A categorical approach to the probability theory

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1 Introduction

Let us look at the classical approach to model mathematically a deterministic process of some sort. One starts with a set (usually with an additional structure) whose points correspond to possible states of the system in question. A change in the state of the system is modeled as a map from this set to itself. A "process" is usually a family of such maps – one for each interval $[t_0, t_1]$ of the line representing time, which satisfy the obvious composition condition for intervals of the form $[t_0, t_1]$, $[t_1, t_2]$ and $[t_0, t_2]$. In particular any (deterministic) computer program which takes t_0, t_1, and the state of the system at time t_0 as an input and produces the state of the system at time t_1 as an output defines a "process" in the sense specified above.

If the program we use is not deterministic but uses a random number generator to compute new values of the variables from the old ones it does not define such a process.

Consider now the case when we have a process whose computer model is based on a randomized algorithm to produce the new values of the variables from the old ones. As an example we may look at a simple population dynamics model where the the state of the system is determined by the number of organisms currently alive, time is discrete and to produce the state at the next moment of time our algorithm uses a random number generator to determine whether a given organism survives (with probability p) or dies (with probability $1 - p$).

Note that all the notions used in the mathematical description of a deterministic process naturally belong to the language of the category theory: we have a set X and a family of morphisms (maps) $f_{[t_0,t_1]} : X \to X$ satisfying the composition condition.

The stochastic category described below allows one to repeat the same description in a randomized case simply by replacing the category of sets with the stochastic category.

2 Expansion categories

1 Measurable spaces

In this section we recall the basic notions related to measurable spaces. Define the category $\mathcal{M}$ of measurable spaces as follows:

Objects are measurable spaces i.e. pairs of the form (X, A) where X is a set and A is a σ-algebra of subsets of X.

Morphisms from (X, A) to (Y, B) are maps of sets $f : X \to Y$ such that for each $V \in B$ one has $f^{-1}(V) \in A$.

Compositions and identities correspond to the compositions of maps of sets and to the identity maps of sets.

2 Stochastic categories

We define the *extended stochastic category* $\mathcal{E}$ as follows. Objects of $\mathcal{E}$ are pairs (X, A) where X is a set and A is a σ-algebra of subsets of X i.e. objects are measurable spaces. Morphisms in $\mathcal{E}$ will be called *kernels*.

Definition 2.1 *[**d1**] A kernel $f = f(x, U)$ from (X, A) to (Y, B) is a function*

$$f(-, -) : X \times B \to [0, \infty]$$

such that for any $x \in X$ the function

$$U \mapsto f(x, U)$$

is a measure on (Y, B) and for any $U \in B$ the function

$$x \mapsto f(x, U)$$

is a measurable function on (X, A).

Remark 2.2 *See [1].*

To define composition of kernels we need a lemma.

Lemma 2.3 *[**comp1**] Let f be a kernel $(X, A) \to (Y, B)$ and $g : Y \to [0, \infty]$ be a non-negative measurable function on Y. Then the function*

$$f^*(g) : x \mapsto \int_Y g df(x, -)$$

is a measurable function on (X, A).

Proof: Consider the class $\mathcal{C}$ of all g such that $f^*(g)$ is measurable. By definition of a value morphism this class contains defining functions I_U of subsets U in B. Hence it contains all non-negative simple functions on (Y, B). The continuity property of the integral (e.g. [**?**, Th.15.1(iii),p.204]) implies that if $0 \leq g_n \uparrow g$ where g_n are in $\mathcal{C}$ then g is in $\mathcal{C}$. By [**?**, Th.13.5, p.185] the smallest class satisfying these two properties contains all measurable functions.

Now let $f : (X, A) \to (Y, B)$, $g : (Y, B) \to (Z, C)$ be two kernels. Consider the function on $X \times C$ of the form

$$[\mathbf{comp2}](x, W) \mapsto \int_Y g(-, W) df(x, -) \tag{1}$$

This function is well defined since $g(-, W)$ is measurable. For each W it is a measurable function on (X, A) by Lemma 2.3. On the other hand for any x the function

$$W \mapsto \int_Y g(-, W) df(x, -)$$

is a measure on (Y, C) by the standard properties of the integral. Therefore, (1) defines a kernel from (X, A) to (Z, C) which we denote by $g \circ f$ and call the composition of f and g.

For every (X, A) the kernel Id which takes x to the measure δ_x concentrated in x is the identity morphism. The following three lemmas imply that our composition is associative and therefore measure spaces, kernels and compositions (1) define a category. We denote this category by $\mathcal{E}$ and call the extended stochastic category.

Lemma 2.4 *[***funcmes***] Let μ be a measure on (X, A) and $f : (X, A) \to (Y, B)$ a kernel. Then the function $f_*(\mu)$ on B of the form*

$$U \mapsto \int f(x, U) d\mu$$

is a measure on (Y, B).

Proof: Obvious.

Lemma 2.5 *[***tudysyudy***] Let $f : (X, A) \to (Y, B)$ be a kernel, μ a measure on (X, A) and g a measurable non-negative function on (Y, B). Then one has*

$$\int f^*(g) d\mu = \int g df_*(\mu)$$

Proof: If g is the simple function corresponding to a subset $U \in B$ then our equality holds by definitions. For a general g the result follows by the same continuity argument as in the proof of Lemma 2.3.

Lemma 2.6 *[***assos***] The composition of kernels defined by (1) is associative.*

Proof: It follows immediately from definitions and Lemma 2.5.

For a topological space X we will write simply X instead of the usual $(X, \mathcal{B})$ for the measure space with the underlying set X and the underlying σ-algebra the Borel σ-algebra on X. We will further consider sets as topological spaces with the discrete topology (all subsets are open). Combining these two conventions we will write X for the measure space with the underlying set X and the underlying σ-algebra of all subsets of X.

Example 2.7 *[***ex0***]*For any (X, A) there is a unique kernel from $\emptyset$ to (X, A). Therefore $\emptyset$ is the initial object of the extended stochastic category. Since there is a unique measure on $\emptyset$ there is also a unique kernel from any (X, A) to the empty set i.e. $\emptyset$ is also the final object.

Example 2.8 *[***ex1***]*We will denote the object of the extended stochastic category corresponding to the one element set by $\mathbf{1}$. A morphism from $\mathbf{1}$ to (X, A) is the same as a measure on (X, A). A morphism from (X, A) to $\mathbf{1}$ is a non-negative measurable function on (X, A). In particular

$$[\mathbf{h11}]Hom(\mathbf{1}, \mathbf{1}) = \mathbf{R}_{\geq 0} \cup \{\infty\} \tag{2}$$

and for any (X, A) the composition pairing

$$Hom(\mathbf{1}, (X, A)) \times Hom((X, A), \mathbf{1}) \rightarrow Hom(\mathbf{1}, \mathbf{1})$$

takes (μ, f) to $\int f\mu$. Note that the composition on (2) is of the form $(a, b) \mapsto ab$ where $0\infty = \infty 0 = 0$ as is usually assumed in the measure theory (cf. []).

Example 2.9 [**matrixex**] Let $\mathbf{n}$ be the measure space with the underlying set $\{1, \ldots, n\}$ and the σ-algebra of all subsets. Then $Hom(\mathbf{n}, \mathbf{n})$ is the set of $n \times n$ matrices with entries from $[0, \infty]$. The composition is given by the product of matrices.

Let (X, A), (Y, B) be measurable spaces and let $f : X \rightarrow Y$ be a measurable map i.e. a map of sets $f : X \rightarrow Y$ such that for $U \in B$ one has $f^{-1}(U) \in A$. Sending $x \in X$ to the measure $\delta_{f(x)}$ on Y concentrated in $f(y)$ defines a morphism from (X, A) to (X, B) in the extended stochastic category. To verify the integrability condition note that for a subset U in Y the function $x \mapsto \delta_{f(x)}(U)$ is the characteristic function of the subset $f^{-1}(U)$. Hence the second condition of Definition 2.1 is equivalent to the condition that f is measurable. This construction defines a functor from the category of measurable spaces and measurable maps to the extended stochastic category. To distinguish morphisms in $\mathcal{E}$ which correspond to maps of measure spaces from the general morphisms we will call the former *deterministic* morphisms.

Example 2.10 *[***ex5***]*Let $\mu : \mathbf{1} \rightarrow (X, A)$ be a measure on (X, A) and $f : (X, A) \rightarrow (Y, B)$ a measurable map considered as a morphism in the value category. Then $f \circ \mu = f_*(\mu)$ is the "direct image" of μ with respect to f.

