,t_n}$ denote the partial projection $X^{[u,v]} \to X_{t_1} \ldots X_{t_n}$. For any $1 < m < n$ and any $t_1 \leq \cdots \leq t_n$ in T consider the disgram

$$\textbf{[prmd]} \quad \begin{array}{ccc} X_{t_1} & \xrightarrow{pr_{t_1,\ldots,t_m} \circ \mu_{t_1}^{t_m}} & X_{t_1} \ldots X_{t_m} \\ \| & & \downarrow Id_{t_1,\ldots,t_{m-1}} \otimes (pr_{t_m,\ldots,t_n} \circ \mu_{t_m}^{t_n}) \\ X_{t_1} & \xrightarrow{pr_{t_1,\ldots,t_n} \circ \mu_{t_1}^{t_n}} & X_{t_1} \ldots X_{t_n} \end{array} \tag{15}$$

Proposition 3.4.10 *[cpthm] Let X_t be a family of measurable spaces given for all $t \in T$ and let $P = \{\mu_u^v\}_{u \leq v}$ be a pre-process on the corresponding canonical path system. Then the following conditions are equivalent:*

1. *P satisfies condition (M),*
2. *diagrams (15) commute for all $1 < m < n$ and all $t_1 \leq \cdots \leq t_n$ in T,*
3. *diagrams (15) commute for $m = 2$, $n > m$ and all $t_1 \leq \cdots \leq t_n$ in T,*
4. *diagrams (15) commute for $n > 2$, $m = n - 1$ and all $t_1 \leq \cdots \leq t_n$ in T.*

Proof: Commutativity of the diagram (15) is equivalent to the commutativity of the diagram

$$
\begin{array}{ccccc}
X_{t_1} & \xrightarrow{\mu_{t_1}^{t_m}} & X^{[t_1,t_m]} & \xrightarrow{pr_{t_1,\ldots,t_m}} & X_{t_1}\ldots X_{t_m} \\
{\scriptstyle \mu_{t_1}^{t_n}}\downarrow & & {\scriptstyle Id\otimes(\mu_{t_m}^{t_n}\circ\xi_{t_m})}\downarrow & & {\scriptstyle Id_{t_1,\ldots,t_{m-1}}\otimes(pr_{t_m,\ldots,t_n}\circ\mu_{t_m}^{t_n})}\downarrow \\
X^{[t_1,t_n]} & \longrightarrow & X^{[t_1,t_m]}\times X^{[t_m,t_n]} & \xrightarrow{pr_{t_1,\ldots,t_{m-1}}\times pr_{t_m,\ldots,t_n}} & X_{t_1}\ldots X_{t_n}
\end{array}
$$

and commutativity of all such diagrams is equivalent to the commutativity of diagrams (7) for the canonical path system since the σ-algebra on $X^{[t_1,t_m]}\times X^{[t_m,t_n]}$ is generated in the strong sense by the pull-backs of the corresponding σ-algebras from finite projections. This proves the equivalence of the first two conditions. It remains to show that conditions (3) and (4) each imply condition (2). In the case of condition (3) proceed by induction on m. For the inductive step consider the following diagram

$$
\begin{array}{ccccc}
X_{t_1} & \xrightarrow{pr\circ\mu_{t_1}^{t_m}} & X_{t_1}\ldots X_{t_m} & \xrightarrow{Id\otimes(pr\circ\mu_{t_m}^{t_n})} & X_{t_1}\ldots X_{t_n} \\
{\scriptstyle Id}\downarrow & & \uparrow{\scriptstyle Id\otimes(pr\circ\mu_{t_2}^{t_m})} & & \downarrow{\scriptstyle Id} \\
X_{t_1} & \xrightarrow[pr\circ\mu_{t_1}^{t_2}]{} & X_{t_1}X_{t_2} & \xrightarrow[Id\otimes(pr\circ\mu_{t_2}^{t_n})]{} & X_{t_1}\ldots X_{t_n} \\
{\scriptstyle Id}\downarrow & & & & \downarrow{\scriptstyle Id} \\
X_{t_1} & & \xrightarrow{(pr\circ\mu_{t_1}^{t_n})} & & X_{t_1}\ldots X_{t_n}
\end{array}
$$

where the maps are such that the upper left square is (15) for $t_1,t_2\ldots,t_m$, the upper right square is equivalent to (15) for $t_2,\ldots,t_m,\ldots,t_n$ multiplied with X_{t_1} and the lower rectangle is equivalent to (15) for $t_1,t_2,\ldots,t_n$. Then the external path of the diagram is equivalent to (15) for $t_1,\ldots,t_m,\ldots,t_n$ which gives the inductive step.

In the case of condition (4) proceed by induction on $n-m$. Consider the diagram

$$
\begin{array}{ccccc}
X_{t_1} & \xrightarrow{pr\circ\mu_{t_1}^{t_m}} & X_{t_1}\ldots X_{t_m} & \xrightarrow{Id\otimes(pr\circ\mu_{t_m}^{t_n})} & X_{t_1}\ldots X_{t_n} \\
{\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle Id\otimes(pr\circ\mu_{t_m}^{t_{n-1}})} & & \downarrow{\scriptstyle Id} \\
X_{t_1} & \xrightarrow[pr\circ\mu_{t_1}^{t_{n-1}}]{} & X_{t_1}\ldots X_{t_{n-1}} & \xrightarrow[Id\otimes(pr\circ\mu_{t_{n-1}}^{t_n})]{} & X_{t_1}\ldots X_{t_n} \\
{\scriptstyle Id}\downarrow & & & & \downarrow{\scriptstyle Id} \\
X_{t_1} & & \xrightarrow{pr\circ\mu_{t_1}^{t_n}} & & X_{t_1}\ldots X_{t_n}
\end{array}
$$

where the maps are such that the upper left square is (15) for $t_1,\ldots,t_m,\ldots,t_{n-1}$, the upper right square is equivalent to (15) for $t_m,\ldots,t_{n-1},t_n$ multiplied with $X_{t_1}\ldots X_{t_{m-1}}$ and the lower rectangle is equivalent to (15) for $t_1,\ldots,t_{n-1},t_n$. Then the external path of the diagram is equivalent to (15) for $t_1,\ldots,t_m\ldots,t_n$ which gives the inductive step.

Corollary 3.4.11 *[**bcase**] If a pre-process on a canonical path system satisfies (Mf) (resp. (Mb)) then it satisfies (M).*

Proof: One can see immediately that in the context of the canonical path systems condition (3) of the proposition implies (Mf) and condition (4) implies (Mb).

Remark 3.4.12 Using Theorem 3.1.2 it is not hard to show that if X is a separable complete topological space with its Borel σ-algebra then for any time window T and any collection of probability kernels satisfying (10) there exists a Markov process on the canonical path system of X over T for which these kernels are the transition kernels. By Lemma 3.4.9 such a process is unique.

Example 3.4.13 One can construct processes on the canonical path system whose transition kernels satisfy (10) and which are not Markov processes. Consider for example the time window $T = \{a, b, c\}$ where $a \le b \le c$. Processes on the canonical path system of X over T correspond to triples of probability kernels

$$X_a \xrightarrow{\nu_a^c} X_b X_c$$
$$X_a \xrightarrow{\nu_a^b} X_b$$
$$X_b \xrightarrow{\nu_b^c} X_c$$

where for $u \le v$ in $\{a, b, c\}$ we write $\nu_u^v = pr\mu_u^v$ for the projection pr which removes X_u. The transition kernels of the process determine $\phi_{ab} = \nu_a^b$, $\phi_{bc} = \nu_b^c$ and $\phi_{ac} = pr_{X_c}\nu_a^c$ and the only non-trivial composition condition asserts that

$$pr_{X_c}\nu_a^c = \nu_b^c \circ \nu_c^b.$$

For a non-trivial X we may choose many different ν_a^c satisfying this condition. By Lemma 3.4.9 all such choices but one will define processes which do not satisfy (M).

Example 3.4.14 Let X be a measurable space. A Markov process on the canonical path system of X over T such that the kernels μ_*^* are deterministic maps is the same as a collection of endomorphism $\phi_{uv} : X \to X$ such that $\phi_{uu} = 1$ and $\phi_{uv} = \phi_{wv}\phi_{uw}$ for $u \le w \le v$.

Let P be a pre-process on the path system X_{**} over T. Consider the canonical path system $X'_{uv} = \prod_{t \in [u,v]} X_t$ over T defined by the family of spaces $X_t = X_{tt}$. For each pair $u \le v$ in T we have a map

$$\xi_{[u,v]} = \prod_{t \in [u,v]} \xi_t : X_{uv} \to \prod_{t \in [u,v]} X_t$$

The compositions $\xi_{[u,v]} \circ \mu_u^v$ form a pre-process P' on X'_{**} which has the same transition kernels and which is called the canonical representation of P. It follows immediately from the definitions that if P is a Markov pre-process then so is P'.

Proposition 3.4.15 *[**eqv3**] Let X_{**} be a path system over T be such that for all $u \le v$ in T one has*

$$[\mathbf{monstr}]\xi_{[u,v]}^{-1}(\mathfrak{S}^{[u,v]}) = \mathfrak{S}_u^v. \tag{16}$$

where

$$\mathfrak{S}^{[u,v]} = cl_\sigma(\cup_{t \in [u,v]} pr_t^{-1}(\mathfrak{S}_t^t))$$

is the σ-algebra of $\prod_{t \in [u,v]} X_t$. Then any pre-process on X_{**} which satisfies (Mf) (resp. (Mb)) satisfies (M) and any two pre-processes which satisfy (M) and have the same transition kernels coincide.

Proof: The equality (16) implies that the maps $\xi_{[u,v]}$ are monomorphisms in the category of kernels and moreover the same holds for the maps $\xi_{[u,w]} \times \xi_{[w,v]}$. Therefore, a process satisfies condition (M), (Mf) or (Mb) if and only if its canonical representation satisfies the corresponding condition. Together with Corollary 3.4.11 this implies the first claim of the proposition. The second claim follows by the same argument from Lemma 3.4.9.

Let us now compare our definition of a Markov process with a classical one from [6, Def.1, p.40]. We will show that any path system over $(0, \infty]$ together with a Markov process on it defines a Markov process in the sense of [6].

The space $(X, \mathfrak{R})$ is called the phase space of the system and the space $(\Omega, \mathfrak{S})$ the trajectory space. We will write Ω_s^t for the measurable space $(\Omega, \mathfrak{S}_s^t)$. For simplicity of notation we will sometimes abbreviate the notation for a path system omitting some of its components.

Proposition 3.4.16 *[**compare1**] Any pair of a path system over $[0, \infty)$ and a process over it satisfying (Mb) defines a Markov process in the sense of [6, Def.1, p.40].*

Proof: For this comparison we will use freely the notations of *loc.cit.*. Note that we write $\mathfrak{S}_s^t$ where they write $\mathfrak{S}_t^s$. Let $\mathfrak{S}^s$ denote the union $\mathfrak{S}_s^t$ for all $t \geq s$. Since we assume (P) we have $pr_{s,u}^{s,t}(\mu_s^u) = \mu_s^t$ and therefore kernels μ_s^t for $t \geq s$ define a kernel

$$P_{s,*} : (X, \mathfrak{R}) \to (\Omega, \mathfrak{S}^s)$$

such that μ_s^t are obtained from it by obvious projections. It is obvious from our definitions that the only condition of [6, Def. 1, p.40] which we have to verify is that for any $x \in X$, any $0 \leq s \leq t \leq u < \infty$ in T and any $B \in \mathfrak{R}$ one has

$$[\mathbf{eqgik1}] P_{s,x}\{\xi_u(\omega) \in B | \mathfrak{S}_s^t\} = P_{t,\xi_t(\omega)}\{\xi_u(\omega) \in B\}. \tag{17}$$

The left hand side $f(\omega)$ of this equation is a real functions on Ω which is defined only up to a subset of measure zero with respect to $P_{s,x}$ and the right hand side $g(\omega)$ is a well defined function on Ω. The definition of conditional expectation tells us that the only thing which we know about the left hand side is that it is $\mathfrak{S}_s^t$-measurable and for any $A \in \mathfrak{S}_s^t$ we have

$$\int_A f dPs, x = P_{s,x}(A \cap \{\xi_u(\omega) \in B\})$$

Hence, the equation (17) really means that for any $A \in \mathfrak{S}_s^t$ one has

$$[\mathbf{eqgik2a}] P_{s,x}(A \cap \{\xi_u(\omega) \in B\}) = \int_A P_{t,\xi_t(\omega)}\{\xi_u(\omega) \in B\} dP_{s,x} \tag{18}$$

which is equivalent to (14). The claim of the lemma follows now from Lemma 3.4.7.

Definition 3.4.17 *[**determmor**] A deterministic morphism of path systems $F : X_{**} \to Y_{**}$ is a collection of measurable maps $X_{uv} \to Y_{uv}$ given for all $u \leq v$ in T which are compatible with the restriction maps for X and Y and such that the maps $X_u \to Y_u$ are isomorphisms.*

Definition 3.4.18 *[**morpath**] A morphism from a path system Y to a path system Z in time window T is a collection of probability kernels $f_u^v : Y_{uv} \to Z_{uv}$ such that for any $u \leq w \leq v$ the square*

$$\begin{array}{ccc} Y_{uv} & \xrightarrow{res \times res} & Y_{uw} \times Y_{wv} \\ {\scriptstyle f_u^v}\downarrow & & \downarrow{\scriptstyle f_u^w \otimes f_w^v} \\ Z_{uv} & \xrightarrow{res \times res} & Z_{uw} \times Z_{wv} \end{array}$$

commutes, i.e. for any $\omega \in Y_{uv}$ one has

$$[\mathbf{2009.04.29.3}](res^{u,v}_{u,w} \times res^{u,v}_{w,v})(f^v_u(\omega)) = f^w_u(res^{u,v}_{u,w}(\omega)) \otimes f^v_w(res^{u,v}_{w,v}(\omega)) \quad (19)$$

If f^*_* and g^*_* are morphisms of path systems such that the compositions $g^*_* f^*_*$ are defined then these composition again form a morphism of path systems.

If f^v_u are defined by measurable maps then the condition of Definition 3.4.18 is equivalent to the requirement that f^v_u commute with the restriction maps i.e. that these maps form a deterministic morphism. In general, the condition of 3.4.18 implies that

$$res^{u,v}_{u',v'} f^v_u = f^{v'}_{u'} res^{u,v}_{u',v'}$$

but does not follow from it. For example, the averaged sum $(f+g)/2$ of two deterministic morphisms between path systems is almost never a morphism of path systems.

Lemma 3.4.19 *[**2009.04.29.1**] Let f^v_u be a collection of probability kernels of the form*

$$f^v_u(\omega) = \sum_{\psi \in Z_{uv}} f^v_u(\omega, \psi)\delta_\psi$$

Then f^v_u is a morphism of path systems if and only if for all $u \le w \le v$, $\omega \in Y_{uv}$, $\psi' \in Z_{uw}$, $\psi'' \in Z_{wv}$ one has

$$[\mathbf{2009.04.29.2}] \sum_{\{\psi \in Z_{uv} \,|\, res^{u,v}_{u,w}(\psi)=\psi' \text{ and } res^{u,v}_{w,v}(\psi)=\psi''\}} f^v_u(\omega, \psi) = f^w_u(res^{u,v}_{u,w}(\omega), \psi') \cdot f^v_w(res^{u,v}_{w,v}(\omega), \psi'') \quad (20)$$

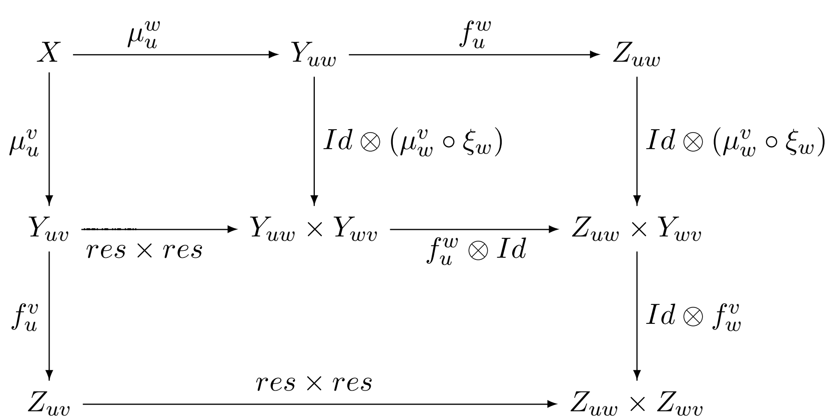
Proof: An easy computation shows that the left and right hand sides of (20) are equal to the coefficient at $\delta_{\psi'} \otimes \delta_{\psi''}$ in the left and right hand sides of (19).

Proposition 3.4.20 *[**2009.04.29.4**] Let Y_{**} and Z_{**} be path systems over X and $f^*_* : Y_{**} \to Z_{**}$ be a morphism over (X, T) i.e. a morphism of path systems in time window T such that for all $u \in T$ one has $\xi_u \circ f^u_u = \xi_u$. Let μ^*_* be a pre-process on Y_{**} over (X, T).*

Then kernels $f^v_u \circ \mu^v_u$ form a pre-process on Z_{**} over (X, T). If μ^*_* is a process (resp. if μ^*_* satisfies (M)) then $f^v_u \circ \mu^v_u$ is a process (resp. satisfies (M)).

Proof: Observe first that since f^*_* is a morphism of path systems and $\xi_w \circ f^w_w = \xi_w$ for all $w \in T$ we conclude that $\xi_w \circ f^v_u = \xi_w$ for all $u \le w \le v$ in T. In particular, $\xi_u \circ f^v_u \circ \mu^v_u = \xi_u \circ \mu^v_u$ which implies that kernels $f^*_* \circ \mu^*_*$ form a pre-process which is a process if μ^*_* is.

Suppose now that μ^*_* satisfies (M). Consider the diagram



The lower pentagon is equivalent to the square of Definition 3.4.18 and therefore commutes. The left hand side upper square commutes since μ_*^* satisfies (M). The right hand side upper square commutes since $\xi_w \circ f_u^v = \xi_w$. We conclude that the ambient square commutes i.e. $f_*^* \circ \mu_*^*$ satisfies (M).

3.5 Right continuous functions

Recall that a function f on $[s,t]$ is called monotone increasing (resp. decreasing) if for $x \leq y$ one has $f(x) \leq f(y)$ (resp. $f(x) \geq f(y)$). A function is called right continuous if for all $u \in [s,t)$ one has

$$\lim_{\epsilon > 0, \epsilon \to 0} f(u+\epsilon) = f(u).$$

The following lemmas give some elementary properties of such functions which will be used below.

Lemma 3.5.1 *[**rcim2**] Let f be a right continuous function on an interval of the form $[u,v)$. Then for any $(a,b) \subset \mathbf{R}$ one has*

$$f^{-1}((a,b)) = \amalg_{\alpha \in A} I_\alpha$$

where A is countable and each I_α is an interval of the form (y_-, y_+) or $[y_{\alpha,-}, y_{\alpha,+})$. In particular, any right continuous function f on $[s,t]$ is measurable.

Proof: For $x \in f^{-1}((a,b))$ consider the sets

$$I_{x,-} = \{y \mid f([y,x]) \subset (a,b)\}$$

$$I_{x,+} = \{y \mid f([x,y]) \subset (a,b)\}$$

$$I_x = I_{x,-} \cup I_{x,+}$$

For any f we have $I_{x,-} = (x_-, x]$ or $I_{x,-} = [x_-, x]$ where $x_- = Inf(I_{x,-})$ and $I_{x,+} = [x, x_+)$ or $I_{x,+} = [x, x_+)$ where $x_+ = Sup(I_{x,+})$ and for $x_1, x_2 \in f^{-1}((a,b))$ one has $I_{x_1} = I_{x_2}$ or $I_{x_1} \cap I_{x_2} = \emptyset$.

Since f is right continuous we have $I_{x,+} = [x, y_+)$ where $y_+ > x$. In particular, the length of each interval I_x is greater than zero which implies that there are at most countably many distinct intervals in this set.

Lemma 3.5.2 *[**pirc**] Let f be a right continuous on $[s,t)$. If f is monotone increasing then for any $a_+ > a$ such that $f^{-1}([a,a_+)) \neq \emptyset$ there exists $b_+ > b$ such that $f^{-1}([a,a_+)) = [b,b_+)$. If f is monotone decreasing then for any $a_+ > a$ such that $f^{-1}((a,a_+]) \neq \emptyset$ there exists $b_- < b$ such that $f^{-1}((a,a_+]) = [b_-, b)$.*

Proof: Consider for example the case of an increasing f. Then if $f^{-1}([a,a_+)) \neq \emptyset$ we have

$$f^{-1}([a,\infty)) = [b,t)$$

and

$$f^{-1}((-\infty,a_+)) = [s,b_+)$$

which implies the claim of the lemma.

Lemma 3.5.3 *[**rcim3**] Let f_n be a countable family of non-negative right continuous functions on $[u,v]$. Suppose that all the functions are monotone decreasing or monotone increasing and that the sum $f = \sum_{n \geq 1} f_n$ exists. Then f is right-continuous.*

Proof: The monotonicity of f_n implies that for each $\epsilon > 0$ there exists $N \geq 1$ such that for all $x \in [u, v]$ one has $\sum_{n>N} f_n(x) < \epsilon$ (in the case of increasing functions one takes $x = v$ and in the case of decreasing ones $x = u$). This easily implies that the sum is right continuous if all summands are.

Example 3.5.4 *[***2009.06.15.1***]*Lemma 3.5.3 is false without the monotonicity assumption on the functions f_n. For example consider a monotone decreasing sequence $a_n \in \mathbf{R}$ converging to $a \in \mathbf{R}$ such that $a < a_n$ for all n. Let f_n be the indicator function of the interval $I_{[a_n, a_{n-1})}$. Then $\sum_{n \geq 1} f_n$ is the indicator function of the open interval (a, a_0) which is not right continuous.

Recall that an ordered set is called well-ordered if any its non-empty subset has a minimal element. For an element a of a well ordered set such that a is not the maximal element one defines the next element a_+ as the minimum of the set of elements greater than a.

Definition 3.5.5 *[***t1subset***] Let I be an interval of $\mathbf{R}$ which is closed from the below (i.e. an interval of the form $[u, v)$ or $[u, v]$). A subset A of I is called a T1 subset if it is closed (in I) , well-ordered (by the induced ordering) and contains $inf(I)$.*

The minimal element of a T1 subset A is necessarily $inf(I)$. If I is closed then a T1 subset has a maximal element a_{max}. If a_{max} exists we will write $(a_{max})+ = sup(I)$. We denote the set of T1 subsets of an interval I by $S_{T1}(I)$. This set is partially ordered by inclusion. For a closed from below subinterval J of I and $A \in S_{T1}(I)$ set $A_J = \{inf(J)\} \cup (A \cap J)$. One observes easily that $A_J \in S_{T1}(J)$. For a function F on $S_{T1}(T)$ we will write $lim_{A \in S_{T1}(T)} F(A) = x$ if for any $\epsilon > 0$ there exists $A' \in S_{T1}(T)$ such that for any $A \in S_{T1}(T)$ such that $A' \subset A$ one has $|F(A) - x| < \epsilon|$.

Lemma 3.5.6 *[***2009.05.16.3***] Let A_1, A_2 be two T1 subsets of I. Then $A_1 \cap A_2$ and $A_1 \cup A_2$ are T1 subsets.*

Proof: Straightforward.

Lemma 3.5.7 *[***2009.05.16.8***] The following conditions on a subset A of I are equivalent:*

1. *A is a T1 subset,*
2. *A contains $inf(I)$ and for any non-empty subset B of A one has $inf(B) \in B$ and if $sup(B) \in I$ then $sup(B) \in A$,*
3. *for any $x \in I$ such that $x \neq sup(I)$ there exists $y > x$ such that $(x, y) \cap A = \emptyset$ and for any $x \in I$ such that x is a limit point of $\{a \in A \,|\, a < x\}$ one has $x \in A$.*

Proof: The equivalence of the first two conditions is straightforward. The third condition clearly implies the second. Suppose that A is a T1 subset. If $x < sup(I)$ is such that for all $y > x$ one has $(x, y) \cap A \neq \emptyset$ then the set $\{a \in A \,|\, a > x\}$ has no minimal element which contradicts the well-orderness of A. If x is such that x is a limit point of $\{a \in A \,|\, a < x\}$ then $a \in A$ since A is closed. We conclude that the first condition implies the third one.

For $a \in A$ set:

$$I(A, a) = \begin{cases} [a, a_+) & \text{if } a \neq a_{max} \\ \{x \in I \,|\, x \geq a_{max}\} & \text{if } a = a_{max} \end{cases}$$

When no confusion is possible we will write $I(a)$ instead of $I(A, a)$.

