Reconstructing population histories of singletons

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1 Singleton paths system

Let us recall first the definition of a paths system given in []. We work in the expansion category $\mathcal{E}$ whose objects are measurable spaces (X, A) and morphisms are kernels (see loc.cit.). For a subset T of the real line, a paths system over T is the following collection of data:

1. for each $s \in T$, a measurable space X_s,

2. for each $s, t \in T$ such that $s < t$, a measurable space X_{st} together with two measurable maps $i : X_{st} \to X_s$ and $f : X_{st} \to X_t$,

3. for each $s, t, u \in T$ such that $s < t < u$, a morphisms in $\mathcal{E}$ of the form $X_{st} \times_{X_t} X_{tu} \to X_{su}$ such that the square

$$\begin{array}{ccc} X_{st} \times_{X_t} X_{tu} & \longrightarrow & X_{su} \\ \downarrow & & \downarrow \\ X_s \times X_t \times X_u & \xrightarrow{pr} & X_s \times X_u \end{array}$$

   commutes.

These data should satisfy the obvious associativity condition for each $s < t < u < v$ in T.

The *singleton paths system* is a paths system on the set of natural numbers (i.e. all X_t's are $\mathbf{N}$) where a path from m to n over $[s, t]$ is a "singleton genealogy" over the time interval $[s, t]$ where the initial population has m members and the final population has n members. "Singleton" here means that we consider genealogies which involve only divisions and deaths but not unions of any kind. The goal of this section is to give precise definitions corresponding to this intuitive idea.

Definition 1.1 *[**hist1**] Let $T = [a, b]$ (where $b > a$) be a closed interval of the real line. A singleton genealogy h (or just a genealogy) over T is the following collection of data:*

1. *a (finite) directed graph $(\bar{V}, E)$ with the set $\bar{V}$ of vertices and the set E of edges*

2. *a finite set F and a map $\phi : F \to \bar{V}$*

3. *a function* $\tau : \bar{V} \to T$

such that the following conditions are satisfied:

1. *if* $\tau(v) \neq a$ *then there exists exactly one edge which ends in* v
2. $\tau(v) = a$ *if and only if there exists exactly one edge which starts in* v
3. *for each* $v \in F$ *one has* $\tau(\phi(v)) = b$
4. *if there is an edge from vertex* v_0 *to vertex* v_1 *then* $\tau(v_1) > \tau(v_0)$

Let I (resp. F_-) denote the subset $\tau^{-1}(a)$ (resp. $\tau^{-1}(b)$ in $\bar{V}$. Condition (4) implies that for $v \in I$ there are no edges ending in v and for $v \in F_-$ there are no edges starting in v. Condition (3) implies that ϕ may be considered as a map $F \to F_-$. Let V_n be the subset of V which consists of vertices v such that one of the following conditions holds:

1. $\tau(v) \neq b$ and there are exactly n edges starting in v
2. $\tau(v) = b$, $n \neq 1$ and $\#\phi^{-1}(v) = n$

Condition (2) of Definition 1.1 implies that $V_1 = \emptyset$. Set $V = \coprod V_n$. Elements of V will be called proper vertices of H.

Intuitively, $I(h)$ is the initial population of our genealogy, $F(h)$ is the final population and $F_-(h)$ is the population in the moments just preceeding the final moment. The elements of V are the events in the genealogy with V_0 being the set of all deaths and for $n > 1$, V_n is the set divisions or other types of events when an individual produces several new individuals. The restriction of τ to V assigns to an event the moment of time when this event occured. We exclude the possibility of events occurring at the initial moment a of the genealogy. Two examples of genealogies are drawn below (**??**).

Define the geometric realization $|h|$ of a genealogy h as follows. For an edge $e \in E(h)$ let $s(e)$ be the vertex where $s(e)$ starts and $t(e)$ the vertex where e ends. Set $|e| = [\tau(s(e)), \tau(t(e))]$. By condition (4), $|e|$ is an interval of non-zero length and we will write $s(|e|)$ and $t(|e|)$ for its beginning and end respectively. Set

$$|S| = (\coprod_{e \in E(h)} |e|)/\sim$$

where $\sim$ is the equivalence relation generated by pairs $(t(|e_0|), s(|e_1|))$ such that $t(e_0) = s(e_1)$. We will consider $|h|$ as a topological space (a cell complex of dimension one) with respect to the quotient topology. Our equivalence relation is such that to each vertex $v \in \bar{V}(h)$ there correspond is a well defined point on $|h|$ which we denote by $|v|$. The inclusions $|e| \to T$ define a continuous map $|\tau(h)| : |h| \to T$ and $|\tau|(|v|) = \tau(v)$. Note that the geometric realization does not contain any information on the set F and the map $F \to F_-$.