Example 2.11 *[***retract***]*Let (X, A) be a measure set and (U, A_U) be a measurable subset of X considered with the induced σ-algebra. Then the embedding $(U, A_U) \rightarrow (X, A)$ can be split by a projection p where $p(x, -)$ is zero for $x \in X - U$ and is the measure concentrated in x for $x \in U$. Hence any measurable subset (including the empty one) of a measure space is canonically a retract of this space in $\mathcal{E}$.

The functor from the category of measurable spaces to the extended stochastic category does not reflect isomorphisms i.e. some morphisms of measurable spaces may become isomorphisms when considered in $\mathcal{E}$. Let (Y, B) be a measurable space and $f : X \rightarrow Y$ a be any surjection of sets. Let $f^{-1}(B)$ be the σ-algebra on X which consists of subsets of the form $f^{-1}(U)$ for $U \in B$. Then measures on $(X, f^{-1}(B))$ are in one-to-one correspondence with measures on (Y, B). In particular for each point $b \in B$ we have a measure f_b on $(X, f^{-1}(A))$ corresponding to the delta measure δ_b on (Y, B). Sending b to f_b gives us a kernel $(Y, B) \rightarrow (X, f^{-1}(A))$ and one verifies easily that it is inverse to the obvious kernel $(X, f^{-1}(B)) \rightarrow (Y, B)$. Hence, from the point of view of the extended stochastic category, the measurable spaces (Y, B) and $(X, f^{-1}(B))$ are indistinguishable.

For measure spaces (X, A), (X', A') the measure space $(X \coprod X', A \coprod A')$ is easily seen to be both a product and a coproduct of (X, A) and (X', A') in $\mathcal{E}$. Together with Example 2.7 it shows that $\mathcal{E}$ has both finite products and finite coproducts which coincide. For any two objects the set of morphisms between them is an abelian semi-group and moreover a "module" over $\mathbf{R}_+ \cup \{\infty\}$. However (since we do not allow negative measures) morphisms can not be subtracted and therefore $\mathcal{E}$ is not an additive category.

Let (X_α, A_α) be a family of measure spaces. Then there are two obvious ways to define a σ-algebra on $\coprod X_\alpha$. Let $A_\alpha^\cap$ denote the σ-algebra generated by elements of the form $U \subset X_\alpha \subset \coprod X_\alpha$ for all α and all U in A_α. Let further $A_\alpha^\cup$ denote the set (which is clearly a σ-algebra) of subsets U in $\coprod X_\alpha$ such that for each α one has $U \cap X_\alpha \in A_\alpha$. Our notations are explained by the following result.

Lemma 2.12 *[***prcopr***] The measure space $(\coprod X_\alpha, A_\alpha^\cap)$ (resp. $(\coprod X_\alpha, A_\alpha^\cup)$) is the product (resp. the coproduct) of the family (X_α, A_α) in V.*

Proof: ???

The families $A_\alpha^\cap$ and $A_\alpha^\cup$ coincide if our family is finite or countable but are different in general. In particular the countable products and coproducts in V coincide.

Example 2.13 [**prcopr2**]The set of natural numbers $\mathbf{N}$ considered with the σ-algebra of all subsets is both the product and the coproduct of a countable number of copies of $\mathbf{1}$. The sets $Hom_V(\mathbf{N}, \mathbf{1})$ and $Hom_V(\mathbf{1}, \mathbf{N})$ can both be identified with the set $[0, \infty]^{\mathbf{N}}$ of infinite sequences of (extended) non-negative real numbers.

Lemma 2.14 *[***l1***] Let G be a finite group of measurable automorphisms of a measure space (X, A). Then the measure space $(X/G, A^G)$ is the categorical quotient of (X, A) in V with respect to the action of G.*

Proof: ???

3 Bounded value category

A morphism $f : (X, A) \to (Y, B)$ is called bounded if the function

$$\beta_f : x \mapsto f(x, Y)$$

is a bounded function on X. Note that this condition means in particular that β_f takes only finite values i.e. that for any x the measure $f(x, -)$ on (Y, B) is finite. The composition of bounded morphisms is bounded and therefore measure spaces and bounded morphisms form a subcategory V_b in V called the bounded value category.

For (X, A), (Y, B) consider the measure space $(X \times Y, A \times B)$ where $A \times B$ is the σ-algebra generated by $U \times V$ with $U \in A$ and $V \in B$. If $f : (X, A) \to (Y, B)$ and $f' : (X', A') \to (Y', B')$ are bounded value morphisms define $f \times f'$ as the family which takes (x, x') to the product measure $f(x, -) \times f'(x', -)$ on $Y \times Y$. Standard results about products of finite measures imply that $f \times f'$ is a morphism in the bounded value category. One can easily see that this construction defines a symmetric monoidal structure on V_b which we will denote by $\otimes$ instead of $\times$ to avoid confusion with the category product. The one element set is the unit of this monoidal structure which is why we denote it by $\mathbf{1}$.

Example 3.1 [**net1**] The standard example of trouble which one can get into if one tries to define the product of two measures one of which is not necessarily finite can be found in [**?**, p.78]. The source of the problem seems to lie in the fact that while all measures are continuous with respect to countable filtered colimits (cf. [**?**, Lemma 1.10(a)]) only finite measures are continuous with respect to countable filtered limits ([**?**, Lemma 1.10(b)]). Since limits are required to produce measurable subsets of the product of two measure spaces (e.g. the diagonal), a pair of measures on the factors can not be canonically extended to a measure on the product.

Remark 3.2 For each (X, A) the diagonal $(X, A) \to (X, A) \otimes (X, A)$ and the projection $(X, A) \to \mathbf{1}$ make (X, A) into a (commutative) comonoid inn V_b with respect to the product $\otimes$. Note however that this structure is not canonical i.e. morphisms in V_b are not morphisms of comonodis.

Definition 3.3 *[**impplem**] Let $f : (X, A) \to (Y, B)$ be a bounded value morphism. An implementation of f is a triple $((\Omega, \mathcal{F}), \mathbf{P}, \xi)$ where $(\Omega, \mathcal{F})$ is a measure space, $\mathbf{P} : \mathbf{1} \to (\Omega, \mathcal{F})$ is a finite measure on $(\Omega, \mathcal{F})$ and $\xi : (\Omega, \mathcal{F}) \times X \to (Y, B)$ is a deterministic morphism such that the diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\mathbf{P} \otimes Id} & (\Omega, \mathcal{F}) \otimes X \\
\downarrow & & \downarrow \xi \\
(X, A) & \xrightarrow{f} & (Y, B)
\end{array}
$$

commutes.

Note that in the definition given above we let X denote the object of the value category corresponding to the set X with the σ-algebra of all subsets. The left vertical arrow in our diagram is the deterministic morphism $X \to (X, A)$ which is the identity on the underlying sets.

Remark 3.4 Explain relation to implementations of randomized algorithms.

Let X be a set and (Y, B) a measure space. Consider the set Y^X of all maps of sets from X to Y. For any V in B and any x in X let $A(x, V)$ be the set of all $g : X \to Y$ such that $g(x) \in V$. Let B^X be the σ-algebra on Y^X generated by the subsets $A(x, V)$. We will denote the measure space (Y^X, B^X) by $(Y, B)^X$. Note that it may be considered as the infinite product of as many copies of (Y, B) as there are elements in X.

Lemma 3.5 (Kolmogorov) *[**kol**] Let $f : X \to (Y, B)$ be a bounded value morphism. Then there exists a unique measure μ_f on $(Y, B)^X$ such that for any finite set of pairwise distinct points $x_1, \ldots, x_n$ of X and any finite set $V_1, \ldots, V_n$ of elements of B one has*

$$\mu_f(\cap_{i=1}^n A_{(x_i, V_i)}) = \prod_{i=1}^n f(x_i, V_i)$$

Proof: ???

Example 3.6 [**paths1**] Let $X = T$ be an interval of real line. Then Y^T is the space of paths in Y. An elementary measurable subset $A(t, V)$ in $(Y, B)^T$ is the subset of all paths γ such that $\gamma(t) \in V$. More generally $\cap_{i=1}^n A_{(t_i, V_i)}$ in Y^T is the subset of all paths which pass through V_i at time t_i. Lemma 3.5 asserts that any non-deterministic path $\phi : T \to (Y, B)$ defines a measure on $(Y, B)^T$ such that the "size" of $\cap_{i=1}^n A_{(t_i, V_i)}$ relative to this measure is the product of the probabilities (determined by ϕ) that t_i lands in V_i.