Proposition 3.5.8 *[***2009.05.16.1***] Let A be a T1 subset of I. Then $I(A,a) \cap I(A,a') = \emptyset$ for $a \neq a'$ and*

$$[\mathbf{2009.05.16.2}] I = \amalg_{a \in A} I(A,a) \tag{21}$$

Proof: The fact that $I(A,a) \cap I(A,a') = \emptyset$ follows immediately from the definition of a_+ and holds for any well-ordered A in I. To see that (21) holds consider an element $x \in I$ and let $A_x = \{a \in A \,|\, a \leq x\}$. Since $inf(I) \in A$ this set is non-empty and since A is closed, $A_x = A \cap I \cap [inf(I), x]$ is a closed subset in $[inf(I), x]$. Therefore it has a maximal element $a(x)$ and one verifies immediately that $x \in I(A, a(x))$.

Corollary 3.5.9 *[***2009.05.16.6***] A T1 subset is countable.*

Lemma 3.5.10 *[***2009.05.16.5***] Let A be a T1 subset. The there exist a T1 subset A' such that $A \subset A'$ and for any $a' \in A$ there exists $a \in A$ such that the closure $\overline{I(A',a')}$ of $I(A',a')$ in $\mathbf{R}$ is contained in $I(A,a)$.*

Proof: For any a in A choose a monotone increasing sequence $a < a_1 < \cdots < a_n < \cdots < a_+$ which converges to a_+. Then the subset $A' = A \cup \{a_n\}_{a \in A, n \geq 1}$ satisfies the condition of the lemma.

If A' satisfies the condition of Lemma 3.5.10 relative to A we will write $A' > A$. The smallest T1 subset of I is $\{inf(I)\}$. A subset A satisfying $A > \{inf(I)\}$ will be called a T2 subset. Equivalently, a T1 subset A is a T2 subset if for each $a \in A$ one has $sup(I(a)) \in I$. If I is closed from the above this condition holds for any T1 subset and if I is open from the above it holds if and only if $sup(A) = sup(I)$. Lemma 3.5.10 implies among other things that the subset $S_{T2}(I)$ of $S_{T1}(I)$ is co-final.

For a function $f : I \to \mathbf{R}$ and $\epsilon > 0$ let $A_0(f, \epsilon)$ be the set of points $a \in I$ such that for all $x < a$ in I there exists x', $x \leq x' \leq a$ such that $|f(a) - f(x')| > \epsilon$.

Proposition 3.5.11 *[***2009.05.16.7***] Let f be a right continuous function. Then for any $\epsilon > 0$, $A_0(f, \epsilon)$ is a T1 subset.*

Proof: Let us verify the third condition of Lemma 3.5.7. Since f is right continuous, for any $x \in I$ such that $x < sup(I)$ there exists $y > x$ such that $f([x,y]) \subset (f(x) - \epsilon/2, f(x) + \epsilon/2)\}$. Let us show that $(x,y) \cap A_0(f,\epsilon) = \emptyset$. Indeed, if $a \in (x,y)$ then for all $x' \in [x,a]$ we have $|f(a) - f(x')| \leq |f(a) - f(x)| + |f(x') - f(x)| \leq \epsilon$ and therefore a is not an element of $A_0(f,\epsilon)$ this proofs the first half of the condition. The second half of the condition follows immediately from the definition of $A_0(f, \epsilon)$.

Corollary 3.5.12 *[***2009.05.16.09***] Let f be a right continuous function on an interval. The the set of its points of discontinuity is countable.*

Proof: It follows from the proposition and Corollary 3.5.9 since f is continuous outside of the subset $\cup_{n \geq 1} A_0(f, 1/n)$.

Proposition 3.5.13 *[***2009.05.16.4***] Let $f : I \to \mathbf{R}$ be a right continuous function. Then for any $\epsilon > 0$ there exists a T1 subset $A = A_1(f, \epsilon)$ such that for any $a \in A_1$ one has*

$$[\mathbf{2009.05.16.10}] sup_{x \in I(a)}(f(x)) - inf_{x \in I(a)}(f(x)) < \epsilon \tag{22}$$

Proof: For any $x \in I$ the set $\{y \mid f([x, y]) \subset (f(x) - \epsilon, f(x) + \epsilon)\}$ is of the form $[x, x_+)$ for some $x_+ > x$ or of the form $[x, sup(I)]$. Set inductively $x_{+0} = x$ and $x_{+(n+1)} = (x_{+n})_+$ assuming that $sup(I)_+ = sup(I)$. For any x, the sequence x_{+n} is monotone increasing and must converge to some x'. One observes immediately that either this sequence stops at $sup(I)$ after a finite number of steps or $x' \in A_0(f, \epsilon)$. Let $A_1(f, \epsilon)$ be the set of points of the form x_{+n} for $x \in A_0$ and $n \geq 0$. Then it is a T1 set which satisfies (22).

Corollary 3.5.14 *[**2009.05.16.15**] Let $f : I \to \mathbf{R}$ be a right continuous function. Then for any $\epsilon > 0$ there exists a T2 subset $A = A_2(f, \epsilon)$ such that*

$$[\mathbf{2009.05.16.11}] sup_{x \in \overline{I(A_2, a)}}(f(x)) - inf_{x \in \overline{I(A_2, a)}}(f(x)) < \epsilon \tag{23}$$

Proof: It follows immediately from the proposition and Lemma 3.5.10.

Proposition 3.5.15 *[**2009.05.16.13**] Let f be a non-negative right continuous function on I. Then for any bounded measure α on I one has*

$$\int_{y \in I} f(y) d\alpha = lim_{A \in S_{T2}(I)} \sum_{a \in A} \alpha(I(a)) f(sup(I(a)))$$

Proof: Follows immediately from Corollary 3.5.14.

3.6 Path system defined by a multi-graph

An important class of path systems arises from multi-graphs. A reflexive multi-graph X is a diagram of sets of the form $(\partial_0, \partial_1 : X_1 \to X_0, \sigma : X_0 \to X_1)$ such that $\partial_0 \circ \sigma = \partial_1 \circ \sigma = Id$. The set X_0 is the set of vertices of X and X_1 is the set of edges. Edges lying in $X_1^{nd} = X_1 \backslash \sigma(X_0)$ are called non-degenerate.

We let $X[u, v]$ denote the set of triples

$$(\{x_1, \ldots, x_n\} \subset [u, v], p : [u, v] \to X_0, e : \{x_1, \ldots, x_n\} \to X_1^{nd})$$

such that:

1. $u < x_1 < \cdots < x_n \leq v$,
2. p is right continuous and continuous outside $\{x_1, \ldots, x_n\}$,
3. for each $i = 1, \ldots, n$ one has $p_-(x_i) = \partial_0(e(x_i))$ and $p(x_i) = \partial_1(e(x_i))$ where $p_-(x) = \lim_{y \uparrow x} p(y)$.

For $u \leq u' \leq v' \leq v$ the obvious restriction defines a map of sets

$$res^{u,v}_{u',v'} : X[u, v] \to X[u', v']$$

such that $res^{u,v}_{u,v} = Id$ and for $u \leq u' \leq u'' \leq v'' \leq v' \leq v$ one has

$$res^{u,v}_{u'',v''} = res^{u',v'}_{u'',v''} res^{u,v}_{u',v'}$$

Let $X_n = X_1 \,{}_{\partial_1}\!\times_{\partial_0} \cdots {}_{\partial_1}\!\times_{\partial_0} X_1$ be the set of paths of length n in X and X_n^{nd} the subset of paths $(e_1, \ldots, e_n)$ such that $e_i \in X_1^{nd}$ for all i.

Let

$$\Delta^n_{(u,v)} = \{u < x_1 < \cdots < x_n < v\}$$

and

$$\Delta^n_{(u,v]} = \{u < x_1 < \cdots < x_n \leq v\}$$

We assume that $\Delta^0_{(u,u)} = \Delta^0_{(u,u]} = pt$ and $\Delta^i_{(u,u)} = \Delta^i_{(u,u]} = \emptyset$ for $i > 0$. For $u > v$ and $n > 0$ we have

$$\Delta^n_{(u,v]} = \Delta^n_{(u,v)} \amalg \Delta^{n-1}_{(u,v)}$$

With this notation $X[u,v]$ can also be written as

$$X[u,v] = \amalg_{n\geq 0} \Delta^n_{(u,v]} \times X^{nd}_n$$

If X is countable then this description provides an obvious choice of σ-algebras on $X[u,v]$. With respect to these σ-algebras the maps $res^{u,v}_{u',v'}$ are measurable and the resulting structure is a path system in the sense of Definition 3.3.1. Our definition implies that $X[u,u] = X_0$ and therefore $X[*,*]$ is naturally a path system over X_0.

If there is at most one edge $e \in X_1$ connecting any pair of vertices $p, p' \in X_0$ then $X[*,*]$ is a sub-system of the canonical path system of X_0 but in general it is not the case.

A (deterministic) morphism $f : X \to X'$ of reflexive multi-graphs defines a morphism of associated path systems as follows. For $(\{x_1, \ldots, x_n\} \subset [u,v], p : [u,v] \to X_0, e : \{x_1, \ldots, x_n\} \to X^{nd}_1)$ in $X[u,v]$ let $\iota : I \subset \{1, \ldots, n\}$ be the subset which consists of i such that $f(x_i) \in (X'_1)^{nd}$. Then

$$f^v_u(\{x_1\}^n_{i=1}, p, e) = (\{x_i\}_{i\in I}, f_0 \circ p, f_1 \circ e \circ \iota)$$

The compatibility of these maps with the restriction maps is obvious and we get a morphism of path systems over the map $X_0 \to X'_0$.

When no confusion is possible we will write $\Delta^e_{(u,v]}$ for the simplex in $X[u,v]$ corresponding to an element $e = (e_1, \ldots, e_n) \in X^{nd}_n$. For any $u \leq w \leq v$ the product $res^{u,v}_{u,w} \times res^{u,v}_{w,v}$ restricted to $\Delta^e_{(u,v]}$ maps it bijectively to

$$(res^{u,v}_{u,w} \times res^{u,v}_{w,v})(\Delta^e_{(u,v]}) = \amalg^n_{i=0}(\Delta^{(e_1,\ldots,e_i)}_{(u,w]} \times \Delta^{(e_{i+1},\ldots,e_n)}_{(w,v]})$$

such that

$$(res^{u,v}_{u,w} \times res^{u,v}_{w,v})^{-1}(\Delta^{(e_1,\ldots,e_i)}_{(u,w]} \times \Delta^{(e_{i+1},\ldots,e_n)}_{(w,v]}) =$$

$$= \{(x_1, \ldots, x_n) \in \Delta^e_{(u,v]} \,|\, u < x_1 < \cdots < x_i \leq w < x_{i+1} < \cdots < x_n \leq v\}$$

In particular, for a reflexive multi-graph X the "cutting maps" $res^{u,v}_{u,w} \times res^{u,v}_{w,v}$ define a bijection

$$res^{u,v}_{u,w} \times res^{u,v}_{w,v} : X[u,v] \cong X[u,w] \times_{X_0} X[w,v]$$

Let us call a sequence of measurable subsets $I_1, \ldots, I_n$ of $\mathbf{R}$ admissible if for all $i = 1, \ldots, n-1$ one has $sup(I_i) < inf(I_{i+1})$. For $u \leq v$ in T, $e \in X^{nd}_n$ and an admissible sequence $I_1, \ldots, I_n$ of measurable subsets in $(u,v]$ let

$$U^v_{e,u}(I_1, \ldots, I_n) = \{(x_1, \ldots, x_n) \in \Delta^n_{(u,v]} \times \{e\} \,|\, x_i \in I_i\}$$

For $n = 0$ we set $U^v_{e,u} = \{e\} = \Delta^0_{(u,v]} \times \{e\}$.

Lemma 3.6.1 *[2009.05.27.2][generation1] For any $u \leq v$ in T and $p \in X_0$ the set of subsets of the form $U^v_{e,u}(I_1, \ldots, I_n)$ where I_i are closed intervals of $(u, v]$ is closed under finite intersections and generates the Borel σ-algebra of $X[u, v]_{p,*}$.*

Proof: We have

$$U^v_{e,u}(I_1, \ldots, I_n) \cap U^v_{e',u}(I'_1, \ldots, I'_{n'}) = \begin{cases} U^v_{e,u}(I_1 \cap I'_1, \ldots, I_n \cap I'_n) & \text{for } e = e' \\ \emptyset & \text{for } e \neq e' \end{cases}$$

which shows that our class of subsets is closed under finite intersections. The fact that it generates the Borel σ-algebra of $X[u, v]_{p,*}$ is equivalent to the assertion that subsets of the form $\{(x_1, \ldots, x_n) \in \Delta^n_{(u,v]} \,|\, x_i \in I_i\}$ generate the Borel σ-algebra of the simplex $\Delta^n_{(u,v]}$. This is a corollary of the theorem which says that the Borel σ-algebra of $\mathbf{R}^n$ coincides with the product of Borel σ-algebras on $\mathbf{R}^1$.

Let T be a time window which we assume to be of the form $T = [T_{min}, T_{max}]$. Let $P = (\mu^v_u \,|\, u \leq v,\ u, v \in T)$ be a pre-process on $(X[*, *], T)$. Let us introduce the following notation. For $u \leq v$ in T and $p \in X_0$ we set

$$h_p(u, v) = \mu^v_{p,u}(U^v_{p,u})$$

and for $e \in X^{nd}_1$ we let $\lambda^v_{e,u}$ denote the co-restriction of $\mu^v_{\partial_0(e),u}$ to $\Delta^1_{(u,v]} \times \{e\}$ considered as a measure on $(u, v]$. For convenience we will often consider $\lambda^v_{e,u}$ as measures on $[u, v]$ which are zero on $\{u\}$.

We will also consider for $p, p' \in X_0$ and u, v as above the functions

$$\phi^{p'}_p(u, v) = \mu^v_{u,p}(X[u, v]_{*,p'})$$

and

$$\upsilon_p(u, v) = \mu^v_{p,u}(X[u, v]_{p,*}) = \sum_{p'} \phi^{p'}_p(u, v)$$

which are defined in the context of any path system.

3.7 Renewal pre-processes on multi-graphs

For a multigraph X and $p \in X_0$ we let $X^{nd}_n(p)$ denote the subset of $e \in X^{nd}_n$ such that $\partial^n_0(e) = p$.

Definition 3.7.1 *[thetagenerator] Let X be a multi-graph and $T = [T_{min}, T_{max}]$ a time window. A generating kernel on (X, T) is a probability kernel*

$$\theta : X_0 \times T \to (X_0 \times \{*\}) \amalg (X^{nd}_1 \times T)$$

such that for any $p \in X_0$ and $u \in T$ the measure $\theta(p, u)$ is supported on $(\{p\} \times \{*\}) \amalg (X^{nd}_1(p) \times T_{>u})$.

For a generating kernel θ, $e \in X^{nd}_1$ and $u \in T$ we let $\theta_{e,u}$ denote the co-restriction of the measure $\theta(\partial_0(e), u)$ to $T = \{e\} \times T$.

Theorem 3.7.2 *[2009.06.30.3] Let X be a countable multi-graph, $T = [T_{min}, T_{max}]$ a time window and θ a generating kernel on (X, T). The there exists a unique pre-process $P = \{\mu^v_{p,u}\}$ on (X, T) such that*

1. for any $u \leq v$ in T and $p \in X_0$ one has

$$\mu^v_{p,u}(U^v_{p,u}) = 1 - \sum_{e \in X^{nd}_1 \,|\, \partial_0(e)=p} \theta_{e,u}(T_{\leq v})$$

2. for any $u \leq v$ in T, $n \geq 1$, $e \in X^{nd}_n$ and any admissible sequence $I_1, \ldots, I_n$ of closed intervals in $(u, v]$ one has

$$\mu^v_{p,u}(U^v_{e,u}(I_1, \ldots, I_n)) = \int_{x_1 \in I_1} \mu^v_{p',x_1}(U^v_{\partial_1(e),x_1}(I_2, \ldots, I_n)) d\theta_{\partial_0^{n-1}(e),u}$$

where

$$p = \partial_0^n(e), \quad p' = \partial_0^{n-1}\partial_1(e)$$

Proof: The uniqueness part follows immediately from Lemma 3.6.1. The proof of the existence part will be finished in Proposition 3.7.10.

Definition 3.7.3 *[**renewal**] A pre-process is called a renewal pre-process if it corresponds according to Theorem 3.7.2 to a generating kernel.*

Definition 3.7.4 *[**2009.06.02.3**] Let X be a multi-graph and $T = [T_{min}, T_{max}]$ a time window. A generating map on (X, T) is a measurable map*

$$[\textbf{egenerator}] E : X_0 \times T \times [0, 1] \to (X_0 \times \{*\}) \amalg (X^{nd}_1 \times T) \tag{24}$$

such that for any $p \in X_0$, $u \in T$ one has

$$Im(E(p, u, -)) \subset (\{p\} \times \{*\}) \amalg (X^{nd}_1(p) \times T_{>u}).$$

A generating map is said to represent a generating kernel θ if for all $p \in X_0$ and $u \in T$ one has

$$E(p, u, -)_*(dx) = \theta(e, u)$$

where dx is the Lebesgue measure on $[0, 1]$.

For a generating map E, $p \in X_0$ and $u \in T$ we let $E_{p,u}$ denote the map $r \mapsto E(p, u, r)$ from $[0, 1]$ to $(\{p\} \times \{*\}) \amalg (X^{nd}_1(p) \times T_{>u})$.

Proposition 3.7.5 *[**2009.06.02.2**] Let θ be a generating kernel for (X, T). Then there exists a generating map*

$$E : X_0 \times T \times [0, 1] \to (X_0 \times \{*\}) \amalg (X^{nd}_1 \times T)$$

which represents θ.

Proof: Let $\Theta_{e,u} : T \to [0, 1]$ be the right continuous distribution function of $\theta_{e,u}$ i.e.

$$\Theta_{e,u}(v) = \theta_{e,u}([u, v]).$$

For each $p \in X_0$ let us choose a linear ordering on $X^{nd}_1(p)$ and let $e(p, n)$ be the n-th element of $X^{nd}_1(p)$ relative to this ordering. Let $x_0(p) = 1$ and for each $n \geq 1$ let

$$x_n(p) = 1 - \sum_{i=1}^{n} \theta_{e(p,n),u}(T).$$

Let further

$$x_\infty(p) = lim_{n\to\infty} x_n(p) = inf\{x_n(p),\ \ n \geq 0\}$$

Then

$$[0,1] = [0, x_\infty] \amalg (\amalg_{n\geq 1}(x_n, x_{n-1}])$$

For $n \geq 1$ define a map $E_{e(p,n),u} : (x_n, x_{n-1}] \to T$ by the rule

$$E_{e(p,n),u}(x) = inf\{v \,|\, \Theta_{e(p,n),u}(v) \geq x - x_n\}$$

Note that $E_{e(p,n),u}$ is a well defined monotone increasing map whose image lies in $\{e(p,n)\} \times T_{>u}$. Let

$$E_{p,u} : [0,1] \to (X_0 \times \{*\}) \amalg (X_1^{nd} \times T)$$

be the map given by the conditions

$$(E_{p,u})_{|[0,x_\infty]} = \{p\} \times \{*\}$$

and

$$(E_{p,u})_{|(x_n,x_{n-1}]} = \{e(p,n)\} \times E_{e(p,n),u}$$

for all $n \geq 1$. We claim that the map

$$E : X_0 \times T \times [0,1] \to (X_0 \times \{*\}) \amalg (X_1^{nd} \times T)$$

given by $E(p,u,x) = E_{p,u}(x)$ is a generating map which represents θ. This is proved in the following two lemmas.

Lemma 3.7.6 *[***2009.05.30.1***] One has $(E_{p,u})_*(dx) = \theta_{p,u}$.*

Proof: (cf. [12, §3.12]) It is clear from the construction that in order to prove the lemma we have to verify that

$$(E_{e(p,n),u})_*(dx^{|(x_n,x_{n-1}]}) = \theta_{p,u}$$

Since $\Theta_{e,u}$ are monotone increasing and right continuous we have

$$\{v \,|\, \Theta_{e(p,n),u}(v) \geq x - x_n\} = [E_{e(p,n),u}(x), T_{max}]$$

Therefore, $E_{e(p,n),u}(x) \leq w$ if and only if $\Theta_{e(p,n),u}(w) \geq x - x_n$ and we have

$$\{x \in (x_n, x_{n-1}] \,|\, E_{e(p,n),u}(x) \leq w\} = \{x \,|\, x > x_n \ \text{ and } \ x \leq x_n + \Theta_{e(p,n),u}(w)\}$$

and

$$dx(\{x \in (x_n, x_{n-1}] \,|\, E_{e(p,n),u}(x) \leq w\}) = \Theta_{e(p,n),u}(w)$$

Lemma 3.7.7 *[***2009.06.01.4***] The map E is a generating map i.e. it satisfies the conditions of Definition 3.7.4.*

Proof: The only non-trivial condition is that E is measurable. Since all subsets of X_0 and X_1^{nd} are assumed to be measurable, in order to prove that E is measurable it is sufficient to show that for any $p \in X_0$ and $n \geq 0$ the map

$$F : T \times [0,1] \to * \amalg T$$

given by

$$F(u,x) = \begin{cases} E_{e(p,n),u}(x) & \text{if } x_{n-1}(p,u) \leq x > x_n(p,u) \\ * & \text{otherwise} \end{cases}$$

is measurable i.e. that for any w the subset $U_w = \{(u,x) \mid F(u,x) \in T \text{ and } F(u,x) \leq w\}$ is measurable. By the same reasoning as in the proof of Lemma 3.7.6 we see that $(u,x) \in U_w$ if and only if $x > x_n(p,u)$ and $x \leq x_n(p,u) + \Theta_{e(p,n),u}(w)$. Since θ is a kernel the functions $\Theta_{e,u}(v)$ are measurable as the functions of $u \in T$. By one of the standard properties of the product σ-algebras the area under the graph of a measurable function is measurable which implies the statement of the lemma.

Let E be a generating map representing θ. Denote the measurable space $[0,1]$ by Rnd and the Lebesgue measure by P_{Rnd}. For $n \geq 1$ define maps

$$E_n : X_0 \times T \times Rnd^n \to (X_0 \times \{*\}) \amalg (X_1^{nd} \times T)$$

inductively as follows:

1. $E_1 = E$
2. If $n > 1$
   (a) if $E_{n-1}(p,u,r_1,\dots,r_{n-1}) = (e,t)$ then $E_n(p,u,r_1,\dots,r_n) = E(\partial_1(e),t,r_n)$,
   (b) else if $E_{n-1}(p,u,r_1,\dots,r_{n-1}) = (p',*)$ then $E_n(p,u,r_1,\dots,r_n) = (p',*)$.

For $u \leq v$ in T, $p \in X_0$ and $n \geq 1$ let

$$\Omega^v_{p,u,n} = \{(r_1,\dots,r_n,\dots) \in Rnd^\infty \mid E_{n+1}(p,u,r_1,\dots,r_{n+1}) \in (X_0 \times \{*\}) \amalg (X_1^{nd} \times T_{>v})\}$$

For convenience we will write $\Omega^v_{p,u,-1}$ for $\emptyset$. We have $\Omega^v_{p,u,n} \subset \Omega^v_{p,u,n+1}$ and $\Omega^{v'}_{p,u,n} \subset \Omega^v_{p,u,n}$ for $v' \geq v$.

For $u \leq v$ in T, $p \in X_0$ and $n \geq 0$ define a map

$$M^v_{p,u,n} : \Omega^v_{p,u,n} \backslash \Omega^v_{p,u,n-1} \to \amalg_{e \in X_n^{nd}(p)} \Delta^n_{(u,v]} \times \{e\}$$

setting

$$M^v_{p,u,0}(\underline{r}) = \{p\}$$

and for $n \geq 1$,

$$M^v_{p,u,n}(\underline{r}) = \{(x_1,\dots,x_n)\} \times \{(e_1,\dots,e_n)\}$$

where $E_i(r_1,\dots,r_i) = (e_i,x_i)$. Let

$$\Omega^v_{p,u,\infty} = \cup_{n\geq 0} \Omega^v_{p,u,n} = \amalg_{n \geq 0}\, \Omega^v_{p,u,n} \backslash \Omega^v_{p,u,n-1} \subset Rnd^\infty$$

Taking the disjoint union over $n \geq 0$ we get a map

$$M^v_{p,u} : \Omega^v_{p,u,\infty} \to X[u,v]_{p.*}$$

and the map

$$\bar{M}^v_{p,u} : Rnd^\infty \to \{*\} \amalg X[u,v]_{p,*}$$

which equals $M^v_{p,u}$ on $\Omega^v_{p,u,\infty}$ and sends $Rnd^\infty\backslash\Omega^v_{p,u,\infty}$ to $*$. This map is clearly measurable and we define measures $\mu^v_{p,u}$ on $X[u,v]_{p,*}$ by the formula

$$\mu^v_{p,u} = ((\bar{M}^v_{p,u})_*(P^{\otimes\infty}_{Rnd}))^{|X[u,v]_{p,*}}$$

Since X_0 is assumed to be countable these measures define kernels $\mu^v_u : X_0 \to X[u,v]$ which form a pre-process $P = P(E)$ on (X,T). We will call it the pre-process defined by a generating map E.