One defines an isomorphism $h_1 \to h_2$ as a triple of bijections $\bar{V}(h_1) \to \bar{V}(h_2)$, $E(h_1) \to E(h_2)$ and $F(h_1) \to F(h_2)$ which respect the incidence relations, functions τ_i and maps ϕ_i.

Lemma 1.2 *[**rig1**] Let* $f : h \to h$ *be an automorphism which is the identity on* $F(h) \coprod V_0(h)$*. Then* $f = Id$*.*

Proof: ???

Lemma 1.3 *[***rig2***] Let $f : |h| \to |h|$ be a continuous automorphism of the topological space $|h|$ which commutes with the function $|\tau|$ and which is identity on the set $|\phi F(h) \coprod V_0(h)|$. Then $f = Id$.*

Proof: ???

An *ordered genealogy* is a genealogy h together with orderings on $F(h)$ and $V_0(h)$. Lemma 1.2 states that ordered genealogies have no automorphisms and that the geometric realizations of ordered genealogies have no automorphisms as spaces over T. Let $D = D_{[a,b]}$ be the set of isomorphism classes of ordered genealogies over $[a, b]$. We are now going to define a topology on D.

Let us write first

$$D = \coprod_{n,m \geq 0} D(* \stackrel{m}{\to} n)$$

where $D(* \stackrel{m}{\to} n)$ is the subset of genealogies such that $\#(F(h)) = n$ and $\#(V_0(h)) = m$. We will inductively construct for each $n, m \geq 0$ the following collection of objects:

1. a topology on $D(* \stackrel{m}{\to} n)$
2. a topological space $\tilde{D}(* \stackrel{m}{\to} n)$ and a continuous map $p : \tilde{D}(* \stackrel{m}{\to} n) \to D(* \stackrel{m}{\to} n)$
3. a continuous map $|\tau| : \tilde{D}(* \stackrel{m}{\to} n) \to T$
4. for each $h \in D(* \stackrel{m}{\to} n)$ an isomorphism $|h| \cong p^{-1}(h)$ such that its composition with $|\tau|$ is $|\tau(h)|$.

We first do induction on n constructing our structures for $m = 0$ and all $n \geq 0$ and then do induction on m.

For $n = m = 0$ there is only the empty genealogy. Hence $D(* \stackrel{0}{\to} 0) = pt$. The geometric realization of the empty genealogy is $\emptyset$ and hence $\tilde{D}(* \stackrel{0}{\to} 0) = \emptyset$. Assume now that everything is defined up to n.

Define subset I in $\tilde{D}(* \stackrel{0}{\to} n)$ setting

$$|I| = \cup_{h \in D(* \stackrel{0}{\to} n)} |I(h)|$$

To get a topology on $D(* \stackrel{0}{\to} n + 1)$ we define a bijection

$$[\textbf{indmap}] D(* \stackrel{0}{\to} n + 1) \to (\tilde{D}(* \stackrel{0}{\to} n) - |I|) \coprod D(* \stackrel{0}{\to} n). \tag{1}$$

Let $h \in D(* \stackrel{0}{\to} n+1)$ be an ordered genealogy and let u be the $(n+1)$-st element in $F(h)$. Let $v = \phi(u)$. If v is the image under ϕ of an element other than u then removing u from F we get a genealogy h' from $D(* \stackrel{0}{\to} n)$ and there is a point $|v|$ in $|h'|$ corresponding to v. In this case we map h to $(h', |v|)$

Assume now that $\phi^{-1}(v) = u$. By condition (1) of Definition 1.1 there exists a unique edge e which ends in v. Let w be the vertex where this edge starts. There are two possibilities. If $\tau(w) = a$ then h is the disjoint union of a genealogy h' in $D(* \xrightarrow{0} n)$ and an edge starting in w and ending in v. In this case we map h to the point of $D(* \xrightarrow{0} n)$ corresponding to h'.