Let $ev : (Y,B)^X \otimes X \to (Y,B)$ be the evaluation morphism $(g,x) \mapsto g(x)$. Our choice of the σ-algebra on Y^X implies immediately that it is a measurable map. Consider μ_f as a morphism $\mathbf{1} \to (Y,B)^X$. Then the diagram

$$\begin{array}{ccc} X & \stackrel{\mu_f \otimes Id}{\longrightarrow} & (Y,B)^X \otimes X \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle ev} \\ X & \stackrel{f}{\longrightarrow} & (Y,B) \end{array}$$

commutes and provides a canonical implementation of the morphism f. The obvious extension of this construction to bounded value morphisms $(X,A) \to (Y,B)$ implies the following result.

Lemma 3.7 *[**hasanimpl**] For any bounded value morphism $f : (X,A) \to (Y,B)$ the diagram*

$$\begin{array}{ccc} X & \stackrel{\mu_f \otimes Id}{\longrightarrow} & (Y,B)^X \otimes X \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle ev} \\ (X,A) & \stackrel{f}{\longrightarrow} & (Y,B) \end{array}$$

where μ_f is the measure of Lemma 3.5, is an implementation of f.

Remark 3.8 Let $f_\alpha : (X_\alpha, A_\alpha) \to (Y,B)$ be a countable family of morphisms in V. Our definitions imply that $\coprod f_\alpha$ is a bounded morphism if and only if the functions β_{f_α} are uniformly bounded. This observation shows in particular that $(\coprod X_\alpha, \coprod A_\alpha)$ is not a coproduct of our family in V.

Similarly for $f_\alpha : (X,A) \to (Y_\alpha, A_\alpha)$, the family which sends x to the measure $\sum f_\alpha(x,-)$ is not a bounded morphism unless this measure is finite i.e. unless

$$\sum \beta_{f_\alpha} < \infty$$

everywhere on X, which shows that $(\coprod Y_\alpha, \coprod B_\alpha)$ is not a product of our family in V.

One can also see (cf. 5.3 below) that sending a family (X_α, A_α) to the coproduct space $(\coprod X_\alpha, \coprod A_\alpha)$ is not a functor from the category of families of objects in V to V. These properties make the bounded value category to be of limited use. Instead one uses the stochastic category considered in the following section.

4 The stochastic category

A morphism $f : (X,A) \to (Y,B)$ is called stochastic if for any x one has $f(x,Y) = 1$ i.e. if the corresponding measures are probability measures. Composition of stochastic morphisms is stochastic. The subcategory generated by stochastic morphisms is called the *stochastic category*. We denote it by S.

Remark 4.1 [**expl1**]Stochastic morphisms from a measure space to itself are known in probability theory as stochastic or Markov kernels.

Example 4.2 [**exsc1**]One obtains an important class of stochastic morphisms as follows. Consider an (idealized) randomized computer algorithm A which takes as an input a sequence of real numbers $r_1, \ldots, r_m$ and produces as an output a sequence of real numbers $s_1, \ldots, s_n$. Let us assume that our computer has access only to the usual (i.e. equally distributed) random numbers on the interval $I = [0,1]$. Then such an algorithm defines a map

$$\tilde{a} : \mathbf{R}^m \times I^\infty \to \mathbf{R}^n$$

where $\tilde{a}(s_1, \ldots, s_m; \rho_1, \ldots)$ is the result our algorithm will produce for the input $r_1, \ldots, r_m$ if its i-th request for a random number gives ρ_i. Consider the usual Lebesgue measure λ on I^∞. Then sending every $(r_1, \ldots, r_m)$ to the push-out of λ with respect to

$$\tilde{a}_{|(r_1,\ldots,r_m)\times I^\infty} : I^\infty \to \mathbf{R}^n$$

we get a stochastic morphism $a : \mathbf{R}^m \to \mathbf{R}^n$ which we call the morphism corresponding to A. This morphism takes (r, U) where $r \in \mathbf{R}^m$ and $U \subset \mathbf{R}^n$ to the probability that our algorithm will produce a result lying in U when given $r = (r_1, \ldots, r_m)$ as an input.

If A and B are two randomized algorithms such that the output of A can be used as an input for B we map consider the composed algorithm $B \circ A$. It is easy to see that the stochastic morphism corresponding to $B \circ A$ is the composition $b \circ a$ of the stochastic morphisms corresponding to A and B. It is also easy to see that the stochastic morphism corresponding to an algorithm is a deterministic morphism if and only if our algorithm is essentially deterministic i.e. while it may request random numbers at some point the output does not depend on which random number it gets.

Note that for a non-empty (X, A) there are no stochastic morphisms from (X, A) to $\emptyset$. Therefore, while $\emptyset$ is an initial object of the stochastic category it is not a finial object. On the other hand for any (X, A) there is exactly one stochastic morphism from (X, A) to $\mathbf{1}$. Therefore, $\mathbf{1}$ is the final object of the stochastic category (but not of the value category).

For (X, A) and (X', A') the coproduct $(X \coprod X', A \coprod A')$ is easily seen too be the coproduct of (X, A) and (X', A') in the stochastic category. However it is not the product of (X, A) annd (X', A') in the stochastic category since the sum of two probability measures is not a probability measure.

For any measurable map of measure spaces $(X, A) \to (Y, B)$ the corresponding morphism in V is stochastic. Therefore the functor from measure spaces to the value category factors through the stochastic category.

Our description of morphisms from infinite coproducts given above implies the following result.

Lemma 4.3 *[**13**] Let (X_α, A_α) be a family of measure spaces. Then $(\coprod X_\alpha, A_\alpha^\cup)$ is a coproduct of this family in the stochastic category.*

Proof: ???

In view of Lemma 4.3 we will write $\coprod(X_\alpha, A_\alpha)$ instead of $(\coprod X_\alpha, A_\alpha^\cup)$.

Note also that the finite group quotients of Lemma 2.14 remain quotients in the stochastic category.

The tensor product of two stochastic morphisms is a stochastic morphism and therefore the symmetric monoidal structure defined above for the bounded value category gives a similar structure on S.

Example 4.4 *[**markov2**]*Let G be a set which is finite or countable. We consider G as a measure space with respect to the σ-algebra which contains all subsets of G. Then $Hom_{V_b}(G, G)$ is the set of matrices $(p_{ij})_{i,j \in G}$ such that $p_{ij} \geq 0$, for any i the sum $p_i = \sum_j p_{ij}$ is finite and the set of numbers p_i is bounded. The set $Hom_S(G, G)$ is the set of stochastic matrices with rows and columns numbered by elements of G. The composition of morphisms corresponds in this description to multiplication of matrices. If P is an element of this set and $f : G \to \mathbf{1}$ a morphism in V (corresponding to a random variable by 2.8) then the sequence of random variables $f_n = f \circ G^n$ is the Markov chain generated by the stochastic matrix P.

For any (X, A) let

$$[\mathbf{tr1}] tr_n = \sum_{i=1}^{n} pr_i : (X, A)^{\otimes n} \to (X, A) \tag{3}$$

be the morphism which sends a point $(x_1, \ldots, x_n)$ to the measure $\sum_{i=0}^{n} \delta_{x_i}$. For $n = 0$ we take tr_0 to be the zero morphism. The following lemma gives an important property of stochastic morphisms.

Lemma 4.5 *[**comm**] For any stochastic morphism $f : (X, A) \to (Y, B)$ and any $n \geq 0$ the diagram*

$$\begin{array}{ccc} (X,A)^{\otimes n} & \xrightarrow{f^{\otimes n}} & (Y,B)^{\otimes n} \\ {\scriptstyle tr_n}\downarrow & & \downarrow{\scriptstyle tr_n} \\ (X,A) & \xrightarrow{f} & (Y,B) \end{array}$$

commutes.