Remark 3.7.8 *[***2009.06.17.1***]* The maps $M^v_{p,u}$ commute with the restriction maps $res^{u,v}_{u,w}$ i.e. for $u \le w \le v$ in T and $p \in X_0$ the square

$$\begin{array}{ccc} \Omega^v_{p,u,\infty} & \xrightarrow{M^v_{p,u}} & X[u,v]_{p,*} \\ \downarrow & & \downarrow{\scriptstyle res^{u,v}_{u,w}} \\ \Omega^w_{p,u,\infty} & \xrightarrow{M^w_{p,u}} & X[u,w]_{p,*} \end{array}$$

commutes. However, the square

$$\begin{array}{ccc} Rnd^\infty & \xrightarrow{\bar{M}^v_{p,u}} & \{*\} \amalg X[u,v]_{p,*} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \{*\}\amalg res^{u,v}_{u,w}} \\ Rnd^\infty & \xrightarrow{\bar{M}^w_{p,u}} & \{*\} \amalg X[u,w]_{p,*} \end{array}$$

does not commute in general since for $\underline{r} \in \Omega^w_{p,u,\infty}\backslash\Omega^v_{p,u,\infty}$ we have $\bar{M}^v_{p,u}(\underline{r}) = \{*\}$ while $\bar{M}^w_{p,u}(\underline{r}) \in X[u,w]_{p,*}$.

For $u \le w \le v$ in T, let

$$j^{u,v}_{w,v} : X[w,v] \to X[u,v]$$

be the embedding which sends $(x_1 \dots, x_n) \in \Delta^n_{(w,v]} \times \{e\}$ to $(x_1 \dots, x_n) \in \Delta^n_{(u,v]} \times \{e\}$. For $v \in T$ and $p \in X_0$ define a map

$$\bar{M}^v_p : T \times Rnd^\infty \to \{*\} \times X[T_{min}, v]_{p,*}$$

by the formula

$$\bar{M}^v_p(x, \underline{r}) = (Id_{\{*\}} \amalg j^{u,v}_{x,v})(\bar{M}^v_{p,x}(\underline{r}))$$

Lemma 3.7.9 *[***2009.06.17.2***] The maps $\bar{M}^v_p$ are measurable.*

Proof: We have to prove that for any $e \in X^{nd}_n(p)$ and any measurable V in $\Delta^n_{(T_{min},v]} \times \{e\}$ the subset $(\bar{M}^v_p)^{-1}(V)$ is measurable. Since the Borel σ-algebra on $\mathbf{R}^n$ coincides with the product of Borel σ-algebras on $\mathbf{R}$ it is sufficient, for $n \ge 1$, to consider subsets of the form $V = \{(x_1, \dots, x_n) \,|\, x_1 \in I,\ (x_2, \dots, x_n) \in U\}$ where I is a measurable subset of $(T_{min}, v]$ and U a measurable subset of $\Delta^{n-1}_{(sup(I),v]}$.

Let us proceed by induction on n. For $n = 0$ we have $X^{nd}_n(p) = \{p\}$ and $\bar{M}^v_p(x, \underline{r}) = \{p\}$ if and only if $E(p, x, r_1) \in (\{p\} \times \{*\}) \amalg (X^{nd}_1 \times T_{>v})$. Therefore $(\bar{M}^v_p)^{-1}(\{p\})$ is measurable since E is measurable. Suppose that $n \ge 1$ and let I and U be as above. Then $\bar{M}^v_p(x, \underline{r}) \in I \times U$ if and only if $E(p, x, r_1) = (\partial^{n-1}_0(e), x')$, $x' \in I$ and $\bar{M}^v_{p'}(x', (r_2, \dots,)) \in U$ where U is considered as a subset of $\Delta^{n-1}_{(Y_{min},v]} \times \{\partial_1(e)\}$, where $p' = \partial^{n-1}_0(\partial_1(e))$. Consider the map

$$f : T \times Rnd^\infty \to ((X_0 \times \{*\}) \amalg (X^{nd}_1 \times T)) \times T \times Rnd^\infty$$

of the form

$$f(x,\underline{r}) = (E(p,x,r_1), y, (r_2,\ldots,))$$

This map is clearly measurable. Let Z_e be the subset of elements of the form $((e,z),z,\underline{r})$ of $((X_0\times\{*\})\amalg(X_1^{nd}\times T))\times T\times Rnd^{\infty}$. It is measurable since the diagonal of $T\times T$ is measurable. On the other hand we have

$$(\bar{M}_p^v)^{-1}(I\times U) = f^{-1}(Z_e\cap((X_0\times\{*\})\amalg(X_1^{nd}\times T))\times((\bar{M}_{p'}^v)^{-1}(U)))$$

which together with the inductive assumption shows that $(\bar{M}_p^v)^{-1}(I\times U)$ is measurable.

The second statement of the following proposition is a reformulation of the second condition of Theorem 3.7.2 and therefore the proof of this proposition finishes the proof of Theorem 3.7.2.

Proposition 3.7.10 *[***2009.06.03.1***] Let P be a process defined by a generating map E. Then for any $p\in X_0$, any $u\le w\le v$ in T and any measurable $U\subset X[w,v]$, the function*

$$[\mathbf{2009.06.03.eq1}] x\mapsto \mu_{p,x}^v((res_{x,w}\times res_{w,v})^{-1}(\{p\}\times U)) \quad (25)$$

on $[u,w]$ is measurable.

Proof: Observe that we have

$$(res_{x,w}\times res_{w,v})^{-1}(\{p\}\times U) = j_{w,v}^{x,v}(U)$$

and

$$(\bar{M}_{p,x}^v)^{-1}(j_{w,v}^{x,v}(U)) = (\bar{M}_p^v)^{-1}(j_{w,v}^{T_{min},v}(U))\cap\{x\}\times Rnd^{\infty}$$

therefore

$$\mu_{p,x}^v((res_{x,w}\times res_{w,v})^{-1}(\{p\}\times U)) = P_{Rnd}^{\otimes\infty}((\bar{M}_p^v)^{-1}(j_{w,v}^{T_{min},v}(U))\cap\{x\}\times Rnd^{\infty})$$

which is a measurable function since $(\bar{M}_p^v)^{-1}(j_{w,v}^{T_{min},v}(U))$ is a measurable subset by Lemma 3.7.9.

Corollary 3.7.11 *[***2009.05.28.1***] For P as above, the functions $h_p(-,v)$ are measurable.*

Proof: The function $h_p(-,v)$ is the function (25) for $w=v$ and $U=\{p\}$.

Lemma 3.7.12 *[***2009.05.27.4***] For $u\le v$ in T, $n\ge 1$, $e\in X_n^{nd}$, and two measurable subsets $I\subset(u,v]$ and $U\subset\Delta_{(sup(I),v]}^{n-1}\times\{\partial_1(e)\}$ let*

$$W_{e,u}^v(I,U) = \{(x_1,\ldots,x_n)\in\Delta_{(u,v]}^n\times\{e\}\,|\,x_1\in I \text{ and } (x_2,\ldots,x_n)\in U\}$$

Let further $e_1=\partial_0^{n-1}(e)$, $p=\partial_0(e_1)$ and $p'=\partial_1(e_1)$. Then for the pre-process defined by a generating map E one has

$$[\mathbf{2009.06.17.eq3}]\mu_{p,u}^v(W_{e,u}^v(I,U)) = \int_{x\in I}\mu_{p',x}^v((res_{x,sup(I)}\times res_{sup(I),v})^{-1}(\{p'\}\times U))d\theta_{e_1,u} \quad (26)$$

Proof: Let $\Omega_{e_1} = E_{p,u}^{-1}(\{e_1\} \times T)$ and let $g : \Omega_{e_1} \to T$ be the map defined by the condition $E_{p,u}(r) = (e_1, g(r))$. We have

$$(\bar{M}_{p,u}^v)^{-1}(W_{e,u}^v(I,U)) =$$

$$= \{\underline{r} \in Rnd^\infty \,|\, E(r_1) = (e_1, x) \in \{e_1\} \times I \text{ and } M_{p',x}^v(r_2, \dots) \in (res_{x,sup(I)} \times res_{sup(I),v})^{-1}(\{p'\} \times U)\}$$

By definition of $\bar{M}_p^v$ we have

$$[\mathbf{2009.06.17.eq2}] (\bar{M}_{p',x}^v)^{-1}((res_{x,sup(I)} \times res_{sup(I),v})^{-1}(\{p'\} \times U)) = (\bar{M}_p^v)^{-1}(U) \cap \{x\} \times Rnd^\infty \quad (27)$$

where on the right hand side, U is considered as a subset of $\Delta^n_{(T_{min},v]} \times \{\partial_1(e)\}$. Therefore,

$$\mu_{p,u}^v(W_{e,u}^v(I,U)) = P_{Rnd}^{\otimes\infty}((\bar{M}_{p,u}^v)^{-1}(W_{e,u}^v(I,U))) =$$

$$= (P_{Rnd}^{|\Omega_{e_1}} \otimes P_{Rnd}^{\otimes\infty})((g \times Id_{Rnd^\infty})^{-1}((I \times Rnd^\infty) \cap (\bar{M}_p^v)^{-1}(U)) =$$

$$= (\theta_{e,u} \otimes P_{Rnd}^{\otimes\infty})((I \times Rnd^\infty) \cap (\bar{M}_p^v)^{-1}(U))$$

By Fubini's theorem we have

$$(\theta_{e,u} \otimes P_{Rnd}^{\otimes\infty})((I \times Rnd^\infty) \cap (\bar{M}_p^v)^{-1}(U)) = \int_{x \in I} P_{Rnd}^{\otimes\infty}((\bar{M}_p^v)^{-1}(U) \cap \{x\} \times Rnd^\infty) d\theta_{e,u}$$

and applying again (27) we conclude that (26) holds.

Proposition 3.7.13 *[***2009.05.27.3***] For the pre-process on (X,T) defined by a generating map E with the underlying kernel θ and $u \leq v$ in T one has*

1. *for any $p \in X_0$,*
$$h_p(u,v) = \mu_{p,u}^v(U_{p,u}^v) = 1 - \sum_{e \in X_1^{nd} \,|\, \partial_0(e) = p} \theta_{e,u}((u,v])$$

2. *for any $n \geq 1$, $e \in X_n^{nd}$ and any sequence of measurable subsets $I_1, \dots, I_n$ in $(u,v]$ such that $sup(I_i) < inf(I_{i+1})$ one has*
$$\mu_{p0,u}^v(U_{e,u}^v(I_1, \dots, I_n)) =$$
$$= \int_{x_1 \in I_1} \dots \int_{x_n \in I_n} h_{p_n}(x_n, v) d\theta_{e_n, x_{n-1}} \dots \theta_{e_1,u}$$
where
$$U_{e,u}^v(I_1, \dots, I_n) = \{x_1, \dots, x_n \in \Delta^n_{(u,v]} \times \{e\} \,|\, x_i \in I_i\},$$
$$e_i = \partial_0^{n-i}\partial_1^{i-1}(e), \;\; p_0 = \partial_0^n(e), \;\; p_n = \partial_1^n(e)$$

Proof: To prove the first part observe that

$$(\bar{M}_{p,u}^v)^{-1}(\{p\}) = \Omega_{p,u,0}^v = \{\underline{r} \in Rnd^\infty \,|\, E(u,p,r_1) \in (T_{>v} \times X_1^{nd}) \amalg (\{*\} \times X_0)$$

and therefore

$$\mu_{p,u}^v(\{p\}) = P_{Rnd}(Rnd \backslash E_{u,p}^{-1}((u,v] \times X_1^{nd}))) = 1 - \sum_{e \in X_1^{nd} \,|\, \partial_0(e) = p} \theta_{e,u}((u,v])$$

To prove the second part observe that for $e \in X_1^{nd}$ one has

$$U_{e,u}^v(I_1) = W_{e,u}^v(I_1, \{p_1\})$$

and

$$(res_{x_1,sup(I_1)} \times res_{sup(I_1),v})^{-1}(\{p_1\} \times \{p_1\}) = \{p_1\}$$

and for $n > 1$

$$U_{e,u}^v(I_1, \ldots, I_n) = W_{e,u}^v(I_1, U_{\partial_1(e),sup(I_1)}^v(I_2, \ldots, I_n))$$

and

$$(res_{x_1,sup(I_1)} \times res_{sup(I_1),v})^{-1}(\{p_1\} \times U_{\partial_1(e),sup(I_1)}^v(I_2, \ldots, I_n)) = U_{\partial_1(e),x_1}^v(I_2, \ldots, I_n)$$

Therefore, by Lemma 3.7.12 we have for $e \in X_1^{nd}$

$$[\mathbf{2009.05.27.eq1}]\mu_{p_0,u}^v(U_{e,u}^v(I_1)) = \mu_{p_0,u}^v(W_{e,u}^v(I_1, \{p_1\})) = \int_{x_1 \in I_1} \mu_{p_1,x_1}^v(\{p_1\}) d\theta_{e,u} \qquad (28)$$

and for $e \in X_n^{nd}$ where $n > 1$,

$$\mu_{p_0,u}^v(U_{e,u}^v(I_1, \ldots, I_n)) = \mu_{p_0,u}^v(W_{e,u}^v(I_1, U_{\partial_1(e),sup(I_1)}^v(I_2, \ldots, I_n))) =$$

$$= \int_{x_1 \in I_1} \mu_{p_1,x_1}^v(U_{\partial_1(e),x_1}^v(I_2, \ldots, I_n)) d\theta_{e_1,u}$$

which by easy induction implies the second part of the proposition.

Corollary 3.7.14 *[**2009.05.27.7**] For any renewal pre-process the functions $h_p(u, -)$ are right continuous.*

Proof: It follows from the first part of the proposition since the distribution function defined through closed intervals is right continuous and the difference of two right continuous functions is right continuous.

Corollary 3.7.15 *[**2009.05.27.5**] Under the assumption of the proposition we have for any $u \le v$ in T and $e \in X_1^{nd}$,*

$$\lambda_{e,u}^v = \theta_{e,u} * h_{\partial_1(e)}(-, v)$$

Proof: It is the equivalent to (28).

Proposition 3.7.16 *[**2009.06.19.1**] Let P be a process defined by a generating kernel θ. Then for any $u \le v$ in T and any $e \in X_1^{nd}$ one has*

$$\theta_{e,u}([u, v]) = lim_{A \in S_{T1}([u,v])} \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a))$$

Proof: Let $p = \partial_0(e)$ and $p' = \partial_1(e)$. We have

$$\lambda_{e,u}^{sup(I(a))}(I(a)) = P_{Rnd}^{\otimes\infty}((\bar{M}_{p,u}^{sup(I(a))})^{-1}(I(a) \subset \Delta^1_{(u,sup(I(a))]} \times \{e\}) =$$

$$= P_{Rnd}^{\otimes\infty}(\{\underline{r} \,|\, E_{p,u}(r_1) = (x, e) \text{ s.t. } x \in I(a) \text{ and } E_{p',x}(r_2) \in \{p' \times *\} \amalg (X_1^{nd} \times T_{>sup(I(a))})\}) =$$

$$= P_{Rnd}^{\otimes 2}(\{(r_1, r_2) \,|\, E_{p,u}(r_1) = (x, e) \text{ s.t. } x \in I(a) \text{ and } E_{p',x}(r_2) \in \{p' \times *\} \amalg (X_1^{nd} \times T_{>sup(I(a))})\})$$

Since intervals $I(a)$ are disjoint so are the subsets $(\bar{M}_{p,u}^{sup(I(a))})^{-1}(I(a))$ and therefore

$$\sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a)) =$$

$$= P_{Rnd}^{\otimes 2}(\{(r_1, r_2) \,|\, E_{p,u}(r_1) = (x, e) \text{ s.t. } x \leq v \text{ and } E_{p',x}(r_2) \in \{p' \times *\} \amalg (X_1^{nd} \times T_{>a(x)})\})$$

where $a(x) = sup(I(a))$ for a such that $x \in I(a)$. On the other hand

$$[\mathbf{2009.0616.eq2}]\theta_{e,u}^v([u, v]) = P_{Rnd}(\{r_1 \,|\, E_{p,u}(r_1) = (e, x) \text{ s.t. } x \leq v\}) \tag{29}$$

Let A be such that for all $a \in A$ one has $sup(I(a)) - inf(I(a)) < \epsilon$. Then

$$|\theta_{e,u}^v([u, v]) - \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a))| \leq P_{Rnd}^{\otimes 2}(\{(r_1, r_2) \,|\, E_{p,u}(r_1) = (x, e) \text{ and } E_{p',x}(r_2) - x < \epsilon\})$$

We have

$$\cap_{n=1}^{\infty} P_{Rnd}^{\otimes 2}(\{(r_1, r_2) \,|\, E_{p,u}(r_1) = (x, e) \text{ and } E_{p',x}(r_2) - x < 1/n\}) = \emptyset$$

and from σ-additivity of $P_{Rnd}^{\otimes 2}$ we conclude that

$$lim_{A \in S_{T1}([u,v])} |\theta_{e,u}^v([u, v]) - \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a))| = 0.$$

Corollary 3.7.17 *[***2009.07.13.1***] The pre-processes defined by generating maps E, E' coincide if and only if the generating kernels θ, θ' defined by E and E' coincide.*

Lemma 3.7.18 *[***2009.05.27.6***] A for a renewal pre-process the functions $\upsilon_p(u, -)$ are right continuous and $\upsilon_p(u, u) = 1$.*

Proof: By construction we have

$$\upsilon_p(u, v) = P_{Rnd}^{\otimes \infty}(\Omega_{p,u,\infty}^v)$$

and one observes easily that

$$[\mathbf{2009.05.27.eq2}] \cup_{\delta \downarrow 0} \Omega_{p,u,n}^{v+\delta} = \Omega_{p,u,n}^v \tag{30}$$

for all $n \geq 1$ and all p. Therefore by σ-additivity of $P_{Rnd}^{\otimes \infty}$ we conclude that

$$lim_{\epsilon \downarrow 0} \upsilon_p(u, v + \epsilon) = \upsilon_p(u, v)$$

i.e. $\upsilon_p(u, -)$ is right continuous. The fact that $\upsilon_p(u, u) = 1$ follows from the obvious equation $\Omega_{p,u,n}^u = Rnd^{\infty}$.

Recall that for $e \in X_1^{nd}$ and $u \leq v$ in T we let $\alpha_{e,u}^v$ denote the measure

$$\alpha_{e,u}^v = ((x_1, e_1)_*(\mu_{\partial_0(e),u}^v))^{[u,v] \times \{e\}}$$

where

$$(x_1, e_1) : X[u, v] \to ([u, v] \times X_1^{nd}) \amalg *$$

is the "first event" map. For $p \in X_0$ set:

$$\alpha_{p,u}^v = \sum_{e \in X_1^{nd} \,|\, \partial_0(e) = p} \alpha_{e,u}^v$$

Proposition 3.7.19 *For any renewal pre-process P, any $e \in X_1^{nd}$, any $u \le w \le v$ in T, any measurable I in $[u,w]$ and any measurable $U \subset X[w,v]$ one has*

$$[\mathbf{2009.05.28.eq1}]\mu^v_{p0,u}((x_1,e_1)^{-1}(U\times\{e\})\cap(res^{u,v}_{w,v})^{-1}(U)) = \int_{x\in I}\mu^v_{p_1,x}((res^{x,v}_{w,v})^{-1}(U))d\theta_{e,u} \quad (31)$$

where $p_0 = \partial_0(e)$ and $p_1 = \partial_1(e)$.

Proof: We have

$$\mu^v_{p_0,u}((x_1,e_1)^{-1}(U\times\{e\})\cap(res^{u,v}_{w,v})^{-1}(U)) = P^{\otimes\infty}_{Rnd}((\bar{M}^v_{p_0,u})^{-1}((x_1,e_1)^{-1}(U\times\{e\})\cap(res^{u,v}_{w,v})^{-1}(U))$$

and

$$(\bar{M}^v_{p_0,u})^{-1}((x_1,e_1)^{-1}(U\times\{e\})\cap(res^{u,v}_{w,v})^{-1}(U) =$$
$$= \{\underline{r}\in Rnd^\infty \,|\, E(p_0,u,r_1)=(x,e)\in I\times\{e\} \text{ and } (r_2,\ldots,)\in\Omega^v_{p_1,x,\infty} \text{ and}$$
$$M^v_{p_1,x}(r_2,\ldots,)\in(res^{x,v}_{w,v})^{-1}(U)\})$$

which implies (31) by Fubini's theorem.

Corollary 3.7.20 *[**2009.05.28.3**] Under the assumptions of the proposition one has*

$$[\mathbf{2009.05.28.eq2}]\alpha^v_{e,u} = \theta_{e,u} * \upsilon_{p_1}(-,v) \quad (32)$$

Proof: The equation (32) is equivalent to equations

$$\int_{x\in[u,w]}\upsilon_{p_1}(x,v)d\theta_{e,u} = \alpha^v_{e,u}([u,v])$$

for all $w\in[u,v]$, which follows from the proposition for $U = X[w,v]$.

Let us consider the following conditions on pre-processes on (X,T):

C1 For any $p\in X_0$, any $u\le w\le v$ in T, any measurable U in $X[u,w]_{p,*}$ one has

$$\mu^v_{p,u}((res^{u,v}_{u,w})^{-1}(U)) \le \mu^w_{p,u}(U),$$

C2a For any $p\in X_0$ the function $h_p(u,v)=\mu^v_{p,u}(\{p\})$ is measurable in u,

C2b For any $p\in X_0$ the function $h_p(u,v)=\mu^v_{p,u}(\{p\})$ is right continuous in v,

C3a For any $p\in X_0$ the function $\upsilon_p(u,v)=\mu^v_{p,u}(X[u,v]_{p,*})$ is measurable in u,

C3b For any $p\in X_0$ the function $\upsilon_p(u,v)=\mu^v_{p,u}(X[u,v]_{p,*})$ is right continuous in v,

C4a For any $e\in X_1^{nd}$ and any $w\le v$ in T the function

$$u\mapsto\lambda^v_{e,u}([w,v])$$

on $[T_{min},w]$ is measurable,

C4b For any $e\in X_1^{nd}$ and any $u\le w$ in T the function

$$v\mapsto\lambda^v_{e,u}([u,w])$$

on $[w,T_{max}]$ is right continuous.

Proposition 3.7.21 *[**2009.06.19.2**] Any renewal process satisfies conditions (C1), (C2a), (C2b), (C4a), (C4b).*

Proof: For any $p \in X_0$ and $u \leq w \leq v$ in T one has

$$(\bar{M}^v_{p,u})^{-1}((res_{u,w})^{-1}(U)) = (M^w_{p,u})^{-1}(U) \cap \Omega^v_{p,u,\infty}$$

and therefore, for any measurable $U \subset X[u,w]$,

$$\mu^v_{p,u}((res_{u,w})^{-1}(U)) = P^{\otimes\infty}_{Rnd}((\bar{M}^v_{p,u})^{-1}((res_{u,w})^{-1}(U)) \leq P^{\otimes\infty}_{Rnd}((\bar{M}^w_{p,u})^{-1}(U)) = \mu^w_{p,u}(U).$$

This shows that a renewal process satisfies (C1). (C2a) is proved in Corollary 3.7.11. (C2b) is proved in Corollary 3.7.14. (C4a) follows from Proposition 3.7.10 since

$$\lambda^v_{e,u}((w,v]) = \mu^v_{p,u}((res_{u,w} \times res_{w,v})^{-1}(\{p\} \times (\Delta^1_{(w,v]} \times \{e\}))$$

where $p = \partial_0(e)$ and for $u < w$

$$\lambda^v_{e,u}([w,v]) = lim_{x\uparrow w}\lambda^v_{e,u}((x,v])$$

To prove (C4b) observe that

$$\lambda^{v+\epsilon}_{e,u}([u,w]) =$$

$$= P^{\otimes\infty}_{Rnd}\{\underline{r} \,|\, E_{p,u}(r_1) = (x,e)\ x \in [u,w]\ E_{p',x}(r_2) \in (X_0 \times \{*\}) \amalg (X^{nd}_1 \times T_{>v+\epsilon})\}$$

and σ-additivity of $Rnd^{\otimes\infty}$ implies immediately that

$$lim_{\epsilon\downarrow 0}\lambda^{v+\epsilon}_{e,u}([u,w]) = \lambda^v_{e,u}([u,w])$$

Lemma 3.7.22 *[**2009.05.28.5**] Let P be a pre-process satisfying (C1). The for any $p \in X_0$ and any $u \leq x_0 < x_1 < \cdots < x_n$ in T one has*

$$\sum_{i=0}^{n-1} \alpha^{x_{i+1}}_{p,u}((x_i, x_{i+1}]) \leq h_p(u,x_0) - h_p(u,x_n)$$

Proof: By obvious induction it is sufficient to show that for $u \leq w \leq v$ one has $\alpha^v_{p,u}((w,v]) \leq h_p(u,w) - h_p(u,v)$. We have

$$\alpha^v_{p,u}((w,v]) + h_p(u,v) = \mu^v_{p,u}(\{p\} \amalg (x_1,e_1)^{-1}((w,v])) =$$

$$= \mu^v_{p,u}((res^{u,v}_{u,w})^{-1}(\{p\})) \leq \mu^w_{p,u}(\{p\}) = h_p(u,w).$$

Proposition 3.7.23 *[**2009.05.28.4**] Let P be a pre-process satisfying (C1). Then for any $e \in X^{nd}_1$ and any $u \leq v$ the limit*

$$[\mathbf{2009.05.28.eq3}]\Theta_{e,u}(v) = lim_{A \in S_{T2}([u,v])} \sum_{a \in A} \lambda^{sup(I(a))}_{e,u}(I(a)) \tag{33}$$

exists and for $p \in X_0$

$$[\mathbf{2009.05.28.eq4}]h_p(u,v) + \sum_{e \in X^{nd}_1(p)} \Theta_{e,u}(v) \leq 1 \tag{34}$$

In particular, $\Theta_{e,u}(-)$ is a non-negative monotone increasing function bounded by 1.