If $\tau(w) \neq a$ let h' be the genealogy obtained from h by removing of the edge e. If $w \in V_2(h)$ then w stops being a vertex in h'. If $w \in V_n(h)$ for $n > 2$ then w becomes an element of $V_{n-1}(h')$. In any case w corresponds to a well defined point $|w|$ on $|h'|$ and we map h to $(h', |w|)$. One verifies immediately that the map (1) defined in this way is a bijection.

Next we need to construct $\tilde{D}(* \xrightarrow{0} n+1)$ and a continuous map $p : \tilde{D}(* \xrightarrow{0} n+1) \to D(* \xrightarrow{0} n+1)$ such that $p^{-1}(h) = |h|$. Let B be the closed subset in $T \times T$ given by the condition that $(t, t') \in D$ if and only if $t \geq t'$. Consider B as a space over T with respect to the second projection $B \to T$ such that the fiber of B over t' is the interval $[t', b]$. Let $s : T \to B$ be the section which sends t' to t'. Set

$$X_n = (\tilde{D}(* \xrightarrow{0} n) - |I|) \times_T B.$$

Then X_n is a space over $\tilde{D}(* \xrightarrow{0} n) - |I|$ whose fiber over $(h, x \in |h|)$ is the interval $[|\tau|(x), b]$ and s defines a section $\tilde{D}(* \xrightarrow{0} n) - |I| \to X_n$ which we also denote by s. Let Y_n be the space defined by the cocartesian square:

$$\text{[\textbf{indeq3}]} \qquad \begin{array}{ccc} \tilde{D}(* \xrightarrow{0} n) - |I| & \xrightarrow{s} & X_n \\ \Gamma \downarrow & & \downarrow \\ (\tilde{D}(* \xrightarrow{0} n) - |I|) \times_{D(* \xrightarrow{0} n)} \tilde{D}(* \xrightarrow{0} n) & \longrightarrow & Y_n \end{array} \qquad (2)$$

where Δ is the graph of the embedding

$$\tilde{D}(* \xrightarrow{0} n) - |I| \to \tilde{D}(* \xrightarrow{0} n).$$

Finally set

$$\tilde{D}(* \xrightarrow{0} n+1) = Y_n \coprod \tilde{D}(* \xrightarrow{0} n) \coprod D(* \xrightarrow{0} n) \times T$$

and define the projection

$$\text{[\textbf{indeq2}]} p : \tilde{D}(* \xrightarrow{0} n+1) \to D(* \xrightarrow{0} n+1) = (\tilde{D}(* \xrightarrow{0} n) - |I|) \coprod D(* \xrightarrow{0} n) \qquad (3)$$

to be the coproduct of the obvious projections

$$Y_n \to \tilde{D}(* \xrightarrow{0} n) - |I|$$

and

$$\tilde{D}(* \xrightarrow{0} n) \coprod D(* \xrightarrow{0} n) \times T \to D(* \xrightarrow{0} n).$$

The definition of $|\tau| : \tilde{D}(* \xrightarrow{0} n+1) \to T$ is obvious. It remains to construct an isomorphism $|h| \to p^{-1}(h)$.

Let u be the $(n+1)$-st element of $F(h)$ and $v = \phi(u)$. Let e be the edge of h ending in v and let w be the starting vertex of this edge. Assume first that $\tau(w) \neq a$ i.e. that

$$h = (h', |w|) \in \tilde{D}(* \xrightarrow{0} n) - |I|.$$

Then the fiber $p^{-1}(h)$ is given by the fiber of (2) over h i.e. by the cofibrant square

$$\begin{array}{ccc} pt & \xrightarrow{\tau(w)} & [\tau(w), b] \\ {\scriptstyle |w|}\downarrow & & \downarrow \\ |h'| & \longrightarrow & p^{-1}(h) \end{array}$$

and we have an obvious isomorphism $|h| \to p^{-1}(h)$. Assume now that $\tau(w) = a$ i.e.

$$h = (h', |w|) \in D(* \xrightarrow{0} n)$$

Then the fiber $p^{-1}(h)$ is $|h'| \coprod T$ and we again have an obvious isomorphism $|h| \to p^{-1}(h)|$.