Proof: In view of the definition of tr_n it is sufficient to verify that $pr_i \circ f^{\otimes n} = f \circ pr_i$ for all i. More generally it is sufficient to see that for a morphism $f : X \to Y$ and a stochastic morphism $f' : X' \to Y'$ one has $pr_Y \circ (f \otimes f') = f \circ pr_X$ i.e. that the square

$$\begin{array}{ccc} X \otimes X' & \xrightarrow{f \otimes f'} & Y \otimes Y' \\ {\scriptstyle pr_X}\downarrow & & \downarrow{\scriptstyle pr_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Let e be the canonical stochastic morphism from an object to the point. We have

$$pr_Y \circ (f \otimes f') = (Id_Y \otimes e) \circ (f \otimes f') = f \otimes (e \circ f') = f \otimes e = f \circ pr_X$$

where the third equality holds since $e \circ f' = e$ eactly means that f' is stochastic.

5 Branching morphisms and branching category

For a measure space (X, A) let $S^n(X, A) = (X, A)^n / \Sigma_n$ be the n-th symmetric power of (X, A). For $n = 0$ we set $S^0(X, A) := \mathbf{1}$ for all (X, A) including the empty set. We further set

$$S^\bullet(X, A) = \coprod_{n \geq 0} S^n(X, A)$$

Example 5.1 [**ex6**]We obviously have:

$$S^\bullet(\emptyset) = \mathbf{1}$$

and

$$S^\bullet(\mathbf{1}) = \mathbf{N}$$

Lemma 2.14 shows that for each n, $S^n(-)$ is a functor from the bounded value category to itself. Since $S^\bullet(X, A)$ is the coproduct of $S^n(X, A)$ in V we conclude that $S^\bullet(-)$ is a functor from the bounded value category to the value category. Finally, since coproduct of stochastic morphisms is stochastic we conclude that both the individual symmetric powers $S^n(X, A)$ and the total symmetric power $S^\bullet(X, A)$ are functors from the stochastic category to itself.

Remark 5.2 For a sufficiently nice space (X, A) the space $S^\bullet(X, A)$ is isomorphic to the space of integer-valued measures $M((X, A), \mathbf{Z}_+)$ on (X, A). This interpretation of the total symmetric power appears in some probabilistic texts on branching processes (e.g. [**?**]). The theory of measure valued branching processes studies the analogs of branching processes with the integer-valued measures replaced by more general measures.

Remark 5.3 *[***ex7***]*One can easily see that the total symmetric power $S^\bullet$ is not a functor from V_b to V_b. Indeed consider a morphism $a : \mathbf{1} \to \mathbf{1}$ where $a > 1$ (see (2)). Then $S^n(a) = a^n$ and $S^\bullet(a)$ is not bounded since the volumes of corresponding measures on $\mathbf{N}$ are $a, a^2, \ldots$ which is an unbounded function on $\mathbf{N}$.

Definition 5.4 *[***d2***] A branching morphism ϕ from (X, A) to (Y, B) is a morphism in S of the form $(X, A) \to S^\bullet(Y, B)$.*

The functor $S^\bullet(-)$ is an extension to S of a functor with the same notation and meaning on the category of measure spaces and measurable maps to itself. In particular the obvious monad structure

$$S^\bullet \circ S^\bullet \to S^\bullet$$

$$Id \to S^\bullet$$

of the total symmetric power functor on sets defines a monad structure on $S^\bullet$ on S. We define the *branching category* B as the category of free algebras over $S^\bullet$. The objects of B are again measure spaces (X, A) and morphisms from (X, A) to (Y, B) are the branching morphisms of Definition 5.4.

Remark 5.5 [**notfree**] In view of Lemma 4.3 algebras over $S^\bullet$ are exactly commutative monoids in S with respect to $\otimes$.

We will write $\phi : [X, A] \to [Y, B]$ for branching morphisms to distinguish them from morphisms in V and S. Let us describe the composition of branching morphisms more explicitly. Observe first that there is a measurable map of measure spaces

$$m : S^\bullet(Y, B) \times S^\bullet(Y, B) \to S^\bullet(Y, B)$$

which makes $S^\bullet(Y, B)$ into a commutative monoid. In view of Lemma 2.14 and the definition of the symmetric product it shows that any morphism ϕ from (X, A) to $S^\bullet(Y, B)$ in V_b defines a family of morphisms of the form

$$\phi_n : S^n(X, A) \to S^\bullet(Y, B)$$

(where we set ϕ_0 to be identically 1). If the original morphism is stochastic so are the morphisms ϕ_n and therefore by Lemma 4.3 they define a morphism

$$\phi_* = \coprod \phi_n : S^\bullet(X, A) \to S^\bullet(Y, B)$$

We can now define the composition of two branching morphisms by the rule:

$$\psi \circ_B \phi := \psi \circ \phi_*$$

Forgetting the $S^\bullet$ algebra structure defines a functor

$$F : B \to S$$

which takes (X, A) to $S^\bullet(X, A)$ and ϕ to the morphism ϕ_* defines above.

Example 5.6 *[***ex8***]*Consider morphisms in the branching category of the form $\phi : [\mathbf{1}] \to [\mathbf{1}]$. Since $S^\bullet(\mathbf{1}) = \mathbf{N}$ we may identify this set with the set of probability measures on $\mathbf{N}$. For any ϕ let $p_\phi = \sum p_i t^i$ be the generating function of this measure. This construction identifies $Hom_B([\mathbf{1}], [\mathbf{1}])$ with formal power series $\sum p_i t^i$ satisfying $p_i \geq 0$ and $\sum p_i = 1$. If ϕ, ψ two endomorphisms of $[\mathbf{1}]$ in B then one has

$$[\textbf{compseries}] p_{\phi\circ\psi} = p_\psi(p_\phi(t)) \tag{4}$$

i.e. in this description the composition of morphisms corresponds to the composition of power series in the reverse order.

Example 5.7 *[***ex10***]*The previous example has an immediate generalization to branching morphisms of the form $\phi : [\mathbf{n}] \to [\mathbf{n}]$ where $\mathbf{n} := \coprod_{i=1}^n \mathbf{1}$ is the set of n elements considered as a measure space with respect to the maximal σ-algebra. Such morphism is a collection of n probability measures on $\mathbf{N}^n$. If we describe these measures through their generating functions we may identify $Hom_B([\mathbf{n}], [\mathbf{n}])$ with the set of n-tuples $(f_1, \ldots, f_n)$ where each f_i is a formal power series in n-variables with non-negative coefficients satisfying the condition $f_i(1, \ldots, 1) = 1$. The composition of morphisms corresponds to the substitution composition for such n-tuples.

The morphism (3) is clearly invariant under the action of the symmetric group and by Lemma 4.3 it defines a bounded value morphism

$$tr_n : S^n(X, A) \to (X, A)$$

which sends the point $x_1, \ldots, x_n$ to the sum of δ-measures $\delta_{x_1} + \ldots + \delta_{x_n}$ (for $n = 0$ our morphism is 0) and which we continue to denote by tr_n. The coproduct of tr_n's is a morphism $tr_* : S^\bullet(X, A) \to (X, A)$ in V. For a stochastic morphism $(X, A) \to S^\bullet(Y, B)$ (i.e. for a branching morphism $\phi : [X, A] \to [Y, B]$) define a value morphism

$$tr(\phi) : (X, A) \to (Y, B)$$

as the composition $tr_* \circ \phi$.

Proposition 5.8 *[***comm2***] For any ϕ as above the diagram*

$$\begin{array}{ccc} S^\bullet(X, A) & \xrightarrow{\phi_*} & S^\bullet(Y, B) \\ {\scriptstyle tr_*}\downarrow & & {\scriptstyle tr_*}\downarrow \\ (X, A) & \xrightarrow{tr(\phi)} & (Y, B) \end{array}$$

commutes.

Proof: By definition of ϕ_* it is sufficient to verify that for any n the diagram

$$\begin{array}{ccccc} (X, A)^{\otimes n} & \xrightarrow{\phi^{\otimes n}} & S^\bullet(Y, B)^{\otimes n} & \xrightarrow{m} & S^\bullet(Y, B) \\ {\scriptstyle tr_n}\downarrow & & {\scriptstyle tr_n}\downarrow & & \downarrow{\scriptstyle tr_*} \\ (X, A) & \xrightarrow{\phi} & S^\bullet(Y, B) & \xrightarrow{tr_*} & (Y, B) \end{array}$$

commutes. The right hand side square consists of morphisms which take a point to the sum of finitely many points and it is easy to verify its commutativity explicitly. The left hand side square commutes by Lemma 4.5.