If P in addition satisfies (C2b) then $\Theta_{e,u}(-)$ is right continuous.

Proof: To show that the limit (33) exists in $\mathbf{R} \cup \{\infty\}$ it is sufficient to show that the expression under the limit is a monotone function of A. Let us show that it is monotone increasing, i.e. for $A' \subset A$ we have

$$\sum_{a\in A} \lambda_{e,u}^{sup(I(A,a))}(I(A,a)) \geq \sum_{a'\in A'} \lambda_{e,u}^{sup(I(A',a'))}(I(A',a'))$$

One observes easily that in order to prove this assertion it is sufficient to show that for an interval I in $[u,v]$ which is closed from the below and $B \in S_{T2}(I)$ we have

$$\sum_{b\in B} \lambda_{e,u}^{sup(I(b))}(I(b)) \geq \lambda_{e,u}^{sup(I)}(I)$$

From condition (C1) we have

$$\lambda_{e,u}^{sup(I(b))}(I(b)) \geq \mu_{p_0,u}^{sup(I)}((res_{u,sup(I(b))}^{u,sup(I)})^{-1}(I(b))) \geq \lambda_{e,u}^{sup(I)}(I(b))$$

and therefore

$$\sum_{b\in B} \lambda_{e,u}^{sup(I(b))}(I(b)) \geq \sum_{b\in B} \lambda_{e,u}^{sup(I)}(I(b)) = \lambda_{e,u}^{sup(I)}(I)$$

This proves that $\Theta_{e,u}$ is well defined. Let us show that it satisfies (34) and in particular that it is bounded.

Lemma 3.7.24 *[***2009.05.29.1***] For any $w \in [u,v]$ and $A \in S_{T2}([w,v])$ one has*

$$h_p(u,v) + \sum_{a\in A} \alpha_{p,u}^{sup(I(a))}(I(a)) \leq h_p(u,w) + \alpha_{p,u}^{w}(\{w\})$$

Proof: Since the infinite sum on the left is the limit over the sums over finite subsets of A it is sufficient to show that for a sufficiently large finite subset $A' \subset A$ one has

$$\sum_{a\in A'} \alpha_{p,u}^{sup(I(a))}(I(a)) \leq h_p(u,w) - h_p(u,v) + \alpha_{p,u}^{w}(\{w\})$$

Without loss of generality we may assume that $A' = \{x_0, \ldots, x_n\}$ where $x_0 = w$ and $x_n = v$. Then

$$\sum_{a\in A'} \alpha_{p,u}^{sup(I(a))}(I(a)) = \sum_{i=0}^{n-2} \alpha_{p,u}^{x_{i+1}}([x_i, x_{i+1})) + \alpha_{p,u}^{x_n}([x_{n-1}, x_n]) =$$

$$= \sum_{i=0}^{n-1} \alpha_{p,u}^{x_{i+1}}((x_i, x_{i+1}]) + \sum_{i=0}^{n-1} \alpha_{p,u}^{x_{i+1}}(\{x_i\}) - \sum_{i=1}^{n-1} \alpha_{p,u}^{x_i}(\{x_i\})$$

By condition (C1) we have $\alpha_{p,u}^{x_i}(\{x_i\}) \geq \alpha_{p,u}^{x_{i+1}}(\{x_i\})$. Therefore

$$\sum_{a\in A'} \alpha_{p,u}^{sup(I(a))}(I(a)) \leq \sum_{i=0}^{n-1} \alpha_{p,u}^{x_{i+1}}((x_i, x_{i+1}]) + \alpha_{p,u}^{x_0}(\{x_0\}) \leq h_p(u,w) - h_p(u,v) + \alpha_{p,u}^{w}(\{w\})$$

where the last inequality holds by Lemma 3.7.22.

Since $\alpha_{e,u}^{w}(\{w\}) = \lambda_{e,u}^{w}(\{w\})$ and

$$\alpha_{e,u}^{sup(I(a))}(I(a)) \geq \lambda_{e,u}^{sup(I(a))}(I(a))$$

we conclude from Lemma 3.7.24 that

$$[\mathbf{2009.05.28.eq5}]h_p(u,v)+\sum_{a\in A}\sum_{e\in X_1^{nd}(p)}\lambda_{e,u}^{sup(I(a))}(I(a))\leq h_p(u,w)+\sum_{e\in X_1^{nd}(p)}\lambda_{e,u}^{w}(\{w\}) \tag{35}$$

which for $w=u$ implies (34).

Let us consider the question of when $\Theta_{e,u}$ is right continuous. For $v'\geq v$ one has

$$\Theta_{e,u}(v')=lim_{A\in S_{T2}([u,v))}\sum_{a\in A}\lambda_{e,u}^{sup(I(a))}(I(a))+lim_{B\in S_{T2}([v,v'])}\sum_{b\in B}\lambda_{e,u}^{sup(I(b))}(I(b))$$

Therefore,

$$\Theta_{e,u}(v+\epsilon)-\Theta_{e,u}(v)=lim_{B\in S_{T2}([v,v+\epsilon])}\sum_{b\in B}\lambda_{e,u}^{sup(I(b))}(I(b))-\lambda_{e,u}^{v}(\{v\})$$

Since all this differences are non-negative it is sufficient to show that their sum over all $e\in X_1^{nd}(p)$ goes to zero with ϵ. By (35)

$$lim_{B\in S_{T2}([v,v+\epsilon])}\sum_{b\in B}\sum_{e\in X_1^{nd}(p)}\lambda_{e,u}^{sup(I(b))}(I(b))-\sum_{e\in X_1^{nd}(p)}\lambda_{e,u}^{v}(\{v\})\leq h_p(u,v)-h_p(u,v+\epsilon)$$

and we conclude that $\Theta_{e,u}(-)$ is right continuous if $h_{\partial_0(e)}(u,-)$ is.

Definition 3.7.25 *[**condn**] Let X be a countable multi-graph and $T=[T_{min},T_{max}]$ a time window. A pre-process on (X,T) is said to satisfy condition (N) if it satisfies condition (C1) and for any $p\in X_0$ and $u\leq v$ in T one has*

$$[\mathbf{2009.05.28.eq6}]h_p(u,v)+\sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p}\Theta_{e,u}(v)=1 \tag{36}$$

Consider a collection of monotone increasing, right continuous functions $\Theta_{e,u}:T_{\geq u}\to\mathbf{R}_{\geq 0}$ such that $\Theta_{e,u}(u)=0$, given for all $e\in X_1^{nd}$ and $u\in T$. Such a collection corresponds to a collection of measures $\theta_{e,u}$ on T determined by the condition

$$\theta_{e,u}([T_{min},v])=\begin{cases}\Theta_{e,u}(v) & \text{if } u\leq v\\ 0 & \text{if } u>v\end{cases}$$

Assume in addition that the condition

$$1-\sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p}\Theta_{e,u}(v)\geq 0$$

holds. Then there is a unique probability measure $\theta_{p,u}$ on $(X_0\times\{*\})\amalg(X_1^{nd}\times T)$ such that

$$(\theta_{p,u})^{|\{e\}\times T}=\begin{cases}\theta_{e,u} & \text{if } \partial_0(e)=p\\ 0 & \text{otherwise}\end{cases}$$

and

$$(\theta_{p,u})^{|\{p'\}\times\{*\}}=\begin{cases}1-\sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p}\theta_{e,u}(T) & \text{for } p'=p\\ 0 & \text{for } p'\neq p\end{cases}$$

Lemma 3.7.26 *[***2009.06.01.2***] Let P be a pre-process satisfying (C1) then for $e \in X_1^{nd}$, $u \leq w \leq v$ in T and U measurable in $[u, v]$ one has*

$$\lambda_{e,u}^{w}(U) \geq \lambda_{e,u}^{v}(U)$$

Proof: We have

$$\lambda_{e,u}^{w}(U) = \mu_{\partial_0(e),u}^{w}(U \subset \Delta_{(u,w]}^{1} \times \{e\}) \geq \mu_{\partial_0(e),u}^{v}((res_{u,w})^{-1}(U \subset \Delta_{(u,w]}^{1} \times \{e\})) \geq$$

$$\geq \mu_{\partial_0(e),u}^{v}(U \subset \Delta_{(u,v]}^{1} \times \{e\})$$

Lemma 3.7.27 *[***2009.06.01.1***] Let P be a pre-process which satisfies condition (C1) and (C4b). Let $e \in X_1^{nd}$, $u \leq w \leq v$ in T and let Q be a dense subset of $[w, v]$ which contains w and v. Then*

$$\Theta_{e,u}(v) - \Theta_{e,u}(w) = lim_{A \in Fin_w(Q)} \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a)) - \lambda_{e,u}^{w}(\{w\})$$

where $Fin_w(Q)$ is the set of finite subsets of Q which contain w, considered as a subset of $S_{T2}([w, v])$.

Proof: It follows easily from the definition of Θ that

$$\Theta_{e,u}(v) - \Theta_{e,u}(w) = lim_{A \in S_{T2}([w,v])} \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a)) - \lambda_{e,u}^{w}(\{w\})$$

It remains to show that

$$lim_{A \in Fin_w(Q)} \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a)) = lim_{A \in S_{T2}([w,v])} \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a))$$

Since the function $F(A) = \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a))$ has the property that $F(A) \geq F(A')$ for $A \subset A'$ the limits on the right and left hand sides are supremums of the sets of values of F on $S_{T2}([w, v])$ and $Fin(Q)$ respectively. In addition since the sum of an infinite set of non-negative numbers is the supremum of the sums over the finite subsets of this set we have

$$lim_{A \in S_{T2}([w,v])} \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a)) = lim_{A \in Fin_w([w,v])} \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a))$$

where $Fin_w([w, v])$ is the set of finite subsets of $[w, v]$ which contain w. It remains to shows that under the assumption of the lemma one has:

$$[\mathbf{2009.06.01.eq1}] lim_{A \in Fin_w(Q)} \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a)) = lim_{A \in Fin_w([w,v])} \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a)) \quad (37)$$

Let $A = \{x_0, \ldots, x_n\}$ be a finite subset of $[w, v]$ such that $x_0 = w$ and $x_n = v$ and let $\epsilon > 0$. We will show that there exists a finite subset $A' = \{x'_0, \ldots, x'_n\}$ such that $x'_0 = w$, $x'_n = v$, $x_i \in Q$ and $F(A') \geq F(A) - \epsilon$, which together with the previous comments implies (37).

Since P satisfies (C4b), the functions $\lambda_{e,u}^{y}([x_i, x_{i+1}]$ and $\lambda_{e,u}^{y}([x_{i+1}, x_{i+1}]$ are right continuous on $y \in [x_{i+1}, T_{max}]$ and therefore $\lambda_{e,u}^{y}([x_i, x_{i+1})$ is right continuous. Therefore, for any $\delta > 0$ there exist $x'_1, \ldots, x'_{n-1}$ such that $x'_i \geq x_i$, $x'_i \in Q$ and for $i = 0, \ldots, n-2$

$$|\lambda_{e,u}^{x'_{i+1}}([x_i, x_{i+1})) - \lambda_{e,u}^{x_{i+1}}([x_i, x_{i+1}))| < \delta$$

Set $x'_0 = w$, $X'_n = v$ and $A' = \{x'_0, x'_1, \ldots, x'_n\}$. Then one has

$$\lambda_{e,u}^{x'_{i+1}}([x'_i, x'_{i+1}]) = \lambda_{e,u}^{x'_{i+1}}([x_i, x_{i+1}]) - \lambda_{e,u}^{x'_{i+1}}([x_i, x'_i]) + \lambda_{e,u}^{x'_{i+1}}([x_{i+1}, x'_{i+1})$$

for $i = 0, \ldots, n-2$ and

$$\lambda_{e,u}^{x'_n}([x'_{n-1}, x'_n]) = \lambda_{e,u}^{x'_n}([x_{n-1}, x_n]) - \lambda_{e,u}^{x'_n}([x_{n-1}, x'_{n-1}))$$

(since $x_n = x'_n$). Therefore,

$$F(A') = \sum_{i=0}^{n-2} \lambda_{e,u}^{x'_{i+1}}([x_i, x_{i+1}]) + \lambda_{e,u}^{x'_n}([x_{n-1}, x_n]) + \sum_{i=0}^{n-2} \Big((\lambda_{e,u}^{x'_{i+1}}([x_{i+1}, x'_{i+1}) - \lambda_{e,u}^{x'_{i+2}}([x_{i+1}, x'_{i+1})) \Big) \geq$$

$$\geq \sum_{i=0}^{n-1} \lambda_{e,u}^{x'_{i+1}}([x_i, x_{i+1}]) \geq F(A) - n \cdot \delta$$

where the first inequality holds by Lemma 3.7.26 and the second one by our choice of x'_i's.

Lemma 3.7.28 *[***2009.05.30.2***] Let P be a pre-process on (X, T) which satisfies (C1), (C2b) and (C4a), (C4b). Then the functions $\Theta_{e,u}(v)$ are measurable as functions of $u \in T_{\leq v}$.*

Proof: It is sufficient to show that for any $y \in \mathbf{R}_{\geq 0}$ the subset $\{u \in T_{\leq v} \,|\, \Theta_{e,u}(v) > y\}$ is measurable. Let us fix v. For $w \in T_{\leq v}$ set

$$F(u, w) = \begin{cases} \Theta_{e,u}(v) - \Theta_{e,u}(w) & \text{if } w \geq v \\ 0 & \text{otherwise} \end{cases}$$

Since $\Theta_{e,u}(u) = 0$ we have

$$\{u \,|\, \Theta_{e,u}(v) > y\} = \{u \,|\, F(u, u) > y\}$$

Since $\Theta_{e,u}(w)$ is right continuous in w, $F(-,-)$ is right continuous in the second argument and for any u such that $F(u, u) > y$ there exists ϵ such that for all $w \in [u, u + \epsilon]$ we have $F(u, w) > y$. In particular, if Q be a dense countable subset in T then

$$\{u \,|\, F(u, u) > y\} = \cup_{q \in Q} \{u \,|\, F(u, q) > y\}.$$

It remains to show that the function $F(u, w)$ is measurable in u for $u < w$. Let Q be a countable, dense subset of $[w, v]$ which contains w and v. Then by Lemma 3.7.27 we have

$$F(u, w) = lim_{A \in Fin_w(Q)} \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a)) - \lambda_{e,u}^{w}(\{w\})$$

For each A the function $u \mapsto \sum_{a \in A} \lambda_{e,u}^{sup(I(a))}(I(a))$ is measurable as the sum of finitely many measurable functions and we conclude that $F(u, w)$ is measurable as the supremum of a countable family of measurable functions.

3.8 Markov pre-processes on multi-graphs

Lemma 3.8.1 *[**2009.06.14.1**] Let X be a multi-graph and $T = [T_{min}, T_{max}]$ a time window. A pre-process $P = \{\mu^*_*\}$ on (X, T) satisfies condition (M) if and only if for any $u \leq w \leq v$ in T and $e \in X^{nd}_n$, $e' \in X^{nd}_m$ such that $\partial^n_1(e) = \partial^m_0(e')$ and any admissible intervals $I_1, \ldots, I_n$ in $(u, w]$, $J_1, \ldots, J_m$ in $(w, v]$ one has*

$$\mu^v_{p,u}(U^w_{e'',u}(I_1, \ldots, I_n, J_1, \ldots, J_m)) = \mu^w_{p,u}(U^w_{e,u}(I_1, \ldots, I_n))\mu^v_{p',w}(U^v_{e',w}(J_1, \ldots, J_m))$$

where $p = \partial^n_0(e)$, $p' = \partial^m_0(e')$ and $e'' \in X^{nd}_{n+m}$ is such that $\partial^m_0(e'') = e$ and $\partial^n_1(e'') = e'$.

Proof: For $u \leq w \leq v$ in T, $p, p' \in X_0$, $e \in X^{nd}_n$, $e' \in X^{nd}_m$ and admissible intervals $I_1, \ldots, I_n$ in $(u, w]$, $J_1, \ldots, J_m$ in $(w, v]$ we have

$$(res_{u,w} \times res_{w,v})^{-1}(U^w_{e,u}(I_1, \ldots, I_n) \times U^v_{e',w}(J_1, \ldots, J_m)) =$$

$$= \begin{cases} U^v_{e'',u}(I_1, \ldots, I_n, J_1, \ldots, J_m) & \text{if } \partial^n_1(e) = \partial^m_0(e') \\ \emptyset & \text{if } \partial^n_1(e) \neq \partial^m_0(e') \end{cases}$$

where $e'' \in X^{nd}_{n+m}$ is such that $\partial^m_0(e'') = e$ and $\partial^n_1(e'') = e'$. Together with Lemma 3.6.1 it implies the claim of the lemma.

Note that for any $U^v_{e,u}(I_1, \ldots, I_n)$ as above, there exist points w_i, $i = 1, \ldots, n-1$ such that $sup(I_i) < w_i < inf(I_{i+1})$ and for any choice of points satisfying these conditions we have

$$[\mathbf{cut1}]U^v_{e,u}(I_1, \ldots, I_n) = (res^{u,v}_{u,w_1} \times \cdots \times res^{u,v}_{w_{n-1},v})^{-1}(U^{w_1}_{e_1,u}(I_1) \times \cdots \times U^v_{e_n,w_{n-1}}(I_n)) \qquad (38)$$

where $e = (e_1, \ldots, e_n)$. This observation immediately implies that it is sufficient to verify conditions of Lemma 3.8.1 for $n \leq 1$ or $m \leq 1$ and that any pre-process on (X, T) satisfying condition (M) is determined by the corresponding functions $h_p(-,-)$ and measures $\lambda^v_{e,u}$.

From property (M) we get for all $p \in X_0$ and all $u \leq w \leq v$ the equations

$$\upsilon_p(u, v) = \sum_{p'} \phi^{p'}_p(u, w)\upsilon_{p'}(w, v)$$

and

$$[\mathbf{ob1}]h_p(u, v) = h_p(u, w)h_p(w, v) \qquad (39)$$

Since $h_p(u, v) \leq \upsilon_p(u, v) \leq 1$ we conclude that $h_p(u, v)$ is monotone increasing in u and monotone decreasing in v and $\upsilon_p(u, v)$ are monotone decreasing in v. We will see from examples below (**??**) that $\upsilon_p(u, v)$ need not be monotone in u.

Lemma 3.8.2 *[**ob00**] Let P be a pre-process on (X, T) satisfying condition (M). Then for any $p, p' \in X_0$ and any $u \leq w < v$ in T one has:*

1. *the function $h_p(u, w + \epsilon)\phi^{p'}_p(w + \epsilon, v)$ is monotone decreasing in ϵ and one has*

$$\lim_{\epsilon \downarrow 0} h_p(u, w + \epsilon)\phi^{p'}_p(w + \epsilon, v) = h^p(u, w)\phi^{p'}_p(w, v)$$

2. *the function $h_p(u, w + \epsilon)\upsilon_p(w + \epsilon, v)$ is monotone decreasing in ϵ and one has*

$$\lim_{\epsilon \downarrow 0} h_p(u, w + \epsilon)\upsilon_p(w + \epsilon, v) = h^p(u, w)\upsilon_p(w, v)$$

3. the function $\phi_p^{p'}(u,w+\epsilon)h_{p'}(w+\epsilon,v)$ is monotone increasing in ϵ and one has

$$\lim_{\epsilon\downarrow 0}\phi_p^{p'}(u,w+\epsilon)h_{p'}(w+\epsilon,v)=\phi_p^{p'}(u,w)h_{p'}(w,v)$$

Proof: Applying property (M) to $U=\{p\}$ and $V=X[w+\epsilon,v]_{p,p'}$ we get

$$h_p(u,w+\epsilon)\phi_p^{p'}(w+\epsilon,v)=\mu_{u,p}^v((res_{u,w+\epsilon}\times res_{w+\epsilon,v})^{-1}(\{p\}\times X[w+\epsilon,v]_{p,p'})).$$

Since for $\epsilon'\geq\epsilon$ one has

$$(res_{u,w+\epsilon'}\times res_{w+\epsilon',v})^{-1}(\{p\}\times X[w+\epsilon',v]_{p,p'})\subset(res_{u,w+\epsilon}\times res_{w+\epsilon,v})^{-1}(\{p\}\times X[w+\epsilon,v]_{p,p'})$$

and

$$\cup_{\epsilon\downarrow 0}(res_{u,w+\epsilon}\times res_{w+\epsilon,v})^{-1}(\{p\}\times X[w+\epsilon,v]_{p,p'})=(res_{u,w}\times res_{w,v})^{-1}(\{p\}\times X[w,v]_{p,p'})$$

we conclude that that the first assertion holds. The second assertion follows by the same argument applied to $U=\{p\}$ and $V=X[w+\epsilon,v]_{p,*}$ and the third to $U=X[u,w+\epsilon]_{p,p'}$ and $V=\{p'\}$.

Recall that P is called non-degenerate $\upsilon_p(u,u)=1$ for all $p\in X_0$ and $u\in T$. Since $X[u,u]=X_0$, a process on $(X[*,*],T)$ is non-degenerate if and only if $h_p(u,u)=1$ for all $p\in X_0$ and $u\in T$. A pre-process satisfying condition (M) is non-degenerate if and only if for any $p\in X_0$ and $u\in T$, $h_p(u,u)\neq 0$ or, equivalently, $\upsilon_p(u,u)\neq 0$.

Proposition 3.8.3 *[**th1**] Let P be a non-degenerate pre-process on $(X[*,*],T)$ satisfying condition (M). Then the following conditions are equivalent:*

1. *for any $p\in X_0$ the function $\upsilon_p(u,v)$ is right continuous on T as a function of u and for all $u<T_{max}$ there exists $v>u$ such that $\upsilon_p(u,v)\neq 0$,*
2. *for any $p\in X_0$ the function $h_p(u,v)$ is right continuous on T as a function of u and for all $u<T_{max}$ there exists $v>u$ such that $\upsilon_p(u,v)\neq 0$,*
3. *for any $p,p'\in X_0$ the function $\phi_p^{p'}(u,v)$ is right continuous on T as a function of u and for all $u<T_{max}$ there exists $v>u$ such that $\upsilon_p(u,v)\neq 0$,*
4. *for any $p\in X_0$ the function $\upsilon_p(u,v)$ is right continuous on T as a function of v,*
5. *for any $p\in X_0$ the function $h_p(u,v)$ is right continuous on T as a function of v,*
6. *for any $p,p'\in X_0$ the function $\phi_p^{p'}(u,v)$ is right continuous on T as a function of v,*

Proof: Observe first that if for $u<T_{max}$ there exits $v>u$ such that $\upsilon_p(u,v)\neq 0$ then, applying Lemma 3.8.2(2) for $w=u$ we get

$$[\textbf{feqp}] lim_{\epsilon\downarrow 0}h_p(u,u+\epsilon)\upsilon_p(u+\epsilon,v)=\upsilon_p(u,v)\neq 0 \quad (40)$$

which implies that there exists $v>u$ such that $h_p(u,v)>0$. For any such v we also have $\upsilon_p(u,v)>0$ and $\phi_p^p(u,v)>0$.