Assume now that the structures related to $D(* \xrightarrow{m} n)$ are constructed and let us construct the structures related to $D_{n,m+1}$. Set as before

$$|I| = \cup_{h \in D(* \xrightarrow{m} n)} |I(h)|$$

$$|S| = \cup_{h \in D(* \xrightarrow{m} n)} |S(h)|.$$

and let

$$|V_0| = \cup_{h \in D(* \xrightarrow{m} n)} |V_0(h)|.$$

Let B° be the subset of points (t, t') in $T \times T$ satisfying the condition $t > t'$ which we consider as a space over T with respect to the second projection. Set

$$U_{n,m} = (\tilde{D}(* \xrightarrow{m} n) - |I| - |V_0|) \times_T B^\circ.$$

Define a bijection

$$D_{n,m+1} \to U_{n,m} \coprod D(* \xrightarrow{m} n) \times (a, b]$$

as follows. Let $h \in D_{n,m+1}$. Let v be the $(m+1)$-st element of $V_0(h)$. Let e be the edge which ends in v and let w be the vertex where e starts. Let h' be the genealogy obtained from h by removing e. Note that w is not in $V_0(h) \cup S(h)$. Assume first that w is not in $I(h)$. Then w gives a well defined point $|w|$ on $h'|$ and we map h to $((h', |w|), (\tau(v), \tau(w))) \in U_{n,m}$. If $w \in I(h)$ then h is the disjoint union of h' and e and we map h to $(h', \tau(v))$. One verifies immediately that our map is indeed a bijection.

To construct $\tilde{D}_{n,m+1}$ proceed as follows. Let C be the set of points (s, t, t') in $T \times B$ such that $t \leq s \leq t'$. Set

$$X_{n,m} = U_{n,m} \times_B C$$

Then $X_{n,m}$ is a space over $U_{n,m}$ whose fiber over $((h, x), t)$ is the closed interval $[\tau(x), t]$. Let $s : U_{n,m} \to X_{n,m}$ be the section which sends $((h, x), t)$ to $(((h, x), t), \tau(x))$. Let $Y_{n,m}$ be the space defined by the cocartesian square

$$\begin{array}{ccc} U_{n,m} & \xrightarrow{s} & X_{n,m} \\ {\scriptstyle\Gamma}\downarrow & & \downarrow \\ U_{n,m} \times \tilde{D}(* \stackrel{m}{\to} n) & \longrightarrow & Y_{n,m} \end{array}$$

where Γ is the graph of the projection $U_{n,m} \to \tilde{D}(* \stackrel{m}{\to} n)$. Let C be the subset of points (t, t') in $T \times (a, b]$ satisfying $t \leq t'$. Set

$$Z_{n,m} = \tilde{D}(* \stackrel{m}{\to} n) \coprod (D(* \stackrel{m}{\to} n) \times C)$$

Finally set $\tilde{D}_{n,m+1} = Y_{n,m} \coprod Z_{n,m}$. We define $p : \tilde{D}_{n,m+1} \to D_{n,m+1}$ in the obvious way.

Let $h \in D_{n,m+1}$ and let us construct an isomorphism $|h| \to p^{-1}(h)$. Let as before v be the $(m+1)$-st element of $V_0(h)$, e the edge which ends in v and w the vertex where e starts. Let further h' be the genealogy obtained from h by removing e. Assume first that w is not in $I(h)$ and let $|w|$ be the corresponding to w point on $|h'$. Then h lies in $U_{n,m}$ and $p^{-1}(h)$ is given by the cocartesian square

$$\begin{array}{ccc} pt & \xrightarrow{\tau(w)} & [\tau(w), \tau(v)] \\ {\scriptstyle |w|}\downarrow & & \downarrow \\ |h'| & \longrightarrow & p^{-1}(h) \end{array}$$

and we get an obvious isomorphism $|h| \to p^{-1}(h)$. Assume now that $w \in I(h)$. Then h lies in $Z_{n,m}$ and the fiber $p^{-1}(h)$ is given by

$$p^{-1}(h) = |h'| \coprod [a, \tau(v)]$$

and we again have an obvious isomorphism $|h| \to p^{-1}(h)$. The construction of $|\tau| : \tilde{D}_{n,m+1} \to T$ is obvious.