Corollary 5.9 *[**main1**] For a pair of branching morphisms $\phi : [X, A] \to [Y, B]$, $\psi : [Y, B] \to [Z, C]$ one has*

$$tr(\psi \circ \phi) = tr(\psi) \circ tr(\phi)$$

Proof: This follows immediately from the explicit description of the composition of branching morphisms given above and Lemma 5.8.

Example 5.10 [**ex11**]Consider a branching morphism $\phi : [\mathbf{1}] \to [\mathbf{1}]$ which we describe through the corresponding probability generating function $p_\phi = \sum p_i t^i$ as in Example 5.6. Then $tr(\phi)$ is a morphism $\mathbf{1} \to \mathbf{1}$ i.e. a non-negative number. One can easily see that

$$tr(\phi) = \sum i p_i = p'_\phi(1)$$

where p'_ϕ is the formal derivative of p_ϕ with respect to t. In other words, $tr(\phi)$ is in this case the expectation value of ϕ. For two morphisms ϕ, ψ of this form Corollary 5.9 asserts that

$$tr(\psi \circ \phi) = tr(\psi) tr(\phi).$$

In view of (4) this follows from the equation

$$(p_\phi \circ p_\psi)'(1) = p'_\psi(1) p'_\phi(p_\psi(1)) = p'_\psi(1) p'_\phi(1)$$

where the last equation holds since the $p_\psi(1) = 1$ because ψ is a stochastic morphism.

Example 5.11 [**ex12**]Consider now branching morphisms $[\mathbf{n}] \to [\mathbf{n}]$ as in Example 5.7. For a morphism ϕ of this form $tr(\phi)$ is a morphism $\mathbf{n} \to \mathbf{n}$ i.e. an $n \times n$-matrix (a_{ij}) with entries from $[0, \infty]$. If we represent ϕ a sequence of power series $(f_1, \dots, f_n)$ in variables $t_1, \dots, t_n$ then one gets

$$a_{ij} = \frac{\partial f_i}{\partial t_j}(1)$$

If $\psi = (g_1, \dots, g_n)$ is another such morphism then the statement of Corollary 5.9 is again equivalent to the formula for the differential of a composition combined with the fact that $g_i(1) = 1$ since ψ is stochastic.

3 A categorical view of Markov processes

1 Processes and Kolmogorov equations

2 Trajectory structures

4 Tonus spaces

1 Tonus spaces

Definition 1.1 *[**conus**] A* conus structure *on a set C is an abelian semi-group structure (with unit 0) together with a map*

$$m : \mathbf{R}_{\geq 0} \times C \to C$$

which makes C into a module over $\mathbf{R}_{\geq 0}$ i.e. such that

$$[\mathbf{eqpo1}] m(r, x + y) = m(r, x) + m(r, y) \tag{5}$$

$$[\mathbf{eqpo3}] m(r+s,x) = m(r,x) + m(s,x) \tag{6}$$

$$[\mathbf{eqpo4}] m(rs,x) = m(r, m(s,x)) \tag{7}$$

$$[\mathbf{eqpo6}] m(1,x) = x \tag{8}$$

$$[\mathbf{eqpo5}] m(0,x) = 0 \tag{9}$$

When no confusion is possible we write rx instead of $m(r,x)$. A set with a conus structure is called a conus space*.*

Definition 1.2 *[**dpo1**] A* tonus structure *on a set C is a topology together with a conus structure such that the addition and the multiplication by scalars are continuous.*

Definition 1.3 *[**dpo2**] Let C_1, C_2 be two conus (resp. tonus) spaces. A morphism $f : C_1 \to C_2$ is a map (resp. a continuous map) which commutes with addition and multiplication by scalars.*

We let T denote the category of tonus spaces.

Proposition 1.4 *[**ppo1**] The category T has all limits. The final object of T is the one point space. For any diagram $\mathcal{D}$ of tonus spaces the underlying topological space of $lim(\mathcal{D})$ is the limit of the corresponding diagram of topological spaces and the same is true for the limit of the underlying diagram of conus spaces and abelian semi-groups.*

Proof: Straightforward.

Proposition 1.5 *[**ppo2**] The category T of tonus spaces has colimits. The initial object of T is the one point space.*

Proof: The statement of the proposition follows from Lemmas 1.6-1.8 below and the usual reduction of general colimits to inductive colimits, reflexive coequalizers and finite coproducts.

Lemma 1.6 *[**lpo5**] Let $(C_\alpha, f_{\alpha\beta} : C_\alpha \to C_\beta)$ be an inductive system of tonus spaces. Let C be the colimit of this sequence in the category of sets which we consider with the colimit topology and the obvious operations of addition and multiplication by elements of $\mathbf{R}_{\geq 0}$. Then C is a tonus space and a colimit of our sequence in T.*

Proof: It follows by direct verification using the fact that inductive colimits commute with finite products in the category of topological spaces.

Lemma 1.7 *[**lpo6**] Let C_1, C_2 be tonus spaces, $f, g : C_1 \to C_2$ two morphisms and $s : C_2 \to C_1$ a common section of f and g (i.e. f, g, s form a reflexive coequalizer diagram). Let C be the coequalizer of f and g in the category of sets which we consider with the coequalizer topology and the obvious operations of addition and multiplication by elements of $\mathbf{R}_{\geq 0}$. Then C is a tonus space and a coequalizer of f and g in T.*

Proof: As in the proof of Lemma 1.6 everything follows by direct verification from the fact that reflexive coequalizers commute with finite products.

Lemma 1.8 *[**lpo7**] Let C_1, C_2 be tonus spaces. Let $C = C_1 \times C_2$ and consider C with the topology of the product and the obvious operations of addition and multiplication by elements of $\mathbf{R}_{\geq 0}$. Then C is a tonus spaces which is both the product and the coproduct of C_1 and C_2 in T.*

Proof: The only non-trivial part of the lemma is that C is the coproduct of C_1 and C_2 i.e. that for any tonus space D the map

$$Hom(C, D) = Hom(C_1, D) \times Hom(C_2, D)$$

given by the composition with the embeddings $C_1 \to C$, $C_2 \to C$ is bijective. It is clearly injective and to verify that it is bijective it is enough to prove that a map $f : C_1 \times C_2 \to D$ which is compatible with the algebraic structures and whose restrictions f_1, f_2 to $C_1 \times \{0\}$ and $\{0\} \times C_2$ are continuous is itself continuous. This follows from the fact that $f = m_D \circ (f_1 \times f_2)$ and the continuity of $m_D : D \times D \to D$.

Definition 1.9 *[**grouplike**] A tonus space C is called group-like if the underlying semi-group is a group.*

For the basic definitions related to the topological vector spaces and pre-ordered vector spaces we follow [**?**].

Lemma 1.10 *[**lpo3**] Let V be a group-like tonus space. Then there exists a unique extension of $m : \mathbf{R}_{\geq 0} \times V \to V$ to a continuous map $m : \mathbf{R} \times V \to V$ satisfying the condition*

$$m(r - s, x) = m(r, x) - m(s, x)$$

and with respect to this map V becomes a topological vector space (over $\mathbf{R}$).

Proof: The uniqueness is obvious. It is also obvious that if m as required exists then it makes V into a topological vector space. To prove the existence consider the map $\tilde{m} : \mathbf{NR} \times \mathbf{R}_{\geq 0} \times V \to V$ of the form $\tilde{m}(r, s, x) = m(r, x) - m(s, x)$. The algebraic properties of m imply that it has a decomposition

$$\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0} \times V \to \mathbf{R} \times V \stackrel{m}{\to} V$$

where the first arrow is defined by $(r, s) \mapsto r - s$. Since the first arrow is a strict topological epimorphism and the composition is continuous we conclude that m is continuous.

Lemma 1.11 *[**lpo4**] Let C be a tonus space and let $C \to V_C$ be the universal map from C as an abelian semi-group to an abelian group. Then V has a unique structure of a tonus space such that $C \to V_C$ is a morphism of tonus spaces. With this structure $C \to V_C$ is the universal morphism from C to a group-like tonus space.*

Proof: By (see e.g. []) we may describe V_C as the set of equivalence classes of pairs (x, y), $x, y \in C$ such that $(x_1, y_1) \cong (x_2, y_2)$ if and only if there exists u such that $x_1 + y_2 + u = x_2 + y_1 + u$. As usual we will write $x - y$ for the equivalence class of (x, y). For $r \in \mathbf{R}_{\geq 0}$ set $r(x, y) = (rx, ry)$. In view of 5 this defines a map $\mathbf{R}_{\geq 0} \times V_C \to V_C$ which takes $x - y$ to $rx - ry$ and one verifies easily that it satisfies the conditions 6-9. Let $\pi : C \times C \to V_C$ be the surjection $(x, y) \mapsto x - y$. Consider V_C as topological space with the topology defined by π i.e. such that U is open in V_C if and only if $\pi^{-1}(U)$ is open in $C \times C$. The universal properties of this topology imply immediately that the addition and multiplication by elements from $\mathbf{R}_{\geq 0}$ are continuous for V and we conclude that V has a structure of a tonus space such that $C \to V_C$ is a morphism of tonus spaces. One can see immediately that such a structure is unique.