(1) $\Rightarrow$ (2) Let $u<T_{max}$ and let v be such that $h_p(u,v)>0$ and $\upsilon_p(u,v)>0$. Since $\upsilon_p(u,v)$ is right continuous in u, the equation (40) implies that

$$(lim_{\epsilon\downarrow 0}h_p(u,u+\epsilon))\upsilon_p(u,v)=\upsilon_p(u,v)$$

and since $\upsilon_p(u,v) > 0$ we conclude that $\lim_{\epsilon\downarrow 0} h_p(u, u+\epsilon) = 1$ for all $u \in T$, $u < T_{max}$. Therefore by (39)

$$lim_{\epsilon\downarrow 0} h_p(u+\epsilon, v) = lim_{\epsilon\downarrow 0} h_p(u, u+\epsilon)^{-1} h_p(u,v) = h_p(u,v)$$

i.e. (2) holds.

(3) $\Rightarrow$ (2) Same as (1) $\Rightarrow$ (2) with the use of Lemma 3.8.2(1) for $u = w$ and $p = p'$.

(2) $\Rightarrow$ (5) Follows from (39).

(5) $\Rightarrow$ (3) Since $h_p(u,u) = 1$, condition (5) implies that for any $u < T_{max}$ there exists $v > u$ such that $h_p(u,v) > 0$ and therefore $\upsilon_p(u,v) > 0$ since $\upsilon_p(u,v) \geq h_p(u,v)$.

By Lemma 3.8.2(3) for $w = u$ we get

$$lim_{\epsilon\downarrow 0} h_p(u, u+\epsilon)\phi_p^{p'}(u+\epsilon, v) = \phi_p^{p'}(u,v)$$

for all $v > u$ which together with condition (5) implies that

$$lim_{\epsilon\downarrow 0} \phi_p^{p'}(u+\epsilon, v) = \phi_p^{p'}(u,v)$$

i.e. that $\phi_p^{p'}(u,v)$ is right continuous in u.

(5) $\Rightarrow$ (1) Same as (5) $\Rightarrow$ (3) using Lemma 3.8.2(2) instead of Lemma 3.8.2(1).

(2) $\Rightarrow$ (6) Let $u < w < T_{max}$ and let v be such that $h_p(w,v) \neq 0$. By Lemma 3.8.2(2) for u, w, v together with the condition that $h_{p'}(-,-)$ is right continuous in the first variable we get

$$(lim_{\epsilon\downarrow 0} \phi_p^{p'}(u, w+\epsilon))h_{p'}(w,v) = lim_{\epsilon\downarrow 0} \phi_p^{p'}(u, w+\epsilon)h_{p'}(w+\epsilon, v) = \phi_p^{p'}(u,w)h_{p'}(w,v)$$

which implies that

$$[\mathbf{seqp}] \lim_{\epsilon\downarrow 0} \phi_p^{p'}(u, w+\epsilon) = \phi_p^{p'}(u,w) \tag{41}$$

or equivalently that $\phi_p^{p'}(u,v)$ is right continuous in v.

(6) $\Rightarrow$ (4) Follows from the fact that $\upsilon_p(u,v) = \sum_{p'} \phi_p^{p'}(u,v)$ by Lemma 3.5.3. ???? wrong argument?

(4) $\Rightarrow$ (2) Since functions $\upsilon_{p'}(u,v)$ are right continuous in v and $\upsilon_{p'}(u,u) = 1$ there exists $v > u$ such that $\upsilon_{p'}(u,v) > 0$ and $h_{p'}(u,v) > 0$. Taking in Lemma 3.8.2(3) $p \neq p'$ and $w = u$ we get

$$lim_{\epsilon\downarrow 0} \phi_p^{p'}(u, u+\epsilon)h_{p'}(u+\epsilon, v) = 0$$

and since $h_{p'}(u+\epsilon, v) \geq h_{p'}(u,v) > 0$ for all ϵ we conclude that

$$[\mathbf{eq020}] lim_{\epsilon\downarrow 0} \phi_p^{p'}(u, u+\epsilon) = 0 \tag{42}$$

i.e. $\phi_p^{p'}(u,v)$ are right continuous in v at $v = u$ for $p' \neq p$. Applying Lemma 3.5.3 we conclude that the same holds for $\sum_{p'\neq p} \phi_p^{p'}(u,v)$ and since it holds for

$$\upsilon_p(u,v) = \phi_p^p(u,v) + \sum_{p'\neq p} \phi_p^{p'}(u,v)$$

we conclude that it holds for $\phi_p^p(u,v)$ i.e. that

$$[\mathbf{teqp}] lim_{\epsilon\downarrow 0} \phi_p^p(u, u+\epsilon) = 1 \tag{43}$$

Applying Lemma 3.8.2(3) for $p' = p$ and $w = u$ together with (43) we conclude that for all $v > u$

$$lim_{\epsilon\downarrow 0} h_p(u+\epsilon, v) = lim_{\epsilon\downarrow 0} \phi_p^p(u, u+\epsilon)h_p(u+\epsilon, v) = h_p(u,v)$$

i.e. that $h_p(u,v)$ is right continuous in u.

Proposition is proved.

Definition 3.8.4 *[**rcont**] A pre-process P on $(X[*,*],T)$ is called right continuous if for any $p \in X_0$ the function $\upsilon_p(u,v)$ is right continuous in v.*

Note that any process on $(X[*,*],T)$ is automatically right continuous.

Example 3.8.5 [**nonrc**] Consider a pre-process P on $X[*,*]$ such that the measures $\mu^v_{p,u}(P)$ are concentrated on $\{p\}$. Such a pre-process is simply a collection of functions $\upsilon_p(u,v)$ on T. It satisfies property (M) if and only if for all $u \le w \le v$ and all $p \in X_0$ one has $\upsilon_p(u,v) = \upsilon_p(u,w)\upsilon_p(w,v)$. From this it is easy to construct an example of a non-degenerate pre-process satisfying (M) such that the functions $\upsilon_p(u,v)$ are right continuous in u but not in v and an example of a degenerate pre-process satisfying (M) for which functions $\upsilon_p(u,v)$ are right continuous in v but not in u.

For a pre-process P on $(X[*,*],T)$ and $p \in X_0$ define $A(P;h_p)$ (abbreviated to $A(h_p)$) as the subset of T which consists of points x such that for all $y \in T$, $y < x$ one has $h_p(y,x) = 0$.

Proposition 3.8.6 *[**ob2**] Let P be a non-degenerate right continuous pre-process satisfying condition (M) and $p \in X_0$. Then for any $x \in T$ one has $h_p(x,T_{max}) > 0$ or there exists a unique element $a(x,h_p)$ in $A(P;h_p)$ such that $a(x,h_p) > x$ and for all $y \in [x, a(x,h_p))$ one has $h_p(x,y) > 0$.*

Proof: We may assume that $h_p(x,T_{max}) = 0$. By Proposition 3.8.3 the function $h_p(x,-)$ is right continuous and since it is non-negative and monotone decreasing the set of zeros of $h_p(x,-)$ is of the form $[a,T_{max}]$ for some $a = a(x,h_p)$ in $(x,T_{max}]$. It remains to show that $a \in A(h_p)$. Let $y < a$. If $y \le x$ then $h_p(y,a) = h_p(y,x)h_p(x,a) = 0$. If $y > x$ then

$$h_p(x,y)h_p(y,a) = h_p(x,a) = 0$$

and since $h_p(x,y) > 0$ we conclude that $h_p(y,a) = 0$ and $a \in A(h_p)$.

Corollary 3.8.7 *[**ob2a**] Let P be as above and $p \in X_0$. Then one has:*

1. *$h_p(u,v) = 0$ if and only if $(u,v]$ contains an element of $A(h_p)$,*
2. *$A(h_p)$ is a T1 subset, in particular it is countable, has a maximal element a_{max} and*

$$T = (\amalg_{x\in A(h_p),\ x\ne a_{max}}[x, a(x,h_p)) \amalg [a_{max}, T_{max}]$$

For any $u < v$ consider the map

$$(x_1,e_1) : X[u,v]\backslash X_0 \to [u,v] \times X_1^{nd}$$

which sends $(x_1,\ldots,x_n) \in \Delta^{(e_1,\ldots,e_n)}_{(u,v]}$ to x_1.

For a pre-process P on (X,T), $e \in X_1^{ng}$ and $u \le v$ denote by $\alpha^v_{e,u}(P)$ the measure

$$\alpha^v_{e,u}(P) = (x_1,e_1)_*((\mu^v_{\partial_0(e),u}(P))^{|X[u,v]\backslash X_0})^{|(u,v]\times\{e\}}$$

For a measurable subset $I \subset [u,v]$ we have

$$(x_1,e_1)^{-1}(I) = (res_{u,sup(I)} \times res_{sup(I),v})^{-1}((I \subset \Delta^e_{(u,sup(I)]}) \times X[sup(I),v]_{\partial_1(e),*}) \amalg D^v_{e,u}(I)$$

where $D^v_{e,u}(I) \subset X[u,v]_{\partial_0(e),*}$ is of the form

$$D^v_{e,u}(I) = \amalg_{n\ge 2} \amalg_{(e_2,\ldots,e_n)\in X^{nd}_{n-1},\ \partial_0(e_2)=\partial_1(e)} \{\underline{x} \in \Delta^{(e,e_2,\ldots,e_n)}_{(u,v]} \,|\, x_1 \in I \,and\, x_2 \in \overline{I}\}$$

and therefore

$$[\mathbf{2009.06.15.18}]\alpha^v_{e,u}(P,I) = \lambda^{sup(I)}_{e,u}(P,I)\upsilon_{\partial_1(e)}(sup(I),v) + \mu^v_{\partial_0(e),u}(D^v_{e,u}(I)) \quad (44)$$

Lemma 3.8.8 *[2009.05.20.2] For any $u \leq v$, $e \in X_1^{nd}$ one has:*

$$[\mathbf{2009.05.20.3}] lim_{A\in S_{T1}([u,v])} \sum_{a\in A} \mu^v_{\partial_0(e),u}(D^v_{e,u}(I(A,a))) = 0 \tag{45}$$

Proof: For any $\delta > 0$ the class of T1 subsets A of $[u,v]$ such that such that for any $a \in A$ one has $sup(I(A,a)) - inf(I(A,a)) < \delta$ is co-final among all subsets. Let

$$sk_{>1,\delta}X[u,v] = \amalg_{n\geq 2} \amalg_{e\in X_n^{nd}} \{\underline{x} \in \Delta^n_{(u,v]} \,|\, x_2 - x_1 < \delta\}$$

Then for A as above

$$\cup_{a\in A} D^v_{e,u}(I(A,a)) \subset sk_{>1,\delta}X[u,v]$$

Since subsets $D^v_{e,u}(I(A,a))$ are disjoint for different a we conclude that

$$\sum_{a\in A} \mu^v_{p_0,u}(D^v_{e,u}(I(A,a))) \leq \mu^v_{p_0,u}(sk_{>1,\delta}X[u,v])$$

On the other hand

$$\cap_{\delta\downarrow 0} sk_{>1,\delta}X[u,v] = \emptyset$$

and by σ-additivity of $\mu^v_{\partial_0(e),u}$ we conclude that (45) holds.

Lemma 3.8.9 *[2009.05.20.4] Let I be closed from the below interval of $T_{\leq u}$ such that $h_{p_1}(inf(I), sup(I)) > 0$. Then*

$$[\mathbf{2009.05.20.5}] lim_{B\in S_{T2}(I)} \sum_{b\in B} \alpha^{sup(I)}_{e,u}(I(b)) \upsilon_{p_1}(sup(I(b)), sup(I))^{-1} = lim_{B\in S_{T1}(I)} \sum_{b\in B} \lambda^{sup(I(b))}_{e,u}(I(b)) \tag{46}$$

i.e. the corresponding limits exist and are equal.

Proof: By (44) we have

$$\alpha^{sup(I)}_{e,u}(I(b)) \upsilon_{p_1}(sup(I(b)), sup(I))^{-1} =$$
$$= \lambda^{sup(I(b))}_{e,u}(I(b)) + \mu^{sup(I)}_{p_0,u}(D^{sup(I)}_{e,u}(I(b))) \upsilon_{p_1}(sup(I(b)), sup(I))^{-1}$$

By our assumption $\upsilon_{p_1}(sup(I(b)), sup(I))^{-1}$ is bounded from the above on $\overline{I}$ which together with Lemma 3.8.8 shows that

$$lim_{B\in S_{T1}(I)} \sum_{b\in B} \mu^{sup(I)}_{p_0,u}(D^{sup(I)}_{e,u}(I(b))) \upsilon_{p_1}(sup(I(b)), sup(I))^{-1} = 0$$

and therefore (3.8.9) holds.

Lemma 3.8.10 *[2009.05.20.6] Let I be an interval closed from the below such that $inf(I) = u$ and $sup(I) = v$ and $h_{p_1}(u,v) > 0$. Then*

$$[\mathbf{2009.05.20.7}] lim_{B\in S_{T1}(I)} \sum_{b\in B} \alpha^v_{e,u}(I(b)) \upsilon_{p_1}(sup(I(b)), v)^{-1} = lim_{B\in S_{T1}(I)} \sum_{b\in B} \alpha^{sup(I(b))}_{e,u}(I(b)) \tag{47}$$

i.e. the corresponding limits exist and are equal.

Proof: In the view of Lemma 3.8.9, it is sufficient to show that

$$lim_{B\in S_{T1}(I)}\sum_{b\in B}(\alpha_{e,u}^{sup(I(b))}(I(b))-\lambda_{e,u}^{sup(I(b))}(I(B,b)))=0$$

By definition,

$$\alpha_{e,u}^{sup(I(b))}(I(b))-\lambda_{e,u}^{sup(I(b))}(I(b))=\mu_{p_0,u}^{sup(I(b))}(D_{e,u}^{sup(I(b))}(I(b)))$$

and since

$$D_{e,u}^{sup(I(B,b))}(I(B,b))$$

Proposition 3.8.11 [**2009.05.16.12**] *Let P be a non-degenerate right continuous pre-process on $T=[T_{min},T_{max}]$ which satisfies (M). Then for any $u\leq w\leq v$ and any $e\in X_1^{nd}$ one has*

$$[\mathbf{2009.05.16.16}](\alpha_{e,u}^{v})^{|[u,w]}*\upsilon_{p_1}(-,w)=\alpha_{e,u}^{w}*\upsilon_{p_1}(-,v)_{|[u,w]} \tag{48}$$

where $p_1=\partial_1(e)$.

Proof: Let $p_0=\partial_0(e)$. Since the measures on both sides of (48) are bounded, their equality is equivalent to the condition that for any $x\in[u,w]$ one has

$$\int_{y\in[u,x]}\upsilon_{p_1}(y,w)d\alpha_{e,u}^{v}=\int_{y\in[u,x]}\upsilon_{p_1}(y,v)d\alpha_{e,u}^{w}$$

In view of Proposition 3.5.15 it is sufficient show that

$$[\mathbf{2009.05.15.17}]lim_{A\in S_{T2}([u,x])}|\sum_{a\in A}\alpha_{e,u}^{w}(I(a))\upsilon_{p_1}(sup(I(a)),v)-\sum_{a\in A}\alpha_{e,u}^{v}(I(a))\upsilon_{p_1}(sup(I(a)),w)|=0 \tag{49}$$

For an interval $I\subset[u,y]$ let $U_{e,u}^{y,p'}(I)$ be the subset in $X[u,y]_{p_0,p'}$ given by

$$U_{e,u}^{y,p'}(I)=(x_1,e_1)^{-1}(I)\cap res^{-1}_{sup(I),sup(I)}(\{p'\})$$

If $sup(I)\leq w$ then by condition (M) we have

$$[\mathbf{2009.05.16.19}]\alpha_{e,u}^{w}(I)=\mu_{p_0,u}^{sup(I)}(U_{e,u}^{sup(I),p_1}(I))\upsilon_{p_1}(sup(I),w)+\mu_{p_0,u}^{w}(\amalg_{p'\neq p_1}U_{e,u}^{w,p'}(I)) \tag{50}$$

and

$$[\mathbf{2009.05.16.20}]\alpha_{e,u}^{v}(I)=\mu_{p_0,u}^{sup(I)}(U_{e,u}^{sup(I),p_1}(I))\upsilon_{p_1}(sup(I),v)+\mu_{p_0,u}^{v}(\amalg_{p'\neq p_1}U_{e,u}^{w,p'}(I)) \tag{51}$$

Therefore

$$|\sum_{a\in A}\alpha_{e,u}^{w}(I(a))\upsilon_{p_1}(sup(I(a)),v)-\sum_{a\in A}\alpha_{e,u}^{v}(I(a))\upsilon_{p_1}(sup(I(a)),w)|\leq$$

$$\leq\sum_{a\in A}\left[\mu_{p_0,u}^{w}(\amalg_{p'\neq p_1}U_{e,u}^{w,p'}(I(a)))\upsilon_{p_1}(sup(I(a)),v)+\mu_{p_0,u}^{v}(\amalg_{p'\neq p_1}U_{e,u}^{v,p'}(I(a)))\upsilon_{p_1}(sup(I(a)),w)\right]\leq$$

$$\leq\sum_{a\in A}\left[\mu_{p_0,u}^{w}(\amalg_{p'\neq p_1}U_{e,u}^{w,p'}(I(a)))+\mu_{p_0,u}^{v}(\amalg_{p'\neq p_1}U_{e,u}^{v,p'}(I(a)))\right]$$

and since we have

$$\amalg_{p'\neq p_1} U^{w,p'}_{e,u}(I) \subset D^w_{e,u}(I)$$

and

$$\amalg_{p'\neq p_1} U^{v,p'}_{e,u}(I) \subset D^v_{e,u}(I)$$

we conclude by Lemma 3.8.8 that

$$lim_{A\in S_{T2}([u,x])}|\sum_{a\in A}\alpha^w_{e,u}(I(a))\upsilon_{p_1}(sup(I(a)),v) - \sum_{a\in A}\alpha^v_{e,u}(I(a))\upsilon_{p_1}(sup(I(a)),w)| = 0$$

For $u\in T$ and $e\in X_1^{nd}$ let A be a T1 subset of $T_{\geq u}$ satisfying the condition of Lemma 3.5.10 relative to the T1 subset $A_{T_{\leq u}}(h_{\partial(e)})$. Then for any $a\in A$ we have

$$\upsilon_{p_1}(inf(I(a)), sup(I(a))) \geq h_{p_1}(inf(I(a)), sup(I(a))) > 0$$

and therefore we may consider the measure

$$\theta_{e,u,A} = \oplus_{a\in A}(\alpha^{sup(I(a))}_{e,u} * \upsilon_{p_1}(-, sup(I(a)))^{-1})^{|I(a)}$$

Proposition 3.8.12 *[**theta**] Let P be a non-degenerate right continuous pre-process which satisfies (M). Then for any $u\in T$ and $e\in X_1^{nd}$ there exists a unique measure $\theta_{e,u}$ on $T_{\geq u}$ such that for any $v\geq u$ one has*

$$[\mathbf{2009.05.17.2}]\alpha^v_{e,u} = (\theta_{e,u})^{|[u,v]} * \upsilon_{\partial_1(e)}(-,v)_{|[u,v]} \tag{52}$$

i.e. for any $u\leq w\leq v$,

$$\alpha^v_{e,u}((u,w]) = \int_{x\in(u,w]}\upsilon_{\partial_1(e)}(x,v)d\theta_{e,u}$$

Proof: By Lemma 3.5.1 and our assumtion that P is right continuous we know that $\upsilon_p(-,v)$ is measurable for all p and v. Let $p_0=\partial_0(e)$ and $p_1=\partial_1(e)$. In view of Corollary 3.8.7 and Lemma 3.5.10 there exists a T1 subset A of $T_{\geq u}$ such that for any $a\in A$ one has $h_{p_1}(inf(I(a)), sup(I(a))) > 0$. Consider the corresponding partition $T_{\geq u}=\amalg_{a\in A}I(A,a)$ of $T_{\geq u}$.

Our condition (52) on $\theta_{e,u}$ implies that for any $a\in A$ we have

$$(\theta_{e,u})^{|I(a)} * \upsilon_{p_1}(-, sup(I(a)))_{|I(a)} = (\alpha^{sup(I(a))}_{e,u})^{|I(a)}$$

and since $\upsilon_{p_1}(-, sup(I(a)))_{|I(a)} > 0$ we conclude that

$$[\mathbf{2009.05.17.4}](\theta_{e,u})^{|I(a)} = (\alpha^{sup(I(a))}_{e,u})^{|I(a)} * (\upsilon_{p_1}(-, sup(I(a)))_{|I(a)})^{-1} \tag{53}$$

One observes immediately that there exists a unique measure $\theta_{e,u}$ on $T_{\geq u}$ which satisfies (53) for all $a\in A$. It remains to check that it satisfies (52) for all $v\geq u$. It follows from Proposition 3.8.11.

Lemma 3.8.13 *[**2009.05.21.1**] Let I be a closed from the below interval of $T_{\geq u}$ such that $h_{p_1}(inf(I), sup(I)) > 0$. Then one has*

$$\theta_{e,u}(I) = \lim_{B\in S_{T2}(I)}\sum_{b\in B}\lambda^{sup(I(b))}_{e,u}(I(b))$$

Proof: Note first that since $h_{p_1}(inf(I), sup(I)) > 0$ we have $\upsilon_{p_1}(-, sup(I)) > 0$ on I and

$$\theta_{e,u}(I) = \int_{x\in I} \upsilon_{p_1}(-, sup(I))^{-1} d\alpha_{e,u}^{sup(I)}$$

By Proposition 3.5.15 we conclude that

$$\theta_{e,u}(I) = lim_{B\in S_{T2}(I)} \sum_{b\in B} \alpha_{e,u}^{sup(I)}(I(b))\upsilon_{p_1}(sup(I(b)), sup(I))^{-1}$$

and by Lemma 3.8.9 that

$$\theta_{e,u}(I) = \lim_{B\in S_{T2}(I)} \sum_{b\in B} \lambda_{e,u}^{sup(I(b))}(I(b)).$$

Lemma 3.8.14 *[**2009.05.21.3**] Let $I = [x, y)$ be an interval of $T_{\le u}$ and $p \in X_0$. Then for any $B \in S_{T2}(I)$ one has*

$$lim_{b'<y} h_p(u, b') + \sum_{b\in B} \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{sup(I(b)}(I(b)) \le h_p(u, x) + \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{x}(\{x\})$$

Proof: Note first that for any $u \ge a < a' \ge y$ we have

$$[\mathbf{2009.05.22.1}]\lambda_{e,u}^{y}((a, a')) = h_{\partial_0(e)}(u, a)\lambda_{e,a}^{y}((a, a')) \tag{54}$$

and

$$[\mathbf{2009.05.22.1b}]\lambda_{e,u}^{y}(\{a'\}) = h_{\partial_0(e)}(u, a)\lambda_{e,a}^{y}(\{a'\}) \tag{55}$$

for any $e \in X_1^{nd}$, and

$$[\mathbf{2009.05.22.2}]h_p(a, a') + \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,a}^{a'}((a, a')) \le \upsilon_p(a, a') - \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,a}^{a'}(a') \tag{56}$$

for any $p \in X_0$.

Consider the function on B given by

$$F(b') = \sum_{b\in B,\ b<b'} \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{sup(I(b)}(I(b)) + h_p(u, b') + \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'}(\{b'\})$$

Then

$$F(x) = h_p(u, x) + \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{x}(\{x\})$$

and using (54), (55) and (56) we get:

$$F(b'_+) - F(b') = \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'_+}([b', b'_+)) + h_p(u, b'_+) + \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'_+}(\{b'_+\})$$

$$-h_p(u, b') - \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'}(\{b'\}) =$$

$$= \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'_+}(\{b'\}) + h_p(u, b') \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'_+}((b', b'_+)) + h_p(u, b')h_p(b', b'_+) +$$

$$\sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'_+}(\{b'_+\}) - h_p(u,b') - \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'}(\{b'\}) \leq$$

$$\leq \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'_+}(\{b'\}) + h_p(u,b')(\upsilon_p(b',b'_+) - \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,b'}^{b'_+}(b'_+))+$$

$$\sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'_+}(\{b'_+\}) - h_p(u,b') - \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'}(\{b'\}) =$$

$$= h_p(u,b')(\upsilon_p(b',b'_+) - 1) + \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b'}(\{b'\})(h_{\partial_1(e)}(b',b'_+) - 1) \leq 0$$

Let now b'' be a limit element of B i.e. $b'' = sup(\{b' \in B\,|\,b' < b\})$. Then a similar computation shows that

$$lim_{b'<b''} F(b') = \sum_{b\in B,\;b<b''} \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{sup(I(b)}(I(b)) + lim_{b'<b''} h_p(u,b')$$

and therefore

$$F(b'') - lim_{b'<b''} F(b') = h_p(u,b'') + \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b''}(\{b''\}) - lim_{b'<b''} h_p(u,b')$$

On the other hand we have

$$h_p(u,b'') + \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{b''}(\{b''\}) = lim_{\epsilon\downarrow 0} h_p(u,b''-\epsilon)\upsilon_p(b''-\epsilon,b'') \leq lim_{b'<b''} h_p(u,b')$$

and therefore $F(b'') - lim_{b'<b''} F(b') \leq 0$. We conclude that for all $b' \in B$ one has

$$F(b') \leq h_p(u,x) + \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{x}(\{x\})$$

On the other hand a simple computation shows that

$$lim_{b'\in B} F(b') = \sum_{b\in B} \sum_{e\in X_1^{nd}\,|\,\partial_0(e)=p} \lambda_{e,u}^{sup(I(b)}(I(b)) + lim_{b'<y} h_p(u,b')$$

which finishes the proof of the lemma.