The change of orderings on $V_0(F)$ and $F(h)$ define actions of the symmetric groups Σ_m and Σ_n on $D(* \stackrel{m}{\to} n)$. These two actions commute and therefore give an action of the product $\Sigma_m \times \Sigma_n$.

Lemma 1.4 *[**iscont**] The action of $\Sigma_m \times \Sigma_n$ on $D(* \stackrel{m}{\to} n)$ specified above is continuous.*

Proof: ???

Set

$$H(* \stackrel{m}{\to} n) := D(* \stackrel{m}{\to} n)/(\Sigma_n \times \Sigma_m).$$

Points of $H(* \stackrel{m}{\to} n)$ are the isomorphism classes of genealogies h such that $\#F(h) = n$ and $\#(V_0(h)) = m$. The universal bundle $\tilde{D}(* \stackrel{m}{\to} n) \to D(* \stackrel{m}{\to} n)$ also has an obvious action of $\Sigma_n \times \Sigma_m$ and taking the quotient we get a universal bundle

$$p : \tilde{H}(* \stackrel{m}{\to} n) \to H(* \stackrel{m}{\to} n)$$

such that for each h we have $p^{-1}(h) = |h|$. Finally, the function $|\tau|$ is clearly invariant under the action which we consider and we get a function

$$|\tau| : \tilde{H}(* \xrightarrow{m} n) \to T.$$

We are ready now to define the singleton paths system $\mathcal{H}$ over $T = [a, b]$. For each $s \in T$ we set $H_s = \mathbf{N}$. For $s < t$ in T we set $H_{st} = H[s, t]$ to be the set of isomorphism classes of singleton genealogies over $[s, t]$ with the Borel σ-algebra corresponding to the topology constructed in the previous section. For $h \in H_{st}$ we set $i(h) = \#I(h)$ and $f(h) = \#F(h)$. For a genealogy h let us denote by $[h]$ its isomorphism class. To define the composition morphisms

$$m : H_{st} \times_{\mathbf{N}} H_{tu} \to H_{su}$$

we have to assign to each pair of genealogies (h_1, h_2) (over $[s, t]$ and $[t, u]$ respectively) such that $\#F(h_1) = \#I(h_2)$ a measure $m(h_1, h_2)$ on H_{su} which depends only on $[h_1]$ and $[h_2]$. Observe that for any h_1 and h_2 as above and any bijection $\alpha : F(h_1) \to I(h_2)$ we have a well defined genealogy $h_1 \cup_\alpha h_2$ over $[s, u]$ obtained by "gluing" h_1 and h_2 using α. Let $n = \#F(h_1) = \#I(h_2)$. We set:

$$[\mathbf{compdef}]m(h_1, h_2) = (1/n!) \sum_\alpha [h_1 \cup_\alpha h_2] \tag{4}$$

where for simplicity we let use the same symbol for a point and the corresponding δ-measure.

Lemma 1.5 *[**isamor**] The formula (4) defines a morphism in $\mathcal{E}$.*

Proof: ???

Lemma 1.6 *[**isassociative**] The composition morphisms constructed above satisfy the associativity condition.*

Proof: ???

2 Processes on the singleton paths system

Recall that a process on a paths system $(X_s, X_{st}, \phi_{stu})$ is a collection of morphisms $\mu_{st} : X_s \to X_{st}$ in $\mathcal{E}$ such that

1. for all $s < t$, μ_{st} is a section of $i : X_{st} \to X_s$,

2. for all $s < t < u$, the following square commutes

$$\begin{array}{ccc} X_{st} & \xrightarrow{f^*(\mu_{tu})} & X_{st} \times_{X_t} X_{tu} \\ \mu_{st} \uparrow & & \downarrow \phi_{stu} \\ X_s & \xrightarrow{\mu_{su}} & X_{su} \end{array}$$

where the upper horizontal arrow is the pull-back of μ_{tu} along $f : X_{st} \to X$.