Definition 1.12 *[**cancellable**] A tonus space C is called pre-group like if the universal map $C \to V_C$ is an injection i.e. if the underlying semi-group is a semi-group with cancellation.*

Definition 1.13 [**reduced**] *A tonus space C is called reduced if it is pre-group like and the topology on C induced by the map $C \to V_C$ coincides with the original topology.*

Definition 1.14 [**closedts**] *A tonus space C is called closed the corresponding universal map $C \to V_C$ is a closed embedding.*

Clearly any closed tonus space is reduced and any reduced is a pre-group like. It is also clear that any group-like tonus space is closed. To produced counter-examples to other implications we will use the following lemma.

Lemma 1.15 [**need1**] *Let $f : C \to V$ be a monomorphism from a tonus space C to a group-like tonus space V and let V_0 be the set of interior points of $f(C)$ in V. Assume that the following two conditions hold:*

1. *the map $C_0 = f^{-1}(V_0) \to V_0$ is a homeomorphism,*
2. *for any $v \in V$ there exist $x, y \in V_0$ such that $v = x - y$.*

Then $V(f) : V_C \to V$ is an isomorphism.

Proof: Clearly $V(f)$ is bijective as a map of sets and continuous. Let us show that it is open. Let V_0 be the set of interior points of $f(C)$ it is open in V and the restriction of f to $C_0 = f^{-1}(V_0)$ is an isomorphism. Consider the diagram:

$$\begin{array}{ccc} C_0 \times C_0 & = & V_0 \times V_0 \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle q_0} \\ V_C & \xrightarrow{V(f)} & V \end{array}$$

where the vertical arrows map (u, v) to $u - v$ and f_0 is the restriction of f to C_0. Our conditions imply that q_0 is surjective. Since V_0 is open in V and the subtraction map $V \times V \to V$ is open (follows from the fact that it is isomorphic to the projection $V \times V \to V$ to one of the factors) we conclude that q_0 is also open. This immediately implies that $V(f)$ is open.

Example 1.16 [**contr2**]Not all reduced tonus spaces are closed. Indeed let C be the subset in $\mathbf{R}^2$ which consists of points (x, y) such that $x \geq 0$ and $y > 0$ and the point $(0, 0)$. Considered with the induced topology and the obvious addition and multiplication by scalars C is a tonus space. Lemma 1.15 implies immediately that the embedding $C \to \mathbf{R}^2$ coincides with the universal embedding to a group-like tonus space. Therefore C is reduced but not closed.

Example 1.17 [**contr1**] Not any pre-group like tonus space is reduced. Consider the subset C in $\mathbf{R}^2$ which consists of (x, y) such that $x, y \geq 0$. Let further U be the subset of elements of C of the form $(x, 0)$ where $x > 0$. Consider the topology on C which is generated by the usual topology coming from $\mathbf{R}^2$ together with the condition that U is open. One verifies immediately that the addition and multiplication by scalars are continuous in this topology. On the other hand Lemma **??** again implies that the embedding $C \to \mathbf{R}^2$ is the universal one. Since in the topology on C induced by this embedding U is not open we conclude that C is pre-group like but not reduced.

Example 1.18 [**expo1**]Not all tonus spaces are pre-group like. Indeed, consider the set $\{0, 1\}$ with the discrete topology, the abelian semi-group structure given by $0+0 = 0$, $0+1 = 1$, $1+1 = 1$ and m given by $m(r, 0) = 0$, $m(r, 1) = 1$ if $r \neq 0$ and $m(0, 1) = 0$. These structures satisfiy all the conditions of Definition 1.2 but the resulting tonus space C is not pre-group like since $V_C = 0$. We will see below (Lemma 1.20) however that all Hausdorf tonus spaces are pre-group like. Note that the spaces in Examples 1.16 and 1.17 are both Hausdorf. Thus a Hausdorf tonus space need not be reduced or closed.

Sending C to $(V_C, C_{red}))$ where C_{red} is the image of C in V_C considered with the topology induced from V_C we get (by Lemmas 1.10, 1.11) a functor from tonus spaces to pairs (V, C) where V is a topological vector space and C is a cone in V. Clearly this functor is a full embedding on the subcategory of reduced tonus spaces and the pair (V, C) is in the image of this embedding if and only if any element of V can be written as $x - y$ where x, y are in C. Recall that a pre-ordered topological vector space is a pair as above such that C is closed in V. Therefore, we get the following result.

Proposition 1.19 *[**embed1**] The category of closed tonus spaces is equivalent to the full subcategory of the category of pre-ordered topological vector spaces (V, C) such that any element of V is of the form $x - y$ for $x, y \in C$.*

Lemma 1.20 *[**lpo1**] Let C be a Hausdorf tonus space then one has:*

1. *C is pre-group like i.e. for any x, y, u in C such that $x + u = y + u$ one has $x = y$*
2. *$m(r, 0) = 0$*

Proof: Let us denote $m(r, x)$ by rx. Consider the first claim. By 8 and 6 for any positive integer n we have $nx = \sum_{i=1}^{n} x$. From this by easy induction we get that for x, y, u as above one has $nx + u = ny + u$. By 5 and 7 we get that

$$x + (1/n)u = y + (1/n)u$$

Since C is Hausdorf a sequence may have only one limit and from the continuity of addition and multiplication by a number and 9 we get

$$x = x + 0u = \lim_{n\to\infty} (x + (1/n)u) = \lim_{n\to\infty} (y + (1/n)u) = y + 0u = y.$$

To get the second claim note that by 5 we have $r0 + r0 = r0 = r0 + 0$ and we conclude from the first part of the proof that $r0 = 0$.

Lemma 1.21 *[**hus**] Let C be a Hausdorf tonus space C. Then V_C is Hausdorf.*

Proof: Consider the natural map $\pi : C \times C \to V_C$. If C is Hausdorf then by Lemma 1.20 we have $\pi^{-1}(0) = \Delta$ where Δ is the diagonal. Since in a Hausdorf space the diagonal is closed and since π is a topological epimorphism we conclude that $\{0\}$ is closed in V_C. Since V_C is a topological vector space this implies in the standard way that V_C is Hausdorf.

Let C be a conus space and let $f_\alpha : C \to C_\alpha$ be a collection of conus maps to tonus spaces C_α. Let $t(f_\alpha)$ be the weakest topology on C which makes all the maps f_α continuous. It is easy to see that with this topology C is a conus space. We will say that the topology on C is defined by the collection f_α.

Lemma 1.22 *[**isred1**] Let C be a pre-group like conus space and let $f_\alpha : C \to C_\alpha$ be a collection of morphisms to reduced tonus spaces. Then C with the induced topology is a reduced tonus space.*

Proof: Let $C \to V_C$ and $C_\alpha \to V_\alpha$ be the universal morphisms to group-like spaces. By universality we get commutative squares

$$\begin{array}{ccc} C & \xrightarrow{f_\alpha} & C_\alpha \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_\alpha} \\ V & \xrightarrow{g_\alpha} & V_\alpha \end{array}$$

such that g_α are continuous. Let $x \in U \subset C$ be an open neighborhood of x in C. We have to show that there is an open neighborhood U' of $p(x)$ in V such that $p^{-1}(U') \subset U$. Since the topology on C is defined by (f_α) there exists a finite set $\alpha_1, \dots, \alpha_n$ and open neighborhoods $W_1, \dots, W_n$ of $f_{\alpha_i}(x)$ in C_α such that U contains $\cap f_{\alpha_i}^{-1}(W_i)$. Since each C_α is assumed to be reduced we have $W_i = p_{\alpha_i}^{-1}(W_i')$ for some W_i' open in V_α. The commutativity of our squares imply now that

$$\cap p^{-1} g_\alpha^{-1}(W_i') \subset U.$$

Remark 1.23 [**impo**] It is important to note that (in the notations of Lemma 1.22) the universal topology on V defined by the topology on C need not coincide with the topology induced by the maps $g_\alpha : V \to V_\alpha$. For an example see **??**.