Proposition 3.8.15 *[***2009.05.22.3***] For any $y \geq x \geq u$ in T and any $e \in X_1^{nd}$ one has*

$$[\mathbf{2009.05.22.4}]\theta_{e,u}([x,y)) = \lim_{A\in S_{T1}([x,y))} \sum_{a\in A} \lambda_{e,u}^{sup(I(a))}(I(a)) \tag{57}$$

and

$$[\mathbf{2009.05.22.5}]\theta_{e,u}([x,y]) = \lim_{A\in S_{T1}([x,y])} \sum_{a\in A} \lambda_{e,u}^{sup(I(a))}(I(a)) \tag{58}$$

Proof: The second equation follows from the first one since

$$\theta_{e,u}(\{y\}) = \lambda_{e,u}^{y}(\{y\})$$

The first one follows from Lemma 3.8.13 since $[x,y]$ has a T1 subset such that the corresponding partition splits it into a disjoint union of countably many intervals which satisfy the condition this lemma.

Corollary 3.8.16 *[***2009.05.22.6***] For any $x \geq u$ and any $p \in X_0$ one has*

$$[\mathbf{2009.05.22.7}] \sum_{e \in X_1^{nd} \,|\, \partial_0(e)=p} \theta_{e,u}([u,x]) \leq 1 - h_p(u,x) \tag{59}$$

Proof: From (3.8.15) and Lemma 3.8.14 we have

$$\sum_{e \in X_1^{nd} \,|\, \partial_0(e)=p} \theta_{e,u}([u,x)) \leq 1 - lim_{A \in S_{T1}([u,x))} lim_{a \in A, a<x} h_p(u,a) = 1 - lim_{a<x} h_p(u,a)$$

and since

$$lim_{a<x} h_p(u,a) \geq h_p(u,x) + \sum_{e \in X_1^{nd} \,|\, \partial_0(e)=p} \lambda^x_{e,u}(\{x\}) = h_p(u,x) + \sum_{e \in X_1^{nd} \,|\, \partial_0(e)=p} \theta^x_{e,u}(\{x\})$$

we get (59).

Proposition 3.8.17 *[***2009.05.22.8***] For any $v \geq u$ in T and $e \in X_1^{nd}$ one has*

$$\lambda^v_{e,u} = \theta_{e,u} * h_{\partial_1(e)}(-,v)$$

Proof: For $v \geq x \geq u$ we have, by Proposition 3.5.15:

$$(\theta_{e,u} * h_{\partial_1(e)}(-,v))([u,x]) = \int_{y \in [u,x]} h_{\partial_1(e)}(y,v) d\theta_{e,u} = lim_{a \in S_{T2}([u,x])} \sum_{a \in A} \theta_{e,u}(I(a)) h_{\partial_1(e)}(sup(I(a)),v)$$

which equals by (57) and (58) to

$$lim_{a \in S_{T2}([u,x])} \sum_{a \in A} \lambda^{sup(I(a))}_{e,u}(I(a)) h_{\partial_1(e)}(sup(I(a)),v) = lim_{a \in S_{T2}([u,x])} \sum_{a \in A} \lambda^v_{e,u}(I(a)) =$$

$$= \lambda^v_{e,u}([u,x])$$

.

Theorem 3.8.18 *[***2009.06.02.5***] Let X be a countable multi-graph, $T = [T_{min}, T_{max}]$ a time window and $P = \{\mu_{p,u}\}$ a pre-process on (X,T) which satisfies condition (M). Then P is a renewal process if and only if it is non-degenerate, right continuous and satisfies condition (N).*

Proof: By Lemma 3.7.18 any renewal process is non-degenerate and right continuous and by Lemma **??** and Proposition 3.7.13(1) it satisfies condition (N). It remains to show that a non-degenerate right continuous pre-process $P = \{\mu^{\}}_{p,u}$ satisfying condition (M) is a renewal pre-process.

3.9 rl-sets

To accommodate a number of important examples the constructions of the previous section has to be generalized from reflexive multi-graphs to objects of a wider class which we call rl-sets.

Let Δ be the usual category of finite non-empty ordered sets and let Θ be the subcategory in Δ with the same objects and morphisms being non-decreasing maps $f : [i] \to [j]$ whose images are segments i.e. such that if $a, b \in Im(f)$ and $a < c < b$ then $c \in Im(f)$. In particular all the non-decreasing surjections are in Θ and for $i < j$ there are exactly $j - i + 1$ injections $[i] \to [j]$

in Θ. One verifies easily that Θ can also be described as the subcategory in Δ generated by the injections $d_n^n : \{0, \ldots, n-1\} \subset \{0, \ldots, n-1, n\}$, $d_n^0 : \{1, \ldots, n\} \subset \{0, 1, \ldots, n\}$ and surjections $s_n^i : \{0, \ldots, n+1\} \to \{0, \ldots, n\}$ where $s^i(i) = s^i(i+1) = i$.

A Θ-set X_* is a contravariant functor from Θ to *Sets*. We let $\partial_0, \partial_1 : X_n \to X_{n-1}$ denote the maps corresponding to d_n^n and d_n^0 and $\sigma_i^n : X_{n-1} \to X_n$ the maps corresponding to s_n^i. In what follows we will write ∂_i instead of ∂_i^n. With this abbreviation, we get the equation

$$\partial_0 \partial_1 = \partial_1 \partial_0$$

which implies that any composition of the maps ∂_*^* going from X_n to X_{n-k} can be written in a unique way as $\partial_0^i \partial_1^j$ where $i + j = k$.

An element $x \in X_n$ is called degenerate if it belongs to the image of one of the degeneracy maps $\sigma_i^{n-1} : X_{n-1} \to X_n$. We denote the subset of non-degenerate maps of X_n by X_n^{nd}. A Θ-set is called an rl-set if it satisfies the Eilenberg-Zilber condition i.e. for any $x \in X_n$ there exists a unique pair (x', s) where $s : [n] \to [m]$ is a surjection, $x' \in X_m^{nd}$ and $s^*(x') = x$.

Example 3.9.1 *[***dgrl***]* Let $(\partial_0, \partial_1 : X_1 \to X_0, id : X_0 \to X_1; \partial_0 \circ id = \partial_1 \circ id = Id)$ be a reflexive multi-graph. Define maps $\partial_0^n, \partial_1^n : X_n \to X_{n-1}$ by $\partial_i^1 = \partial_i$ and

$$\partial_0^n(e_1, \ldots, e_n) = (e_1, \ldots, e_{n-1})$$

$$\partial_1^n(e_1, \ldots, e_n) = (e_2, \ldots, e_n)$$

for $n > 1$ and maps $\sigma_i^n : X_{n-1} \to X_n$ by

$$\sigma_i^n(e_1, \ldots, e_{n-1}) = (e_1, \ldots, e_{i-1}, \sigma(p_i), e_i, \ldots, e_{n-1})$$

where $p_i = \partial_0(e_i)$ for $i = 1, \ldots, n$ and $p_{n+1} = \partial_1(e_n)$.

One verifies easily that the resulting system of sets and maps is an rl-set.

Example 3.9.2 The first two terms X_1, X_0 of any rl-set form a reflexive-multigraph. We denote the rl-set defined by this multi-graph by $\underline{X}_*$ i.e. $\underline{X}_0 = X_0$, $\underline{X}_1 = X_1$ and for $n > 1$ we have $\underline{X}_n = X_1 \,{}_{\partial_1}\!\times_{\partial_0} \ldots {}_{\partial_1}\!\times_{\partial_0} X_1$. The rl-set generated by this multi-graph coincides with the original one if and only if the segments of non-degenerate elements are non-degenerate, non-degenerate elements are determined by their one dimensional segments and any finite sequence of one dimensional segments with compatible end-points defines an n-dimensional segment.

Example 3.9.3 *[***srl***]* Let Y_* be a simplicial set. A collection of subsets X_n of Y_n which is closed under the boundaries ∂_n^n and ∂_0^n and such that $(\sigma_n^i)^{-1}(X_n) \subset X_{n-1}$ defines an rl-set. In the rest of this section we show that any rl-set can be obtained by this construction from an rl-set.

Let $\phi : \Theta \to \Delta$ be the obvious functor. It defines a pair of adjoint functors (ϕ^*, ϕ_*) between Θ-sets and simplicial sets. For any simplicial set Y the Θ-set $\phi_* Y$ is an rl-set by the Eilenberg-Zilber Lemma. The following proposition shows that any rl-set can be obtained from a simplicial set by the construction of Example 3.9.3.

Proposition 3.9.4 *[***dlin***] Let X be an rl-set. Then the natural morphism $X \to \phi_* \phi^* X$ is a monomorphism and an element $x \in X_n$ is degenerate in X if and only if its image in $\phi^* X_n$ is degenerate in $\phi^* X$. In addition, any non-degenerate element of $\phi^* X_n$ is the boundary of a non-degenerate element of X_m for some $m \geq n$.*

Proof: Let Δ_{surj} be the subcategory of surjective maps in Δ. Since all morphisms in Δ_{surj} are epimorphisms the category $[m]\backslash\Delta_{surj}$ of arrows $[m]\to[n]$ under a given object $[m]$ is a partially ordered set. This partially ordered set is isomorphic to the cube $(0\to 1)^m$ or, equivalently, to the partially ordered set of subsets of $\{1,\dots,m\}$. To establish this isomorphism, a surjection $[m]\to[n]$ is viewed as a partition of $\{0,\dots,m\}$ into n sequential non-empty segments which is obtained by "erasing" $n-m$ of the elementary intervals $[i,i+1]$, $i=0,\dots,m-1$. For example, the standard generating surjections $s_n^i:[n+1]\to[n]$ correspond to the subsets $\{i\}$ of $\{0,\dots,n\}$.

For a surjection $p:[n]\to[m]$ in Δ define the minimal section $s_p:[m]\to[n]$ of p setting $s_p(i)=min\{p^{-1}(i)\}$. One verifies easily that for a composable pair of surjections $[n]\stackrel{p}{\to}[m]\stackrel{q}{\to}[k]$ one has

$$[\mathbf{2009.04.20.1}]s_ps_q=s_{qp} \tag{60}$$

Let $p:[n]\to[m]$ be a surjection and $f:[m']\to[m]$ a morphism. Define commutative square

$$\begin{array}{ccc} [n(p,f)] & \stackrel{p_f}{\longrightarrow} & [m'] \\ {\scriptstyle f_p}\downarrow & & \downarrow{\scriptstyle f} \\ [n] & \stackrel{p}{\longrightarrow} & [m] \end{array}$$

by the condition that for $k\in\{0,\dots,m\}$ such that $p^{-1}(k)\cong\{1,\dots,i_k\}$ and $f^{-1}=\{1,\dots,j_k\}$ where $j_k\ge 1$ one has $(fp_f)^{-1}(k)=(f_pp)^{-1}(k)=\{1,\dots,j\}\amalg_{\{j_k\}}\{j_k,\dots,j_k+i_k\}$, the map to $\{1,\dots,i_k\}$ maps the first segment to $\{1\}$ and the rest bijectively and the map to $\{1,\dots,j_k\}$ maps the first segment bijectively and the second one to j_k:

$$\begin{array}{ccc} \{1,\dots,j_k\}\amalg_{\{j_k\}}\{j_k,\dots,j_k+i_k\} & \longrightarrow & \{1,\dots,j_k\}\amalg_{\{j_k\}}\{j_k\} \\ \downarrow & & \downarrow \\ \{1\}\amalg_{\{1\}}\{1,\dots,i_k\} & \longrightarrow & \{k\} \end{array} \tag{61}$$

Then

$$[\mathbf{msec}]f_ps_{p_f}=s_pf \tag{62}$$

We have $n+1=\sum_{k=0}^m i_k$, $m'+1=\sum_{k=0}^m j_k$. Therefore, if f is a surjection then

$$n(p,f)=-1+\sum_{k=0}^m(i_k+j_k-1)=-1+n+1+m'+1-m-1=n+m'-m.$$

Let X_* be a Θ-set. A pair (p,x) where $p:[n]\twoheadrightarrow[m]$ is a surjection and $x\in X_n$ defines an element $s_p^*(x)\in\phi^*(X)_m$. In view of (60) this construction defines a map

$$[\mathbf{phiuppereq1}]\lim_{[n]\twoheadrightarrow[m]}X_n\to\phi^*(X)_m \tag{63}$$

where the limit is taken over the category of surjections over $[m]$ in Δ.

Lemma 3.9.5 *[**phiupper1**] For any Θ-set X the maps (63) are bijections. For $(p,x)\in\phi^*(X)_m$ and $f:[m']\to[m]$ in Δ one has*

$$[\mathbf{phiuppereq2}]f^*(p,x)=(p_f,f_p^*(x)) \tag{64}$$

Proof: The set $\phi^*(X)_m$ is the quotient set of the set $\amalg_{[m]\to[n]}X_n$ by the equivalence relation defined by the condition that for $[m] \stackrel{f}{\to} [n_0] \stackrel{g}{\to} [n_1]$ where g is in Θ and $x_0 \in X_{n_0}$, $x_1 \in X_{n_1}$ one has $(f, x_0) \backsim (gf, x_1)$.

Let $f : [m] \to [n]$ be a morphism in Δ. One can easily see that it can be written as $[m] \stackrel{s_p}{\to} [n'] \stackrel{g}{\to} [n]$ where g a morphism in Θ and $p : [n'] \to [m]$ is a surjection such that $p^{-1}(m) = \{n'\}$. This shows that the map (63) is surjective. To show that it is injective it is sufficient to verify that for two such representations $f = g_1 s_{p_1} = g_2 s_{p_2}$ of a morphism f of this form there exist surjections $q_1 : [n''] \to [n'_1]$, $q_2 : [n''] \to [n'_2]$ such that $q_1 g_1 = q_2 g_2$. We have

$$g_1(0) = g_1(s_{p_1}(0)) = g_2(s_{p_2}(0)) = g_2(0)$$

$$g_1(n'_1) = g_1(s_{p_1}(m)) = g_2(s_{p_2}(m)) = g_2(n'_2)$$

Since g_1, g_2 are in Θ we conclude that $Im(g_1) = In(g_2)$ and therefore surjections with the required property exist.

The equation 64 follows immediately from (62).

Lemma 3.9.6 *[**maintd**] Let X be a Θ-set. Then X is an rl-set if and only if the maps $X_m \to \phi^*(X)_m$ are injective and any $x \in X_m$ which is degenerate in $\phi^*(X)$ is degenerate in X.*

Proof: "If" Follows immediately from the fact that any simplicial set has Eilenberg-Zilber property.

"Only if" By Lemma 3.9.5 we have $\phi(X)_m = lim_{[n]\twoheadrightarrow[m]} X_n$. Suppose that X is an rl-set. Then all maps $X_m \to X_n$ for surjections $[n] \twoheadrightarrow [m]$ are injective and therefore $X_m \to \phi^*(X)_m$ is injective.

Let $f : [m] \twoheadrightarrow [k]$ be a surjection and $(p : [n] \twoheadrightarrow [k], x \in X_n) \in \phi^*(X)_k$. By Lemma 3.9.5 we have $f^*(p : [n] \twoheadrightarrow [k], x \in X_n) = (p_f, f_p^*(x))$. Suppose that this element lies in X_m i.e. $f_p^*(x) = p_f^*(y)$ for $y \in X_m$. We need to show that y is degenerate in X_*. If y is non-degenerate then there is a map (necessarily surjective one) $h : [m'] \to [n]$ such that $f_p = hp_f$. From the construction of f_p we see that it is possible only if p is an isomorphism in which case $y = f^*(x)$ which is impossible since f is a surjection and we assumed y to be non-degenerate.

To finish the proof of the proposition it remains to show that any non-degenerate simplex of ϕ^*X is the boundary of a non-degenerate simplex of X. It follows easily from the fact that any morphism in Δ can be represented by an injection followed by a morphism from Θ and also by a surjection followed by an injection.

Example 3.9.7 Let Δ_{surj} be the subcategory of surjective maps in Δ. A Θ-set X is an rl-set if and only if it takes push-out squares in Δ_{surj} to pull-back squares in sets. In view of the description of $[m]\backslash\Delta_{surj}$ given below we know that any push-out square is a "composition" of squares which are push-outs of pairs of maps of the form σ_n^i. Therefore, X is an rl-set if and only if the maps $s_n^i : X_n \to X_{n+1}$ are injective and for $0 \le i \le j \le n$ one has

$$Im(s_{n+1}^i s_n^j = s_{n+1}^{j+1} s_n^i) = Im(s_{n+1}^i) \cap Im(s_{n+1}^{j+1})$$

Example 3.9.8 [**univrl**] Since any representable functor takes push-out squares to pull-back squares the Θ-sets Θ^n represented by objects $[n]$ of Θ are rl-sets. These sets correspond by the construction of Example 3.9.1 to the "linear" reflexive graph with non-degenerate base of the form $1 \to 2 \to \ldots \to n$. These rl-sets are universal in the same way as simplexes Δ^n are universal among simplicial sets. There is a unique element $y_n \in (\Theta^n)_n^{nd}$ and for any rl-set and any element $x \in X_n$ there exists a unique morphism $f : \Theta^n \to X$ such that $f(y_n) = x$.

The rl-sets Θ^n can be obtained by the construction of Example 3.9.3 as follows. Take $Y = \Delta^n$. The non-degenerate simplexes of Δ^n correspond to non-empty subsets of $\{0, \dots, n\}$. The non-degenerate simplexes of Θ^n consists of non-empty segments $\{i, \dots, i+j\}$ of $\{0, \dots, n\}$.

3.10 Path systems defined by countable rl-sets

Note: that X_* is a path system over X_0, notation $X[u,v]_{p,*}$, $X[u,v]_{*,p'}$ and $X[u,v]_{p,p'}$.

Let X_* be an rl-set and

$$X[u,v] = \amalg_{n \geq 0} \Delta^n_{(u,v]} \times X_n^{nd}$$

(note that for $u = v$ we get $X[u,v] = X_0$). For $u \leq u' \leq v' \leq v$ define maps $res^{u,v}_{u',v'}$ as follows. Consider a point $(x_1, \dots, x_n; r)$ of $X[u,v]$. Then we have

$$\{x_1, \dots, x_n\} \cap (u, u'] = \{x_1, \dots, x_i\}$$

$$\{x_1, \dots, x_n\} \cap (v', v] = \{x_{j+1}, \dots, x_n\}$$

for some well defined values of i and j (note that $j \geq i$ and for $u' = v'$ we get $i = j$). Since X_* satisfies the Eilenberg-Zilber condition there exist a unique pair (r', s) where $s : [j-i] \to [m]$ is a surjection and $r' \in X^{nd}_m$ is an element such that $\partial_1^i \partial_0^{n-j}(r) = s^*(r')$. As noted below order preserving surjections $[m'] \to [m]$ correspond to $(m'-m)$-element subsets I in an ordered set with m' elements. In particular, s corresponds to a subset I of $\{i+1, \dots, j\}$. We set $J = \{i+1, \dots, j\} \backslash I$ and

$$res^{u,v}_{u',v'}(x_1, \dots, x_n; r) = (\{x_i\}_{i \in J}; r')$$

In the case when X_* is generated by a reflexive multi-graph this construction agrees with the one given above in terms of right continuous maps. Together with the universal property of the rl-sets Θ^n given in Example 3.9.8 it implies that for $u \leq u' \leq u'' \leq v'' \leq v' \leq v$ we have

$$res^{u',v'}_{u'',v''} res^{u,v}_{u',v'} = res^{u,v}_{u'',v''}$$

In the case when X is countable, the sets $X[u,v]$ have obvious structures of measurable spaces, the maps res are measurable for these structures $(X[*,*], res^{*,*}_{*,*})$ is a path system.

Example 3.10.1 Let X be a simplicial set. Define the $[u,v]$-geometric realization of X as

$$|X|_{[u,v]} = \amalg_n \Delta^n_{(u,v)} \times X^{nd}_n$$

for $u < v$ and $|X|_{[u,u]} = \pi_0(X)$ for $v = u$. For $[u,v] = [0,1]$ we get the usual geometric realization considered as the disjoint union of the open simplexes of its canonical triangulation.

Let EX be the simplicial set which is the composition of X with the functor $\{0, \dots, n\} \mapsto \{0, \dots, n\} \amalg \{n+1\}$ such that $(EX)_n = X_{n+1}$ and $(EX)^{nd}_n = X^{nd}_{n+1} \amalg X^{nd}_n$ (cf. [**?**]).

Then $\phi_* X[u,v] = |EX|_{[u,v]}$. Indeed, for $u < v$ we get

$$|EX|_{[u,v]} = \amalg_n \Delta^n_{(u,v)} \times (EX)^{nd}_n = \amalg_n \Delta^n_{(u,v)} \times X^{nd}_{n+1} \amalg X^{nd}_n =$$

$$= \amalg_n (\Delta^n_{(u,v)} \amalg \Delta^{n-1}_{(u,v)}) \times X^{nd}_n = \amalg_n \Delta^n_{(u,v]} \times X^{nd}_n = \phi_* X[u,v]$$

and for $v = u$

$$|EX|_{[u,u]} = \pi_0(EX) = X_0 = \phi_* X[u,u]$$

Example 3.10.2 [**univrlp**] For $u < v$ we have

$$\Theta^n[u,v] = \Delta^n_{(u,v]} \amalg 2\cdot\Delta^{n-1}_{(u,v]} \amalg \dots \amalg (n+1)\cdot\Delta^0_{(u,v]}$$

For any rl-set X such that the boundaries of non-degenerate elements are non-degenerate, any $e \in X^{nd}_n$ and any $u \le w \le v$ the product $res^{u,v}_{u,w} \times res^{u,v}_{w,v}$ restricted to $\Delta^e_{(u,v]}$ maps it bijectively to

$$(res^{u,v}_{u,w} \times res^{u,v}_{w,v})(\Delta^e_{(u,v]}) = \amalg_{i=0}^n (\Delta^{\partial_0^{n-i}(e)}_{(u,w]} \times \Delta^{\partial_1^i(e)}_{(w,v]})$$

such that

$$(res^{u,v}_{u,w} \times res^{u,v}_{w,v})^{-1}(\Delta^{\partial_0^{n-i}(e)}_{(u,w]} \times \Delta^{\partial_1^i(e)}_{(w,v]}) \cap \Delta^e_{(u,v]} = V^v_{e,u}(w,i)$$

where

$$V^v_{e,u}(w,i) = \{x_1,\dots,x_n \,|\, u < x_1 < \dots < x_i \le w < x_{i+1} < \dots < x_n \le v\}$$

For $e \in X^{nd}_n$, $e' \in X^{nd}_{n'}$, $u \le w \le v$ and again under the assumption that boundaries of non-degenerate elements are non-degenerate, we get

$$[\mathbf{preim1}](res^{u,v}_{u,w} \times res^{u,v}_{w,v})^{-1}(\Delta^e_{(u,w]} \times \Delta^{e'}_{(w,v]}) = \amalg_{\{f\in X^{nd}_{n+n'} \,|\, \partial_0^{n'}(f)=e,\ \partial_1^n(f)=e'\}} V^v_{f,u}(w,n) \qquad (65)$$

As in the case of X corresponding to a directed multi-graph let

$$U^v_{e,u}(I_1,\dots,I_n) = \{(x_1,\dots,x_n) \in \Delta^e_{(u,v]} \,|\, x_i \in I_i\}$$

where I_i is a sequence of sub-intervals $[y_{i,-}, y_{i,+}]$ of $(u,v]$ such that $y_{i,+} < y_{i+1,-}$. For any such sequence there exist points w_i, $i = 1,\dots,n-1$ such that $y_{i,+} < w_i < y_{i+1,-}$ and for any choice of points satisfying these conditions we have

$$(res^{u,v}_{u,w_1} \times \dots \times res^{u,v}_{w_{n-1},v})^{-1}(U^{w_1}_{e_1,u}(I_1) \times \dots \times U^v_{e_n,w_{n-1}}(I_n)) = \amalg_{\{f\in X^{nd}_n \,|\, \partial_0^{n-i}\partial_1^{i-1}(f)=e_i\}} U^v_{f,u}(I_1,\dots,I_n)$$

The following lemma is straightforward.