Let us expand this definition in the case of the singleton paths system. Let $H(m\to *)_{st}$ (resp. $H(m\to n)_{st}$) be the subset of genealogies h in H_{st} such that $i(h)=m$ (resp. $i(h)=m$ and $f(h)=n$). The composition morphisms can be written as a collection of morphisms in $\mathcal{E}$ of the form

$$\phi_k:\coprod_m H(k\to m)_{st}\times H(m\to *)_{tu}\to H(k\to *)_{su}.$$

Then a process on $\mathcal{H}$ is a collection of measures $\mu(m\to n)_{st}$ on $H(m\to n)_{st}$ such that

1. for all $s<t$ and all $m\in\mathbf{N}$

$$\mu(m\to *)_{st}:=\oplus_m\mu(m\to n)_{st}$$

   is a probability measure on $H(m\to *)_{st}$,

2. for all $s<t<u$ and all $k\in\mathbf{N}$ one has

$$\sum_m \phi_k(\mu(k\to m)_{st}\otimes\mu(m\to *)_{tu})=\mu(k\to *)_{su}.$$

Let us define a pre-process on $\mathcal{H}$ as a collection of σ-finite measures $\mu(k\to m)_{st}$ on $H(k\to m)_{st}$ which satisfies the second of these two conditions. Our first goal is to provide a description of pre-processes on $\mathcal{H}$ in terms of “generators and relations”.

For $s<t_1<\ldots<t_n<u$ and $i_0,\ldots,i_n\in\mathbf{N}$ denote by $H^{i_0,\ldots,i_n}_{st_1\ldots t_nu}$ the set of genealogies h over $[s,u]$ such that $\phi(V(h))$ has i_0 points between s and t_1, i_1 points between t_1 and t_2 etc. In other words, we consider genealogies where events occur in i_0 different moments of time on $(s,t_1]$, i_1 diffrenent moments of time on $(t_1,t_2]$ etc.

Theorem 2.1 *[th1.0] There is a bijection between pre-processes on $\mathcal{H}$ over T and collections of σ-finite measures $\mu^0(k\to *)_{st}$, $\mu^1(k\to *)_{st}$ on $H^0_{st}(k\to *)$ and $H^1_{st}(k\to *)$ respectively such that for $s<t<u$ one has*

$$\mu^0(k\to *)_{su}=\sum_m\phi_k(\mu^0(k\to m)_{st}\otimes\mu^0(m\to *)_{tu})$$

and

$$\mu^1(k\to *)_{su}=\sum_m\phi_k(\mu^0(k\to m)_{st}\otimes\mu^1(m\to *)_{tu}+\mu^1(k\to m)_{st}\otimes\mu^0(m\to *)_{tu})$$

Let us start with the following lemmas.

Lemma 2.2 *[l1] Let X be a measurable space and $X=\cup X_i$ a covering of X by a countable set of measurable subsets. Then there is a bijection between the set of σ-finite measures on X and collections of σ-finite measures μ_i on X_i such that for each i,j one has*

$$(\mu_i)_{|X_i\cap X_j}=(\mu_j)_{|X_i\cap X_j}$$

Proof: ???

Lemma 2.3 *[l2] Let $s < t_1 < \ldots < t_n < u$ be in T, $i_0, \ldots, i_n \in \mathbf{N}$ and let $t \in (s,u)$. If $t_m < t < t_{m+1}$ then there is a commutative diagram of the form*

$$\begin{array}{ccc} H^{i_0,\ldots,i_m}_{st_1\ldots t} \times_{\mathbf{N}} H^{i_m,\ldots,i_n}_{tt_{m+1}\ldots u} & \longrightarrow & H^{i_0,\ldots,i_n}_{st_1\ldots u} \\ \downarrow & & \downarrow \\ H_{st} \times_{\mathbf{N}} H_{tu} & \xrightarrow{\phi} & H_{su} \end{array}$$

where the vertical arrows are the inclusions. In addition for each m we have a commutative diagram of the form:

$$\begin{array}{ccc} H^{i_0,\ldots,i_{m-1}}_{st_1\ldots t_m} \times_{\mathbf{N}} H^{i_m,\ldots,i_n}_{t_m t_{m+1}\ldots u} & \longrightarrow & H^{i_0,\ldots,i_n}_{st_1\ldots u} \\ \downarrow & & \downarrow \\ H_{st_m} \times_{\mathbf{N}} H_{t_m u} & \xrightarrow{\phi} & H_{su} \end{array}$$

Proof: ???