In the following lemma we keep the notations of Lemma 1.22.

Lemma 1.24 *[**isclosed**] Let C be a pre-group like conus space and $f_\alpha : C \to C_\alpha$ a collection of maps to closed tonus spaces such that if $x \in V$ is an element satisfying $g_\alpha(x) \in C_\alpha$ for all α then $x \in C$. Then with the topology defined by (f_α), C is a closed tonus space.*

Proof: By Lemma 1.22 C is reduced. It remains to check that the image of C in V is closed. Let $x \in V$ be an element outside of C. Then by our assumption there exists α such that $g_\alpha(x)$ is outside C_α. Since C_α are closed this implies that there is a neighborhood W of $g_\alpha(x)$ which does not intersect C_α. Then $g_\alpha^{-1}(W)$ is a neighborhood of x which does not intersect C.

2 Embedding $\mathcal{E}^{op} \to T$

Let (X, A) be a measure space and $M^+(X, A)$ the set of non-negative measurable functions on (X, A). It has an obvious structure of a conus space. Define the standard topology on $M^+(X, A)$ by the condition that a set Z is closed if and only if for any sequence f_n of elements of Z such that $f_n \uparrow f$ we have $f \in Z$.

3 Embedding $\mathcal{E} \to T$

Let (X, A) be a measurable space and let $M_+(X, A)$ be as above the set of all bounded measures on (X, A). Any (bounded, non-negative) measurable function $f \in M^+(X, A)$ defines a map

$$f_* : M_+(X, A) \to \mathbf{R}_{\geq 0}$$

Define the *standard topology* on $M_+(X, A)$ as the weakest topology which makes all maps of the form f_* continuous.

Lemma 3.1 *[**lem4**] A map u from a topological space T to $M_+(X, A)$ is continuous with respect to the standard topology if and only if for any $f \in M^+(X, A)$ the composition*

$$f_* \circ u : T \to \mathbf{R}_{\geq 0}$$

is continuous.

Lemma 3.2 *[**lem1**] The set $M_+(X, A)$ considered with the standard topology and the addition and multiplication by elements of $\mathbf{R}_{\geq 0}$ defined in the obvious is a closed, Hausdorff tonus space.*

Proof: The continuity of the addition and multiplication by scalars follow from Lemma 3.1. To see that the standard topology is Hausdorff consider two measures μ_1 and μ_2 such that $\mu_1 \neq \mu_2$. Then there is a measurable subset $U \in A$ such that $\mu_1(U) \neq \mu_2(U)$. Let f be the indicator function of U. Then for any μ, $f_*(\mu) = \mu(U)$ and if V_1, V_2 are two non-intersecting neighborhoods of $\mu_1(U)$ and $\mu_2(U)$ respectively then $f_*^{-1}(V_i)$ give us two non-intersecting neighborhoods of μ_1 and μ_2.

To see that $C = M_+(X, A)$ is closed in the corresponding vector space V we need to check that if μ_1, μ_2 are two measures such that $x = \mu_1 - \mu_2$ is not in C then there exists a neighborhood N of x in V such that $N \cap C = \emptyset$. By Lemma 1.11, V is universal and therefore any map of the form f_* extends to a continuous map $f_* : V \to \mathbf{R}$. Since x is not in C there exists a measurable subset $U \in A$ such that $x(U) = \mu_1(U) - \mu_2(U) < 0$. Let W be a neighborhood of $x(U)$ which lies in $(-\infty, 0)$. Taking f to be the indicator function of U we get a neighborhood $f_*^{-1}(W)$ of x which does not intersect C.

Remark 3.3 [**dense**]Unless A is finite the image of $C = M_+(X, A)$ in the corresponding universal group-like tonus space V has no internal points i.e. the complement to C in V is dense.

Lemma 3.4 *[**lem2**] Let $\phi : (X, A) \to (Y, B)$ be a bounded kernel. Then the composition with ϕ defines a map*

$$\phi_* : M_+(X, A) \to M_+(Y, B)$$

which is a morphism of tonus spaces.

Proof: Follows from Lemma 3.1.

Remark 3.5 [**rem1**]Consider the metric on $M_+(X, A)$ given by

$$[\mathbf{eqem1}]\nu(\mu_1, \mu_2) = sup_{U \in A}|\mu_1(U) - \mu_2(U)| \tag{10}$$

Remark 3.6 The proof of Lemma 3.4 implies that if ϕ is a (sub-)stochastic kernel then the corresponding map $M_+(\phi)$ does not increase the distances between measures.

Remark 3.7 [**rem1**]For any point x of (X, A) we have the δ-measure δ_x concentrated in x. Evaluating ϕ_* on δ_x we get a measure $\phi_*(\delta_x)$ on (Y, B) and one verifies easily that it is exactly $\phi(x, -)$. This shows that for any (X, A), (Y, B) the map

$$Hom_{\mathcal{E}}((X, A), (Y, B)) \to Hom_T(M_+(X, A), M_+(Y, B))$$

is a monomorphism. We will see below in Theorem 3.12 that it is in fact a bijection.

Let μ be a measure on (X, A) and let $X = \coprod_{i=1}^n X_i$ be a partition of X into a disjoint union of measurable subsets. For any $\delta > 0$ denote by $U(\mu, \delta, (X_i))$ the set of all measures λ on (X, A) such that for each $i = 1, \ldots, n$ one has

$$|\mu(X_i) - \lambda(X_i)| < \delta.$$

Clearly $U(\mu, \delta, (X_i))$ is an open neighborhood of μ in the standard topology.

Lemma 3.8 *[**lem55**] Subsets of the form $U(\mu, \delta, (X_i))$ form a fundamental system of open neighborhoods of μ in the standard topology.*

Proof: If $X = \coprod_{i=1}^n X_i$ and $X = \coprod_{j=1}^m Y_j$ are two measurable partitions of X then $X = \coprod (X_i \cap Y_j)$ is also a measurable partition of X. Let $\delta > 0$ be a real number and k be an integer such that $k \geq n$ and $k \geq m$. Let λ be an element of $U(\mu, \delta/k, (X_i \cap Y_j))$. Then

$$|\mu(X_i) - \lambda(X_i)| = |\sum_{j=1}^m (\mu(X_i \cap Y_j) - \lambda(X_i \cap Y_j)| \leq \sum_{j=1}^m |(\mu(X_i \cap Y_j) - \lambda(X_i \cap Y_j)| \leq (m/k)\delta \leq \delta$$

i.e. $\lambda \in U(\mu, \delta, (X_i))$. Similarly $\lambda \in U(\mu, \delta, (Y_j))$ and we conclude that the intersection of two subsets of the type we consider contains a third subset of the same type.

The standard topology is generated by the maps $f_* : \mu \mapsto \int f d\mu$ for bounded non-negative measurable functions f. In particular for any μ finite intersections of subsets of the form

$$U(\mu, \epsilon, f) = \{\lambda : |\int f d\mu - \int f d\lambda| < \epsilon\}$$

form a fundamental system of open neighborhoods of μ. It remains to show that any neighborhood of the form $U(\mu, \epsilon, f)$ contains a neighborhood of the form $U(\mu, \delta, (X_i))$ i.e. that for any f and any $\epsilon > 0$ there exists a partition $X = \coprod X_i$ and $\delta > 0$ such that for any λ satisfying

$$|\mu(X_i) - \lambda(X_i)| < \delta$$

we have

$$|\int f d\mu - \int f d\lambda| < \epsilon.$$

Without loss of generality we may assume that $f(x) < 1$ for all $x \in X$. Let $n > 0$ be an integer. For $k = 0, \ldots, n-1$ set $I_k = [k/n, (k+1)/n)$. Then

$$[0, 1) = \coprod_{k=0}^{n-1} I_k$$

is a measurable partition of the interval $[0, 1)$. Let further $X_k = f^{-1}(I_k)$ and let

$$f_n = \sum_{k=0}^{n-1} k/nF_k$$

where F_k is the indicator function of X_k. By construction we have $f(x) \geq f_n(x)$ and $f(x) - f_n(x) < 1/n$ for all $x \in X$. For any λ we have

$$|\int f d\mu - \int f d\lambda| \leq |\int (f - f_n) d\mu - \int (f - f_n) d\lambda| + |\int f_n d\mu - \int f_n d\lambda| \leq$$

$$\leq |\int (f - f_n) d\mu| + |\int (f - f_n) d\lambda| + \sum_{k=0}^{n-1} k/n |\mu(X_k) - \lambda(X_k)| \leq$$

$$\leq \mu(X)/n + \lambda(X)/n + \sum_{k=0}^{n-1} k/n |\mu(X_k) - \lambda(X_k)| \leq$$