Lemma 3.10.3 *[**2009.04.28.1**] For any countable rl-set X and any $u \le v$ the σ-algebra on $X[u,v]$ is generated in the strong sense by points $\{p\} \in X_0$ and subsets $U^v_{e,u}(I_1,\dots,I_n)$ where $n > 0$ and $e \in X^{nd}_n$.*

Let $f_* : X_* \to Y_*$ be a morphism of rl-sets. Then f defines a deterministic morphism of the corresponding path systems as follows. Let $u \le v$. By definition we have

$$X[u,v] = \amalg_{n\ge 0} \Delta^n_{(u,v]} \times X^{nd}_n$$

Since Y_* satisfies the Eilenberg-Zilber condition, for any $y \in Y_n$ there exist a unique surjection $s_y : [n] \twoheadrightarrow [m]$ and $y^{nd} \in Y^{nd}_m$ such that $f(x) = s^*(y)$. As explained in the proof of Proposition 3.9.4 surjections $[n] \twoheadrightarrow [m]$ are in a bijective correspondence with $n-m$ element subsets of $\{1,\dots,n\}$. Let I_s be the subset corresponding to s and CI_s be its complement. For a sequence $(x_1,\dots,x_n)$ denote by $s^*(x_1,\dots,x_n)$ the sequence which consists of x_i with $i \in CI_s$.

For $r = (x_1,\dots,x_n;x) \in X[u,v]$ define $f^v_u(r)$ as the element $(s^*_y(x_1,\dots,x_n);y^{nd})$ of $\Delta^m_{(u,v]} \times \{y^{nd}\}$ in $Y[u,v]$.

Lemma 3.10.4 *[**2009.04.29.6**] The family of maps f^v_u corresponding to a map of rl-sets $f : X_* \to Y_*$ is a deterministic morphism of path systems.*

Proof: ???

Lemma 3.10.5 *[2009.04.29.7] If $f_* : X_* \to Y_*$, $g_* : Y_* \to Z_*$ is a composable pair of maps of rl-sets then $(g\circ f)_u^v = g_u^v \circ f_u^v$.*

Proof: ???

Let now $f_* : X_* \to Y_*$ be a family of probability kernels of the form

$$f(x) = \sum_{y\in Y_n} f(x,y)\delta_y$$

Define kernels $f_u^v : X[u,v] \to Y[u,v]$ setting

$$f_u^v(x_1,\dots,x_n;x) = \sum_{y\in Y_n} f(x,y)\cdot \delta_{(s_y^*(x_1,\dots,x_n);y^{nd})} =$$

$$= \sum_{m\le n}\sum_{s:[n]\twoheadrightarrow[m]}\sum_{y\in Y_m^{nd}} f(x,s^*(y))\cdot\delta_{(s^*(x_1,\dots,x_n);y)}$$

The first representation of $f_u^v(x_1,\dots,x_n;x)$ shows that it is a probability measure on $Y[u,v]$ and therefore f_u^v is a probability kernel.

Let us re-write f_u^v in the form

$$f_u^v(x_1,\dots,x_n;x) = \sum_{(y_1,\dots,y_m;y)\in Y[u,v]} f_u^v((x_1,\dots,x_n;x),(y_1,\dots,y_m;y))\cdot\delta_{(y_1,\dots,y_m;y)}$$

where $f_u^v((x_1,\dots,x_n;x),(y_1,\dots,y_m;y)) \ne 0$ only if there exists a surjection $[n]\twoheadrightarrow[m]$ such that $(y_1,\dots,y_m) = s^*(x_1,\dots,x_n)$ in which case

$$[\mathbf{2009.04.30.1}] f_u^v((x_1,\dots,x_n;x),(y_1,\dots,y_m;y)) = f(x,s^*(y)) \tag{66}$$

Let us assume that Y is regular i.e. boundaries of a non-degenerate element are non-degenerate. Substituting our definition of f_*^* definition into (20) we see that the collection f_*^* is a morphism of path systems if and only if for all $u\le w\le v$, $x\in X_m^{nd}$, $y'\in Y_{n'}^{nd}$, $y''\in Y_{n''}^{nd}$, $(x_1\dots,x_i)\in\Delta^i_{(u,w]}$, $(x_{i+1},\dots,x_m)\in\Delta^{m-i}_{(w,v]}$, $(y'_1,\dots,y'_{n'})\in\Delta^{n'}_{(u,w]}$, $(y''_1,\dots,y''_{n''})\in\Delta^{n''}_{(w,v]}$ one has:

$$\sum_{\{y\in Y^{nd}_{n'+n''}\,|\,\partial_0^{n''}(y)=y' \ and\ \partial_1^{n'}(y)=y''\}} f_u^v((x_1,\dots,x_m;x),(y'_1,\dots,y'_{n'},y''_1,\dots,y''_{n''};y)) =$$

$$= f_u^w((x_1,\dots,x_i;\partial_0^{m-i}(x)),(y'_1,\dots,y'_{n'};y'))f_w^v((x_{i+1},\dots,x_m;\partial_1^i(x)),(y''_1,\dots,y''_{n''};y''))$$

Using (66) we see that f_u^v is a morphism of path systems if and only if for all $u\le w\le v$, $x\in X^{nd}_{i+j}$, $s' : [i]\twoheadrightarrow[n']$, $s'' : [j]\twoheadrightarrow[n'']$ one has

$$\sum_{\{y\in Y^{nd}_{n'+n''}\}} f_u^v(x,(s'+s'')^*(y)) = f_u^w(\partial_0^j(x),(s')^*\partial_0^{n'}(y))f_w^v(\partial_1^i(x),(s'')^*\partial_1^{n''}(y))$$

where $s'+s'' : [i+j]\twoheadrightarrow[n'+n'']$ is the surjection which corresponds to the subset $C(CI_{s'}\cup CI_{s''})$. Since

$$(s')^*\partial_0^{n'}(y) = \partial_0^j(s'+s'')^*(y)$$

$$(s'')^*\partial_1^{n''}(y) = \partial_1^i(s'+s'')^*(y)$$

and Y satisfies the Eilenberg-Zilber property we get the following result

Lemma 3.10.6 *[**2009.04.30.2**] The kernels f_u^v defined above form a morphism of path systems if and only if for all $u \le w \le v$, $i, j \ge 0$, $x \in X_{i+j}^{nd}$, $y_1 \in Y_i$, $y_2 \in Y_j$ one has*

$$[\mathbf{2009.04.30.3}] \quad \sum_{\{y\in Y_{i+j}\,|\,\partial_0^j(y)=y_1 \text{ and } \partial_1^i(y)=y_2\}} f(x,y) = f(\partial_0^j(x), y_1) f(\partial_1^i(x), y_2) \tag{67}$$

Applying the condition (3.10.6) for $i = j = 0$ we conclude that for all $x \in X_0$ and $y \in Y_0$ we have

$$f(x,y) = f(x,y)^2$$

Together with the condition that $\sum_y f(x,y) = 1$ this implies that for each $x \in X_0$ there is exactly one $y \in Y_0$ for which $f(x,y) \ne 0$ and for this y we have $f(x,y) = 1$ i.e. $f_0 : X_0 \to Y_0$ is a deterministic map.

Applying (3.10.6) for $i = 0$ and for $j = 0$ we further conclude that for $x \in X_n^{nd}$ and $y \in Y_n$ we have $f(x,y) \ne 0$ only if

$$f_0(\partial_0^n(x)) = \partial_0^n(y)$$

and

$$f_0(\partial_1^n(x)) = \partial_1^n(y)$$

Summing up the equations (3.10.6) over y_1 for a given y_2 and over y_2 for a given y_1 we get

$$\sum_{\{y\in Y_n\,|\,\partial_0^j(y)=y_1\}} f(x,y) = f(\partial_0^j(x), y_1)$$

and

$$\sum_{\{y\in Y_n\,|\,\partial_1^i(y)=y_2\}} f(x,y) = f(\partial_1^i(x), y_2)$$

which is equivalent to commutativity of the squares

$$\begin{array}{ccc} X_n^{nd} & \xrightarrow{f_n} & Y_n \\ \partial_0^j \downarrow & & \downarrow \partial_0^j \\ X_{n-j} & \xrightarrow{f_{n-j}} & Y_{n-j} \end{array} \qquad \begin{array}{ccc} X_n^{nd} & \xrightarrow{f_n} & Y_n \\ \partial_1^i \downarrow & & \downarrow \partial_1^i \\ X_{n-i} & \xrightarrow{f_{n-i}} & Y_{n-i} \end{array}$$

Consider now two families of probability kernels

$$X_* \stackrel{f_*}{\to} Y_* \stackrel{g_*}{\to} Z_*$$

For $u \le v$, $x \in X_n^{nd}$ and $(x_1, \ldots, x_n) \in \Delta_{(u,v]}^n$ we have

$$g_u^v f_u^v(x_1, \ldots, x_n; x) = \sum_{y\in Y_n} f(x,y) \sum_{z\in Z_{m(y)}} g(y^{nd}, z)\delta_{(s_z^*(s_y^*(x_1,\ldots,x_n));z^{nd})}$$

where $y^{nd} \in Y_{m(y)}$, and

$$(g\circ f)_v^u(x_1, \ldots, x_n; x) = \sum_{z\in Z_n}\sum_{y\in Y_n} f(x,y)g(x,y)\delta_{(s_z^*(x_1,\ldots,x_n);z^{nd})}$$

Therefore, the equation

$$[\mathbf{2009.05.01.1}] g_u^v f_u^v = (g\circ f)_v^u \tag{68}$$

is equivalent to the condition that for all $n \geq k$, $x^{nd} \in X^{nd}_n$, $z^{nd} \in Z^{nd}_k$ and $s : [n] \twoheadrightarrow [k]$ we have

$$\sum_{n\geq m\geq k} \sum_{\{J\subset\{1,\ldots,n\}\,|\,CI_s\subset J \ and \ \#J=m\}} \sum_{y^{nd}\in Y^{nd}_m} f(x, s^*_{CJ}(y))g(y^{nd}, r^*(z^{nd})) =$$

$$[\mathbf{2009.05.01.2}] = \sum_{y\in Y_n} f(x, y)g(y, s^*_{CI}(z^{nd})) \tag{69}$$

where $r : [m] \twoheadrightarrow [k]$ corresponds to the inclusion $CI_s \subset J$.

Let us look for the conditions on g which would imply that (68) holds for deterministic morphisms of rl-sets f.

Consider the case when $x \in X^{nd}_n$ and $f_n(x) = y$ for a given $y \in Y_n$ (e.g. when f is the universal morphism $\Theta^n \to Y$ corresponding to y). Let $y = s^*_y(y^{nd})$ for $y^{nd} \in Y^{nd}_m$ and let $J = CI_{s_y}$. Then (69) implies that for all $k \leq m$, $I \subset J$ such that $\#I = k$ and all $z \in Z^{nd}_k$ one has

$$[\mathbf{2009.05.01.4}]g(y^{nd}, r^*(z^{nd})) = g(y, s^*_y(r^*(z^{nd}))) \tag{70}$$

where $r : [m] \twoheadrightarrow [k]$ corresponds to the inclusion $I \subset J$ and for all I which are not contained in J one has

$$[\mathbf{2009.05.01.5}]g(y, s^*_{CI}(z)) = 0 \tag{71}$$

Since Z has the Eilenberg-Zilber property these conditions are equivalent to the commutativity of the squares

$$\begin{array}{ccc} Y_n & \xrightarrow{g_n} & Z_n \\ {\scriptstyle s^*}\uparrow & & \uparrow{\scriptstyle s^*} \\ Y^{nd}_m & \xrightarrow{g_m} & Z_m \end{array}$$

for all $s : [n] \twoheadrightarrow [m]$ and therefore to the commutativity of the squares

$$[\mathbf{2009.05.01.3}] \begin{array}{ccc} Y_n & \xrightarrow{g_n} & Z_n \\ {\scriptstyle s^*}\uparrow & & \uparrow{\scriptstyle s^*} \\ Y_m & \xrightarrow{g_m} & Z_m \end{array} \tag{72}$$

Assume that (70) and (71) hold. By (70), the left hand side of (69) equals

$$\sum_{y\in Y_{n,I}} f(x, y)g(y, s^*_{CI}(z^{nd}))$$

where $Y_{n,I}$ is the subset of Y_n which consists of y such that $I \subset CI_{s_y}$ and by (71) the same hold for the right hand side.

Example 3.10.7 *[***srl2***]*Let $\mathcal{C}$ be a small category and E be a class of morphisms in $\mathcal{C}$ which is closed under compositions with isomorphisms. Each pair like that defines an rl-set $X = X(\mathcal{C}, E)$ as follows. Consider the nerve $N(\mathcal{C})$ of $\mathcal{C}$. The set of n-simplexes $N_n(\mathcal{C})$ is the set of composable sequences $(f_1, \ldots, f_n)$ of morphisms of $\mathcal{C}$ of length n. Define an equivalence relation $\backsimeq$ on $N(\mathcal{C})$ saying that $(f_1, \ldots, f_n)$ is equivalent to $(f'_1, \ldots, f'_n)$ if there is a sequence of isomorphisms $(g_1, \ldots, g_n)$ such that $g_{i+1}f_i = f'_i g_i$.

One can easily see that this equivalence relation is compatible with the face and degeneracy maps and the quotient sets $N_n(\mathcal{C})/\backsimeq$ form a simplicial set $NN(\mathcal{C})$. Consider the subsets $X(E)_n$ of $NN_n(\mathcal{C})$ which consist of equivalence classes which consist of elements $(e_1, \ldots, e_n)$ with $e_i \in E$. These subsets satisfy the conditions of Example 3.9.3 and therefore define an rl-set $X(\mathcal{C}, E)$.

Example 3.10.8 [**slr3**]More generally, one can define an rl-set starting with a "gadget" which ... A gadget can be defined in a several equivalent ways:

1. A gadget is a list of data of the form:
   (a) sets X_1 and X_0,
   (b) mappings $\partial_0, \partial_1 : X_1 \to X_0$ and $id : X_0 \to X_1$ such that $\partial_0 \circ id = \partial_1 \circ id = Id_{X_0}$ (for $p, p' \in X_0$ we let $X_1(p, p')$ denote the set $\partial_0^{-1} \times \partial_1^{-1}(\{(p, p')\})$),
   (c) a mapping $G : X_0 \to Groups$,
   (d) a mapping which assigns to each $p, p' \in X_0$ a right action $X_1(p, p') \times G(p) \to X_1(p, p')$ and a left action $G(p') \times X_1(p, p') \to X_0(p, p')$ such that for all $g \in G(p)$ one has

$$[\mathbf{centereq}] g \cdot id(p) = id(p) \cdot g \tag{73}$$

2. A gadget is a pair of a (small) groupoid A and a functor $F : A^{op} \times A \to Sets$ together with a mapping which assigns to any $p \in A$ an element $id(p) \in F(p, p)$ such that for any $g \in Aut(p)$ one has $F(g^{-1}, g)(id(p)) = id(p)$.

To a groupoid A and a functor $F : A \times A^{op} \to Sets$ one associates a gadget in the sense of the first definition in the following way. Note that this construction is well defined only up to an isomorphism of the resulting gadget. Let X_0 be a set of objects of A which contains exactly one representative of each isomorphism class. Let $X_1 = \amalg_{(p,p') \in X_0 \times X_0} F(p, p')$. The maps ∂_0, ∂_1 and id are defined in the obvious way. For $p \in X_0$ set $G(p) = Aut_A(p)$. The left and right actions are given by

1. for $g \in G(p)$ and $x \in X_0(p, p')$ one sets $x \cdot g = X_0(g, Id)(x)$,
2. for $g \in G(p')$ and $x \in X_0(p, p')$ one sets $g \cdot x = X_0(Id, g)(x)$.

A large class of gadgets arises from pairs (C, S) where C is a category and S is a class of morphisms in C which contains identities and is closed under compositions with isomorphisms. The corresponding gadget is the one associated with the groupoid C_{iso} of isomorphisms in C and the functor S which maps $p, p' \in ob(C)$ to

$$S(p, p') = Hom_C(p, p') \cap S$$

4 Tonus spaces

4.1 Tonus spaces

Definition 4.1.1 *[**conus**] A* conus structure *on a set C is an abelian semi-group structure (with unit 0) together with a map*

$$m : \mathbf{R}_{\geq 0} \times C \to C$$

which makes C into a module over $\mathbf{R}_{\geq 0}$ i.e. such that

$$[\mathbf{eqpo1}] m(r, x + y) = m(r, x) + m(r, y) \tag{74}$$

$$[\mathbf{eqpo3}] m(r + s, x) = m(r, x) + m(s, x) \tag{75}$$

$$[\mathbf{eqpo4}] m(rs, x) = m(r, m(s, x)) \tag{76}$$

$$[\mathbf{eqpo6}] m(1, x) = x \tag{77}$$

$$[\mathbf{eqpo5}] m(0, x) = 0 \tag{78}$$

When no confusion is possible we write rx instead of $m(r, x)$. A set with a conus structure is called a conus space*.*

Definition 4.1.2 *[**dpo1**] A* tonus structure *on a set C is a topology together with a conus structure such that the addition and the multiplication by scalars are continuous.*

Definition 4.1.3 *[**dpo2**] Let C_1, C_2 be two conus (resp. tonus) spaces. A morphism $f : C_1 \to C_2$ is a map (resp. a continuous map) which commutes with addition and multiplication by scalars.*

We let T denote the category of tonus spaces.

Proposition 4.1.4 *[**ppo1**] The category T has all limits. The final object of T is the one point space. For any diagram $\mathcal{D}$ of tonus spaces the underlying topological space of $lim(\mathcal{D})$ is the limit of the corresponding diagram of topological spaces and the same is true for the limit of the underlying diagram of conus spaces and abelian semi-groups.*

Proof: Straightforward.

Proposition 4.1.5 *[**ppo2**] The category T of tonus spaces has colimits. The initial object of T is the one point space.*

Proof: The statement of the proposition follows from Lemmas 4.1.6-4.1.8 below and the usual reduction of general colimits to inductive colimits, reflexive coequalizers and finite coproducts.

Lemma 4.1.6 *[**lpo5**] Let $(C_\alpha, f_{\alpha\beta} : C_\alpha \to C_\beta)$ be an inductive system of tonus spaces. Let C be the colimit of this sequence in the category of sets which we consider with the colimit topology and the obvious operations of addition and multiplication by elements of $\mathbf{R}_{\geq 0}$. Then C is a tonus space and a colimit of our sequence in T.*

Proof: It follows by direct verification using the fact that inductive colimits commute with finite products in the category of topological spaces.

Lemma 4.1.7 *[lpo6] Let C_1, C_2 be tonus spaces, $f, g : C_1 \to C_2$ two morphisms and $s : C_2 \to C_1$ a common section of f and g (i.e. f, g, s form a reflexive coequalizer diagram). Let C be the coequalizer of f and g in the category of sets which we consider with the coequalizer topology and the obvious operations of addition and multiplication by elements of $\mathbf{R}_{\geq 0}$. Then C is a tonus space and a coequalizer of f and g in T.*

Proof: As in the proof of Lemma 4.1.6 everything follows by direct verification from the fact that reflexive coequalizers commute with finite products.

Lemma 4.1.8 *[lpo7] Let C_1, C_2 be tonus spaces. Let $C = C_1 \times C_2$ and consider C with the topology of the product and the obvious operations of addition and multiplication by elements of $\mathbf{R}_{\geq 0}$. Then C is a tonus spaces which is both the product and the coproduct of C_1 and C_2 in T.*

Proof: The only non-trivial part of the lemma is that C is the coproduct of C_1 and C_2 i.e. that for any tonus space D the map

$$Hom(C, D) = Hom(C_1, D) \times Hom(C_2, D)$$

given by the composition with the embeddings $C_1 \to C$, $C_2 \to C$ is bijective. It is clearly injective and to verify that it is bijective it is enough to prove that a map $f : C_1 \times C_2 \to D$ which is compatible with the algebraic structures and whose restrictions f_1, f_2 to $C_1 \times \{0\}$ and $\{0\} \times C_2$ are continuous is itself continuous. This follows from the fact that $f = m_D \circ (f_1 \times f_2)$ and the continuity of $m_D : D \times D \to D$.

Definition 4.1.9 *[grouplike] A tonus space C is called group-like if the underlying semi-group is a group.*

For the basic definitions related to the topological vector spaces and pre-ordered vector spaces we follow [**?**].

Lemma 4.1.10 *[lpo3] Let V be a group-like tonus space. Then there exists a unique extension of $m : \mathbf{R}_{\geq 0} \times V \to V$ to a continuous map $m : \mathbf{R} \times V \to V$ satisfying the condition*

$$m(r - s, x) = m(r, x) - m(s, x)$$

and with respect to this map V becomes a topological vector space (over $\mathbf{R}$).

Proof: The uniqueness is obvious. It is also obvious that if m as required exists then it makes V into a topological vector space. To prove the existence consider the map $\tilde{m} : \mathbf{NR} \times \mathbf{R}_{\geq 0} \times V \to V$ of the form $\tilde{m}(r, s, x) = m(r, x) - m(s, x)$. The algebraic properties of m imply that it has a decomposition

$$\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0} \times V \to \mathbf{R} \times V \stackrel{m}{\to} V$$

where the first arrow is defined by $(r, s) \mapsto r - s$. Since the first arrow is a strict topological epimorphism and the composition is continuous we conclude that m is continuous.

Lemma 4.1.11 *[lpo4] Let C be a tonus space and let $C \to V_C$ be the universal map from C as an abelian semi-group to an abelian group. Then V has a unique structure of a tonus space such that $C \to V_C$ is a morphism of tonus spaces. With this structure $C \to V_C$ is the universal morphism from C to a group-like tonus space.*

Proof: By (see e.g. []) we may describe V_C as the set of equivalence classes of pairs (x, y), $x, y \in C$ such that $(x_1, y_1) \cong (x_2, y_2)$ if and only if there exists u such that $x_1 + y_2 + u = x_2 + y_1 + u$. As usual we will write $x - y$ for the equivalence class of (x, y). For $r \in \mathbf{R}_{\geq 0}$ set $r(x, y) = (rx, ry)$. In view of 74 this defines a map $\mathbf{R}_{\geq 0} \times V_C \to V_C$ which takes $x - y$ to $rx - ry$ and one verifies easily that it satisfies the conditions 75-78. Let $\pi : C \times C \to V_C$ be the surjection $(x, y) \mapsto x - y$. Consider V_C as topological space with the topology defined by π i.e. such that U is open in V_C if and only if $\pi^{-1}(U)$ is open in $C \times C$. The universal properties of this topology imply immediately that the addition and multiplication by elements from $\mathbf{R}_{\geq 0}$ are continuous for V and we conclude that V has a structure of a tonus space such that $C \to V_C$ is a morphism of tonus spaces. One can see immediately that such a structure is unique.

Definition 4.1.12 *[***cancellable***] A tonus space C is called pre-group like if the universal map $C \to V_C$ is an injection i.e. if the underlying semi-group is a semi-group with cancellation.*

Definition 4.1.13 *[***reduced***] A tonus space C is called reduced if it is pre-group like and the topology on C induced by the map $C \to V_C$ coincides with the original topology.*

Definition 4.1.14 *[***closedts***] A tonus space C is called closed the corresponding universal map $C \to V_C$ is a closed embedding.*

Clearly any closed tonus space is reduced and any reduced is a pre-group like. It is also clear that any group-like tonus space is closed. To produced counter-examples to other implications we will use the following lemma.

Lemma 4.1.15 *[***need1***] Let $f : C \to V$ be a monomorphism from a tonus space C to a group-like tonus space V and let V_0 be the set of interior points of $f(C)$ in V. Assume that the following two conditions hold:*

1. *the map $C_0 = f^{-1}(V_0) \to V_0$ is a homeomorphism,*
2. *for any $v \in V$ there exist $x, y \in V_0$ such that $v = x - y$.*

Then $V(f) : V_C \to V$ is an isomorphism.

Proof: Clearly $V(f)$ is bijective as a map of sets and continuous. Let us show that it is open. Let V_0 be the set of interior points of $f(C)$ it is open in V and the restriction of f to $C_0 = f^{-1}(V_0)$ is an isomorphism. Consider the diagram:

$$\begin{array}{ccc} C_0 \times C_0 & = & V_0 \times V_0 \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle q_0} \\ V_C & \xrightarrow{V(f)} & V \end{array}$$

where the vertical arrows map (u, v) to $u - v$ and f_0 is the restriction of f to C_0. Our conditions imply that q_0 is surjective. Since V_0 is open in V and the subtraction map $V \times V \to V$ is open (follows from the fact that it is isomorphic to the projection $V \times V \to V$ to one of the factors) we conclude that q_0 is also open. This immediately implies that $V(f)$ is open.

Example 4.1.16 [**contr2**]Not all reduced tonus spaces are closed. Indeed let C be the subset in $\mathbf{R}^2$ which consists of points (x, y) such that $x \geq 0$ and $y > 0$ and the point $(0, 0)$. Considered with the induced topology and the obvious addition and multiplication by scalars C is a tonus space. Lemma 4.1.15 implies immediately that the embedding $C \to \mathbf{R}^2$ coincides with the universal embedding to a group-like tonus space. Therefore C is reduced but not closed.

Example 4.1.17 [**contr1**] Not any pre-group like tonus space is reduced. Consider the subset C in $\mathbf{R}^2$ which consists of (x, y) such that $x, y \geq 0$. Let further U be the subset of elements of C of the form $(x, 0)$ where $x > 0$. Consider the topology on C which is generated by the usual topology coming from $\mathbf{R}^2$ together with the condition that U is open. One verifies immediately that the addition and multiplication by scalars are continuous in this topology. On the other hand Lemma **??** again implies that the embedding $C \to \mathbf{R}^2$ is the universal one. Since in the topology on C induced by this embedding U is not open we conclude that C is pre-group like but not reduced.

Example 4.1.18 [**expo1**]Not all tonus spaces are pre-group like. Indeed, consider the set $\{0, 1\}$ with the discrete topology, the abelian semi-group structure given by $0+0 = 0$, $0+1 = 1$, $1+1 = 1$ and m given by $m(r, 0) = 0$, $m(r, 1) = 1$ if $r \neq 0$ and $m(0, 1) = 0$. These structures satisfiy all the conditions of Definition 4.1.2 but the resulting tonus space C is not pre-group like since $V_C = 0$. We will see below (Lemma 4.1.20) however that all Hausdorf tonus spaces are pre-group like. Note that the spaces in Examples 4.1.16 and 4.1.17 are both Hausdorf. Thus a Hausdorf tonus space need not be reduced or closed.

Sending C to $(V_C, C_{red}))$ where C_{red} is the image of C in V_C considered with the topology induced from V_C we get (by Lemmas 4.1.10, 4.1.11) a functor from tonus spaces to pairs (V, C) where V is a topological vector space and C is a cone in V. Clearly this functor is a full embedding on the subcategory of reduced tonus spaces and the pair (V, C) is in the image of this embedding if and only if any element of V can be written as $x - y$ where x, y are in C. Recall that a pre-ordered topological vector space is a pair as above such that C is closed in V. Therefore, we get the following result.

Proposition 4.1.19 [**embed1**] *The category of closed tonus spaces is equivalent to the full subcategory of the category of pre-ordered topological vector spaces (V, C) such that any element of V is of the form $x - y$ for $x, y \in C$.*

Lemma 4.1.20 [**lpo1**] *Let C be a Hausdorf tonus space then one has:*

1. *C is pre-group like i.e. for any x, y, u in C such that $x + u = y + u$ one has $x = y$*
2. *$m(r, 0) = 0$*

Proof: Let us denote $m(r, x)$ by rx. Consider the first claim. By 77 and 75 for any positive integer n we have $nx = \sum_{i=1}^{n} x$. From this by easy induction we get that for x, y, u as above one has $nx + u = ny + u$. By 74 and 76 we get that

$$x + (1/n)u = y + (1/n)u$$

Since C is Hausdorf a sequence may have only one limit and from the continuity of addition and multiplication by a number and 78 we get

$$x = x + 0u = \lim_{n\to\infty}(x + (1/n)u) = \lim_{n\to\infty}(y + (1/n)u) = y + 0u = y.$$

To get the second claim note that by 74 we have $r0 + r0 = r0 = r0 + 0$ and we conclude from the first part of the proof that $r0 = 0$.

Lemma 4.1.21 [**hus**] *Let C be a Hausdorf tonus space C. Then V_C is Hausdorf.*

Proof: Consider the natural map $\pi : C \times C \to V_C$. If C is Hausdorf then by Lemma 4.1.20 we have $\pi^{-1}(0) = \Delta$ where Δ is the diagonal. Since in a Hausdorf space the diagonal is closed and since π is a topological epimorphism we conclude that $\{0\}$ is closed in V_C. Since V_C is a topological vector space this implies in the standard way that V_C is Hausdorf.

Let C be a conus space and let $f_\alpha : C \to C_\alpha$ be a collection of conus maps to tonus spaces C_α. Let $t(f_\alpha)$ be the weakest topology on C which makes all the maps f_α continuous. It is easy to see that with this topology C is a conus space. We will say that the topology on C is defined by the collection f_α.

Lemma 4.1.22 *[***isred1***] Let C be a pre-group like conus space and let $f_\alpha : C \to C_\alpha$ be a collection of morphisms to reduced tonus spaces. Then C with the induced topology is a reduced tonus space.*

Proof: Let $C \to V_C$ and $C_\alpha \to V_\alpha$ be the universal morphisms to group-like spaces. By universality we get commutative squares

$$\begin{array}{ccc} C & \xrightarrow{f_\alpha} & C_\alpha \\ p\downarrow & & \downarrow p_\alpha \\ V & \xrightarrow{g_\alpha} & V_\alpha \end{array}$$

such that g_α are continuous. Let $x \in U \subset C$ be an open neighborhood of x in C. We have to show that there is an open neighborhood U' of $p(x)$ in V such that $p^{-1}(U') \subset U$. Since the topology on C is defined by (f_α) there exists a finite set $\alpha_1, \ldots, \alpha_n$ and open neighborhoods $W_1, \ldots, W_n$ of $f_{\alpha_i}(x)$ in C_α such that U contains $\cap f_{\alpha_i}^{-1}(W_i)$. Since each C_α is assumed to be reduced we have $W_i = p_{\alpha_i}^{-1}(W_i')$ for some W_i' open in V_α. The commutativity of our squares imply now that

$$\cap p^{-1} g_\alpha^{-1}(W_i') \subset U.$$

Remark 4.1.23 [**impo**] It is important to note that (in the notations of Lemma 4.1.22) the universal topology on V defined by the topology on C need not coincide with the topology induced by the maps $g_\alpha : V \to V_\alpha$. For an example see **??**.

In the following lemma we keep the notations of Lemma 4.1.22.

Lemma 4.1.24 *[***isclosed***] Let C be a pre-group like conus space and $f_\alpha : C \to C_\alpha$ a collection of maps to closed tonus spaces such that if $x \in V$ is an element satisfying $g_\alpha(x) \in C_\alpha$ for all α then $x \in C$. Then with the topology defined by (f_α), C is a closed tonus space.*

Proof: By Lemma 4.1.22 C is reduced. It remains to check that the image of C in V is closed. Let $x \in V$ be an element outside of C. Then by our assumption there exists α such that $g_\alpha(x)$ is outside C_α. Since C_α are closed this implies that there is a neighborhood W of $g_\alpha(x)$ which does not intersect C_α. Then $g_\alpha^{-1}(W)$ is a neighborhood of x which does not intersect C.

4.2 Embedding $\mathcal{K}^{op} \to T$

Let $(X, \mathfrak{R})$ be a measure space and $M^+(X, \mathfrak{R})$ the set of non-negative measurable functions on $(X, \mathfrak{R})$. It has an obvious structure of a conus space. Define the standard topology on $M^+(X, \mathfrak{R})$ by the condition that a set Z is closed if and only if for any sequence f_n of elements of Z such that $f_n \uparrow f$ we have $f \in Z$.

4.3 Embedding $\mathcal{K} \to T$

Let $(X, \mathfrak{R})$ be a measurable space and let $M_+(X, \mathfrak{R})$ be as above the set of all bounded measures on $(X, \mathfrak{R})$. Any (bounded, non-negative) measurable function $f \in M^+(X, \mathfrak{R})$ defines a map

$$f_* : M_+(X, \mathfrak{R}) \to \mathbf{R}_{\geq 0}$$

Define the *standard topology* on $M_+(X, \mathfrak{R})$ as the weakest topology which makes all maps of the form f_* continuous.

Lemma 4.3.1 *[**lem4**] A map u from a topological space T to $M_+(X, \mathfrak{R})$ is continuous with respect to the standard topology if and only if for any $f \in M^+(X, \mathfrak{R})$ the composition*

$$f_* \circ u : T \to \mathbf{R}_{\geq 0}$$

is continuous.

Lemma 4.3.2 *[**lem1**] The set $M_+(X, \mathfrak{R})$ considered with the standard topology and the addition and multiplication by elements of $\mathbf{R}_{\geq 0}$ defined in the obvious is a closed, Hausdorf tonus space.*

Proof: The continuity of the addition and multiplication by scalars follow from Lemma 4.3.1. To see that the standard topology is Hausdorf consider two measures μ_1 and μ_2 such that $\mu_1 \neq \mu_2$. Then there is a measurable subset $U \in \mathfrak{R}$ such that $\mu_1(U) \neq \mu_2(U)$. Let f be the indicator function of U. Then for any μ, $f_*(\mu) = \mu(U)$ and if V_1, V_2 are two non-intersecting neighborhoods of $\mu_1(U)$ and $\mu_2(U)$ respectively then $f_*^{-1}(V_i)$ give us two non-intersecting neighborhoods of μ_1 and μ_2.

To see that $C = M_+(X, \mathfrak{R})$ is closed in the corresponding vector space V we need to check that if μ_1, μ_2 are two measures such that $x = \mu_1 - \mu_2$ is not in C then there exists a neighborhood N of x in V such that $N \cap C = \emptyset$. By Lemma 4.1.11, V is universal and therefore any map of the form f_* extends to a continuous map $f_* : V \to \mathbf{R}$. Since x is not in C there exists a measurable subset $U \in \mathfrak{R}$ such that $x(U) = \mu_1(U) - \mu_2(U) < 0$. Let W be a neighborhood of $x(U)$ which lies in $(-\infty, 0)$. Taking f to be the indicator function of U we get a neighborhood $f_*^{-1}(W)$ of x which does not intersect C.

Remark 4.3.3 [**dense**]Unless $\mathfrak{R}$ is finite the image of $C = M_+(X, \mathfrak{R})$ in the corresponding universal group-like tonus space V has no internal points i.e. the complement to C in V is dense.

Lemma 4.3.4 *[**lem2**] Let $\phi : (X, \mathfrak{R}) \to (Y, \mathfrak{S})$ be a bounded kernel. Then the composition with ϕ defines a map*

$$\phi_* : M_+(X, \mathfrak{R}) \to M_+(Y, \mathfrak{S})$$

which is a morphism of tonus spaces.

Proof: Follows from Lemma 4.3.1.

Remark 4.3.5 [**rem1**]Consider the metric on $M_+(X, \mathfrak{R})$ given by

$$[\mathbf{eqem1}]\nu(\mu_1, \mu_2) = sup_{U \in \mathfrak{R}} |\mu_1(U) - \mu_2(U)| \tag{79}$$

Remark 4.3.6 The proof of Lemma 4.3.4 implies that if ϕ is a (sub-)stochastic kernel then the corresponding map $M_+(\phi)$ does not increase the distances between measures.

Remark 4.3.7 *[rem1]* For any point x of $(X, \mathfrak{R})$ we have the δ-measure δ_x concentrated in x. Evaluating ϕ_* on δ_x we get a measure $\phi_*(\delta_x)$ on $(Y, \mathfrak{S})$ and one verifies easily that it is exactly $\phi(x, -)$. This shows that for any $(X, \mathfrak{R})$, $(Y, \mathfrak{S})$ the map

$$Hom_{\mathcal{K}}((X, \mathfrak{R}), (Y, \mathfrak{S})) \to Hom_T(M_+(X, \mathfrak{R}), M_+(Y, \mathfrak{S}))$$

is a monomorphism. We will see below in Theorem 4.3.12 that it is in fact a bijection.

Let μ be a measure on $(X, \mathfrak{R})$ and let $X = \coprod_{i=1}^n X_i$ be a partition of X into a disjoint union of measurable subsets. For any $\delta > 0$ denote by $U(\mu, \delta, (X_i))$ the set of all measures λ on $(X, \mathfrak{R})$ such that for each $i = 1, \ldots, n$ one has

$$|\mu(X_i) - \lambda(X_i)| < \delta.$$

Clearly $U(\mu, \delta, (X_i))$ is an open neighborhood of μ in the standard topology.

Lemma 4.3.8 *[lem55] Subsets of the form $U(\mu, \delta, (X_i))$ form a fundamental system of open neighborhoods of μ in the standard topology.*

Proof: If $X = \coprod_{i=1}^n X_i$ and $X = \coprod_{j=1}^m Y_j$ are two measurable partitions of X then $X = \coprod (X_i \cap Y_j)$ is also a measurable partition of X. Let $\delta > 0$ be a real number and k be an integer such that $k \geq n$ and $k \geq m$. Let λ be an element of $U(\mu, \delta/k, (X_i \cap Y_j))$. Then

$$|\mu(X_i) - \lambda(X_i)| = |\sum_{j=1}^m (\mu(X_i \cap Y_j) - \lambda(X_i \cap Y_j)| \leq \sum_{j=1}^m |(\mu(X_i \cap Y_j) - \lambda(X_i \cap Y_j)| \leq (m/k)\delta \leq \delta$$

i.e. $\lambda \in U(\mu, \delta, (X_i))$. Similarly $\lambda \in U(\mu, \delta, (Y_j))$ and we conclude that the intersection of two subsets of the type we consider contains a third subset of the same type.

The standard topology is generated by the maps $f_* : \mu \mapsto \int f d\mu$ for bounded non-negative measurable functions f. In particular for any μ finite intersections of subsets of the form

$$U(\mu, \epsilon, f) = \{\lambda : |\int f d\mu - \int f d\lambda| < \epsilon\}$$

form a fundamental system of open neighborhoods of μ. It remains to show that any neighborhood of the form $U(\mu, \epsilon, f)$ contains a neighborhood of the form $U(\mu, \delta, (X_i))$ i.e. that for any f and any $\epsilon > 0$ there exists a partition $X = \coprod X_i$ and $\delta > 0$ such that for any λ satisfying

$$|\mu(X_i) - \lambda(X_i)| < \delta$$

we have

$$|\int f d\mu - \int f d\lambda| < \epsilon.$$

Without loss of generality we may assume that $f(x) < 1$ for all $x \in X$. Let $n > 0$ be an integer. For $k = 0, \ldots, n-1$ set $I_k = [k/n, (k+1)/n)$. Then

$$[0, 1) = \coprod_{k=0}^{n-1} I_k$$

is a measurable partition of the interval $[0, 1)$. Let further $X_k = f^{-1}(I_k)$ and let

$$f_n = \sum_{k=0}^{n-1} k/nF_k$$

where F_k is the indicator function of X_k. By construction we have $f(x) \geq f_n(x)$ and $f(x) - f_n(x) < 1/n$ for all $x \in X$. For any λ we have

$$|\int f d\mu - \int f d\lambda| \leq |\int (f - f_n) d\mu - \int (f - f_n) d\lambda| + |\int f_n d\mu - \int f_n d\lambda| \leq$$

$$\leq |\int (f - f_n) d\mu| + |\int (f - f_n) d\lambda| + \sum_{k=0}^{n-1} k/n |\mu(X_k) - \lambda(X_k)| \leq$$

$$\leq \mu(X)/n + \lambda(X)/n + \sum_{k=0}^{n-1} k/n |\mu(X_k) - \lambda(X_k)| \leq$$

We also have:

$$\lambda(X) = \sum_{k=0}^{n-1} \lambda(X_k)| \leq \sum_{k=0}^{n-1} |\mu(X_k) - \lambda(X_k)| + \sum_{k=0}^{n-1} \mu(X_k) = \sum_{k=0}^{n-1} |\mu(X_k) - \lambda(X_k)| + \mu(X)$$

and therefore

$$|\int f d\mu - \int f d\lambda| \leq 2\mu(X)/n + \sum_{k=0}^{n-1} (k+1)/n |\mu(X_k) - \lambda(X_k)| \leq$$

$$\leq 2\mu(X)/n + (1 + 1/n) \sum_{k=0}^{n-1} |\mu(X_k) - \lambda(X_k)|$$

To find n, δ such that $U(\mu, \delta, (X_k)_{k=0}^{n-1})$ is contained in $U(\mu, \epsilon, f)$ it is sufficient now to choose n such that $2\mu(X)/n < \epsilon$ and then choose δ such that $(n+1)\delta < \epsilon - 2\mu(X)/n$.

Let $M_*(X, \mathfrak{R})$ be the universal group-like tonus space associated with $M_+(X, \mathfrak{R})$ i.e. the space of signed measures on $(X, \mathfrak{R})$ with the topology defined by the canonical map

$$p : M_+(X, \mathfrak{R}) \times M_+(X, \mathfrak{R}) \to M_*(X, \mathfrak{R})$$

For any $f \in M^+(X, \mathfrak{R})$ the map $f_* : M_+(X, \mathfrak{R}) \to \mathbf{R}_{\geq 0}$ defines a map $M_*(X, \mathfrak{R}) \to \mathbf{R}$ which we will also denote by f_*.

Lemma 4.3.9 *[***imp1***] The topology on $M_*(X, \mathfrak{R})$ coincides with the topology defined by the linear functionals f_* for $f \in M^+(X, \mathfrak{R})$.*

Proof: Let $\mu = \mu_+ - \mu_-$ be an element of $M_*(X, \mathfrak{R})$ and U be a subset in $M_*(X, \mathfrak{R})$ which contains μ and such that $p^{-1}(U)$ is open in $M_+(X, \mathfrak{R}) \times M_+(X, \mathfrak{R})$. We need to verify that there exists a finite set $f_1, \ldots, f_n$ of elements of $M^+(X, \mathfrak{R})$ and $\delta > 0$ such that for any $\lambda = \lambda_+ - \lambda_-$ in $M_*(X, \mathfrak{R})$ satisfying

$$|\int f_i d\lambda - \int f_i d\mu| < \delta$$

for all $i = 1, \ldots, n$, we have $\lambda \in U$. The condition that $p^{-1}(U)$ is open together with Lemma 4.3.8 implies that there exists $\epsilon > 0$ and a measurable partition $X = \coprod_{i=1}^m X_i$ such that for any pair of measures λ_+, λ_- satisfying

$$|\lambda_+(X_i) - \mu_+(X_i)| < \epsilon|$$

$$|\lambda_-(X_i) - \mu_-(X_i)| < \epsilon|$$

one has $\lambda_+ - \lambda_- \in U$.

Proposition 4.3.10 *[**tem1**] The map $f \mapsto f_*$ gives a bijection*

$$M^+(X, \mathfrak{R}) \to Hom_T(M_+(X, \mathfrak{R}), \mathbf{R}_{\geq 0}).$$

Its inverse takes a map ϕ of tonus spaces to the function f such that for each $x \in X$ one has $f(x) = \phi(\delta_x)$.

Proof: Let $\phi : M_+(X, \mathfrak{R}) \to \mathbf{R}_{\geq 0}$ be a morphism.

Corollary 4.3.11 *[**definedby**] Let $f, g : M_+(X, \mathfrak{R}) \to \mathbf{R}_{\geq 0}$ be two morphisms of tonus spaces which coincide on measures of the form δ_x for all $x \in X$. Then $f = g$.*

Theorem 4.3.12 *[**t1**] The functor $\mathcal{K} \to T$ sending $(X, \mathfrak{R})$ to $M_+(X, \mathfrak{R})$ is a full embedding. I.e. For any measurable spaces $(X, \mathfrak{R})$, $(Y, \mathfrak{S})$ the map*

$$[\mathbf{mm}] Hom_{\mathcal{K}}((X, \mathfrak{R}), (Y, \mathfrak{S})) \to Hom_T(M_+(X, \mathfrak{R}), M_+(Y, \mathfrak{S})) \tag{80}$$

is a bijection. Its inverse takes a map ϕ of tonus spaces to the kernel ψ such that for each $x \in X$ the measure $\phi(x, -)$ is $f(\delta_x)$.

Proof: We already noted in Remark 4.3.7 that the map (80) is injective. To show that it is surjective consider a morphism $\phi : M_+(X, \mathfrak{R}) \to M_+(Y, \mathfrak{S})$ of tonus spaces. Let U be a measurable subset of Y and let I_U be its indicator function. The composition of ϕ with the morphism $M_+(Y, \mathfrak{S}) \to \mathbf{R}_{\geq 0}$ defined by I_U is, by Proposition 4.3.10 a measurable function on $(X, \mathfrak{R})$ whose value on $x \in X$ is $\phi(\delta_x)(U)$. Therefore, a map $\psi : X \times \mathfrak{S} \to \mathbf{R}_{\geq 0}$ of the form $\psi(x, U) = \phi(\delta_x)(U)$ is a kernel. It remains to show that the map $\psi_* : M_+(X, \mathfrak{R}) \to M_+(Y, \mathfrak{S})$ defined by this kernel is ϕ. We know that it coincides with ϕ on delta measures. Since the measurable functions on $(Y, \mathfrak{S})$ distinguish elements of $M_+(Y, \mathfrak{S})$ it is sufficient to check that the compositions of ϕ and ψ_* with any map $M_+(Y, \mathfrak{S}) \to \mathbf{R}_{\geq 0}$ coincide. This follows from Corollary 4.3.11.

4.4 Radditive functors on $\mathcal{K}$

Recall that a contravariant functor F from a category C with finite coproducts and initial object 0 is called radditive if $F(0) = pt$ and $F(X \coprod Y) = F(X) \times F(Y)$. We let $R(C)$ denote the full subcategory in the category of all contravariant functors formed by radditive functors. For general properties of radditive functors see [], [].

Lemma 4.4.1 *[**lrf1**] Let C be a category as above and assume that finite coproducts in C coincide with finite products (in particular $pt = 0$). Then $R(C)$ is equivalent to the category of contravariant functors F from C to the category of abelian semi-groups such that $F(X \coprod Y) = F(X) \times F(Y)$.*

Proof: In the case of an additive C (i.e. under the additional assumption that morphisms in C can be subtracted) the statement is proved in []. The same proof works without subtraction.

4.5 Accessible spaces

4.6 Accessible enrichment of $\mathcal{K}$

Let $(X, \mathfrak{R})$, $(Y, \mathfrak{S})$ be measurable spaces. For any bounded measure μ on $(X, \mathfrak{R})$ and a bounded measurable function f on $(Y, \mathfrak{S})$ consider the map

$$\eta(\mu, f) : Hom_{\mathcal{K}}((X, \mathfrak{R}), (Y, \mathfrak{S})) \to \mathbf{R}_{\geq 0}$$

sending ϕ to

$$f \circ \phi \circ \mu : \mathbf{1} \to (X, \mathfrak{R}) \to (Y, \mathfrak{S}) \to \mathbf{1}.$$

Define the *standard topology* on $Hom_{\mathcal{K}}((X, \mathfrak{R}), (Y, \mathfrak{S}))$ as the weakest topology with respect to which all maps $\eta(\mu, f)$ are continuous.

Lemma 4.6.1 *[**lae1**] The set $Hom_{\mathcal{K}}((X, \mathfrak{R}), (Y, \mathfrak{S}))$ with the standard topology and the obvious operations of addition and multiplication by scalar is a closed, Hausdorf tonus space.*

Proof: ???

Lemma 4.6.2 *[**lem0**] The composition of morphisms in $\mathcal{K}$ defines maps of tonus spaces of the form*

$$Hom_{\mathcal{K}}((X, \mathfrak{R}), (Y, \mathfrak{S})) \otimes Hom_{\mathcal{K}}((Y, \mathfrak{S}), (Z, \mathfrak{T})) \to Hom_{\mathcal{K}}((X, \mathfrak{R}), (Z, \mathfrak{T})).$$

Proof: ???

Remark 4.6.3 [**nottopen**] Note that the maps of topological spaces

$$Hom_{\mathcal{K}}((X, \mathfrak{R}), (Y, \mathfrak{S})) \times Hom_{\mathcal{K}}((Y, \mathfrak{S}), (Z, \mathfrak{T})) \to Hom_{\mathcal{K}}((X, \mathfrak{R}), (Z, \mathfrak{T}))$$

defined by composition of morphisms need not be continuous if we take the standard topology on the right and the product of the standard topologies on the left.

4.7 Notes

To the relativistic Brownian motion. A physical formulation of the problem. There is a particle p moving according to the Brownian motion pattern on a physical line L with a marked Borel subset B. There are three observers X, N_1, N_2 all moving inertially relative to each other. Observer X fixes the act of observation of the particle by N_1 and the result of the observation (particle is in point $l_1 \in L$). He further fixes an act of observation of the same particle by N_2 and bets that N_2 observed the particle in B. What is the probability that he won?

The relative velocities of the observers with respect to each other and to the line are known. Observer X has a clock. For simplicity assume that all the observers are moving along the line L.

Here is another version. There is a physical line L with a 'Brownian motion field' F. An experimenter X which is located at point 0 of L and has a clock T creates an apparatus A which moves along L with a constant speed v. At time $s \in T$ the experimenter emits a light signal. When A receives this signal it places a particle p at its current location on L. From this point on the movement of p is controlled by F. At time $t \in T$ the experimenter emits a second light signal. When A receives this signal it emits a light signal along L which when it reaches p reflects back. When A receives the reflected signal it emits a light signal to X who notices the time of its arrival.

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