We also have:

$$\lambda(X) = \sum_{k=0}^{n-1} \lambda(X_k)| \leq \sum_{k=0}^{n-1} |\mu(X_k) - \lambda(X_k)| + \sum_{k=0}^{n-1} \mu(X_k) = \sum_{k=0}^{n-1} |\mu(X_k) - \lambda(X_k)| + \mu(X)$$

and therefore

$$|\int f d\mu - \int f d\lambda| \le 2\mu(X)/n + \sum_{k=0}^{n-1}(k+1)/n|\mu(X_k) - \lambda(X_k)| \le$$

$$\le 2\mu(X)/n + (1+1/n)\sum_{k=0}^{n-1}|\mu(X_k) - \lambda(X_k)|$$

To find n, δ such that $U(\mu, \delta, (X_k)_{k=0}^{n-1})$ is contained in $U(\mu, \epsilon, f)$ it is sufficient now to choose n such that $2\mu(X)/n < \epsilon$ and then choose δ such that $(n+1)\delta < \epsilon - 2\mu(X)/n$.

Let $M_*(X, A)$ be the universal group-like tonus space associated with $M_+(X, A)$ i.e. the space of signed measures on (X, A) with the topology defined by the canonical map

$$p : M_+(X, A) \times M_+(X, A) \to M_*(X, A)$$

For any $f \in M^+(X, A)$ the map $f_* : M_+(X, A) \to \mathbf{R}_{\ge 0}$ defines a map $M_*(X, A) \to \mathbf{R}$ which we will also denote by f_*.

Lemma 3.9 *[**imp1**] The topology on $M_*(X, A)$ coincides with the topology defined by the linear functionals f_* for $f \in M^+(X, A)$.*

Proof: Let $\mu = \mu_+ - \mu_-$ be an element of $M_*(X, A)$ and U be a subset in $M_*(X, A)$ which contains μ and such that $p^{-1}(U)$ is open in $M_+(X, A) \times M_+(X, A)$. We need to verify that there exists a finite set $f_1, \dots, f_n$ of elements of $M^+(X, A)$ and $\delta > 0$ such that for any $\lambda = \lambda_+ - \lambda_-$ in $M_*(X, A)$ satisfying

$$|\int f_i d\lambda - \int f_i d\mu| < \delta$$

for all $i = 1, \dots, n$, we have $\lambda \in U$. The condition that $p^{-1}(U)$ is open together with Lemma 3.8 implies that there exists $\epsilon > 0$ and a measurable partition $X = \coprod_{i=1}^m X_i$ such that for any pair of measures λ_+, λ_- satisfying

$$|\lambda_+(X_i) - \mu_+(X_i)| < \epsilon|$$

$$|\lambda_-(X_i) - \mu_-(X_i)| < \epsilon|$$

one has $\lambda_+ - \lambda_- \in U$.

Proposition 3.10 *[**tem1**] The map $f \mapsto f_*$ gives a bijection*

$$M^+(X, A) \to Hom_T(M_+(X, A), \mathbf{R}_{\ge 0}).$$

Its inverse takes a map ϕ of tonus spaces to the function f such that for each $x \in X$ one has $f(x) = \phi(\delta_x)$.

Proof: Let $\phi : M_+(X, A) \to \mathbf{R}_{\ge 0}$ be a morphism.

Corollary 3.11 *[**definedby**] Let $f, g : M_+(X, A) \to \mathbf{R}_{\ge 0}$ be two morphisms of tonus spaces which coincide on measures of the form δ_x for all $x \in X$. Then $f = g$.*

Theorem 3.12 *[***t1***] The functor $\mathcal{E} \to T$ sending (X, A) to $M_+(X, A)$ is a full embedding. I.e. For any measurable spaces (X, A), (Y, B) the map*

$$[\mathbf{mm}] Hom_{\mathcal{E}}((X, A), (Y, B)) \to Hom_T(M_+(X, A), M_+(Y, B)) \tag{11}$$

is a bijection. Its inverse takes a map ϕ of tonus spaces to the kernel ψ such that for each $x \in X$ the measure $\phi(x, -)$ is $f(\delta_x)$.

Proof: We already noted in Remark 3.7 that the map (11) is injective. To show that it is surjective consider a morphism $\phi : M_+(X, A) \to M_+(Y, B)$ of tonus spaces. Let U be a measurable subset of Y and let I_U be its indicator function. The composition of ϕ with the morphism $M_+(Y, B) \to \mathbf{R}_{\geq 0}$ defined by I_U is, by Proposition 3.10 a measurable function on (X, A) whose value on $x \in X$ is $\phi(\delta_x)(U)$. Therefore, a map $\psi : X \times B \to \mathbf{R}_{\geq 0}$ of the form $\psi(x, U) = \phi(\delta_x)(U)$ is a kernel. It remains to show that the map $\psi_* : M_+(X, A) \to M_+(Y, B)$ defined by this kernel is ϕ. We know that it coincides with ϕ on delta measures. Since the measurable functions on (Y, B) distinguish elements of $M_+(Y, B)$ it is sufficient to check that the compositions of ϕ and ψ_* with any map $M_+(Y, B) \to \mathbf{R}_{\geq 0}$ coincide. This follows from Corollary 3.11.

4 Radditive functors on $\mathcal{E}$

Recall that a contravariant functor F from a category C with finite coproducts and initial object 0 is called radditive if $F(0) = pt$ and $F(X \coprod Y) = F(X) \times F(Y)$. We let $R(C)$ denote the full subcategory in the category of all contravariant functors formed by radditive functors. For general properties of radditive functors see [], [].

Lemma 4.1 *[***lrf1***] Let C be a category as above and assume that finite coproducts in C coincide with finite products (in particular $pt = 0$). Then $R(C)$ is equivalent to the category of contravariant functors F from C to the category of abelian semi-groups such that $F(X \coprod Y) = F(X) \times F(Y)$.*

Proof: In the case of an additive C (i.e. under the additional assumption that morphisms in C can be subtracted) the statement is proved in []. The same proof works without subtraction.

5 Accessible spaces

6 Accessible enrichment of $\mathcal{E}$

Let (X, A), (Y, B) be measurable spaces. For any bounded measure μ on (X, A) and a bounded measurable function f on (Y, B) consider the map

$$\eta(\mu, f) : Hom_{\mathcal{E}}((X, A), (Y, B)) \to \mathbf{R}_{\geq 0}$$

sending ϕ to

$$f \circ \phi \circ \mu : \mathbf{1} \to (X, A) \to (Y, B) \to \mathbf{1}.$$

Define the *standard topology* on $Hom_{\mathcal{E}}((X, A), (Y, B))$ as the weakest topology with respect to which all maps $\eta(\mu, f)$ are continuous.

Lemma 6.1 *[***lae1***] The set $Hom_{\mathcal{E}}((X, A), (Y, B))$ with the standard topology and the obvious operations of addition and multiplication by scalar is a closed, Hausdorff tonus space.*

Proof: ???

Lemma 6.2 *[***lem0***] The composition of morphisms in $\mathcal{E}$ defines maps of tonus spaces of the form*

$$Hom_{\mathcal{E}}((X,A),(Y,B)) \otimes Hom_{\mathcal{E}}((Y,B),(Z,C)) \to Hom_{\mathcal{E}}((X,A),(Z,C)).$$

Proof: ???

Remark 6.3 [**nottopen**] Note that the maps of topological spaces

$$Hom_{\mathcal{E}}((X,A),(Y,B)) \times Hom_{\mathcal{E}}((Y,B),(Z,C)) \to Hom_{\mathcal{E}}((X,A),(Z,C))$$

defined by composition of morphisms need not be continuous if we take the standard topology on the right and the product of the standard topologies on the left.

Extended Lawvere category

References

[1] Michèle Giry. A categorical approach to probability theory. In *Categorical aspects of topology and analysis (Ottawa, Ont., 1980)*, volume 915 of *Lecture Notes in Math.*, pages 68–85. Springer, Berlin, 1982.