Note that the upper horizontal arrow in the sqaure of Lemma 2.3 is uniquely defined and is simply the restriction of the composition morphism to the corresponding subset.

Lemma 2.4 *[l3] Let T' be a dense subset in T. Then for $s < u$ in T and $n \geq 1$ one has*

$$H^{n+1}_{su} = \cup_{t \in T' \cap (s,u)} H^{1,n}_{stu}$$

In addition for $t < t'$ one has

$$H^{1,n}_{stu} \cap H^{1,n}_{st'u} = H^{1,0,n}_{stt'u}.$$

Proof: ???

Proof of the theorem: Let us prove the existence part of the theorem. The uniqueness will be clear. We have

$$H_{st}(k \to *) = \coprod_{n \geq 0} H^n_{st}(k \to *).$$

By Lemma 2.3 the composition of genealogies defines for all $s < t < u$ morphisms

$$\phi : \coprod_{i+j=n} \coprod_m H^i_{st}(k \to m) \times H^j_{tu}(m \to *) \to H^n_{su}(k \to *)$$

and a collection of measures $\mu^n_{st}(k \to *)$ on $H^n_{st}(k \to *)$ defines a pre-process if and only if for all $s < t < u$ and all n one has

$$[\mathbf{rela}]\phi(\oplus_{i+j=k} \oplus_m \mu^i_{st}(k \to m) \otimes \mu^j_{tu}(m \to *)) = \mu^n_{stu}(k \to *). \tag{5}$$

We will construct measures $\mu^n_{st}(k \to *)$ satisfying (5) by induction on n.

By assumption we already have measures on $H^0_{st}(k \to *)$ and $H^1_{st}(k \to *)$ for all k and all $s < t$ in T and these measures satisfy (5).

Let us choose a countable dense subset T' in T. By Lemma 2.4 and Lemma 2.2 in order to construct μ^n_{su} it is sufficient to construct measures $\mu^{1,n-1}_{stu}(k \to *)$ on $H^{1,n-1}_{stu}(k \to *)$ for all $t \in T'$ such that for $t < t'$ their restrictions to $H^{1,0,n-1}_{stt'u}(k \to *)$ coincide. Consider the composition morphisms

$$\phi : \coprod_m H^1_{st}(k \to m) \times H^{n-1}_{tu}(m \to *) \to H^{1,n-1}_{stu}(k \to *)$$

from the second part of Lemma 2.3. Set

$$\mu^{1,n-1}_{stu}(k \to *) = \phi(\oplus_m \mu^1_{st}(k \to m) \otimes \mu^{n-1}_{tu}(m \to *)).$$

The fact that for $t < t'$ the restrictions of our measures to $H^{1,0,n-1}_{stt'u}(k \to *)$ coincide follow from the associativity square: and the inductive assumption that

Let us say that an elementary genealogy of multiplicity k over $[s,t]$ is a genealogy h such that $\#V(h) \leq 1$, $i(h) = 1$ and $f(h) = k$. One can easily see that for $k = 1$ there is exactly one such genealogy and for any $k \neq 1$ the set of isomorphism classes of such genealogies is $(s,t]$ where $x \in (s,t]$ corresponds to the genealogy h with $\tau(V(h)) = \{x\}$ (see **??**).

Let us consider now the set $X^1(n \to *)_{st}$ of isomorphism classes of genealogies h in $H(n \to *)_{st}$ such that $\tau(V(h))$ consists of one point. Since vertices with the same value of τ can not be connected by an edge any such genealogy is a disjoint union of elementary genealogies. Let $H(n; k_1, \ldots, k_n)_{st}$ be the subset of $X^1(n \to *)_{st}$ which consists of genealogies h such that the multiplicities of the corresponding elementary genealogies are $k_1, \ldots, k_n$ (we want here to consider $k_1, \ldots, k_n$ as an un-ordered set of numbers rather than as a sequence). As before, we have

$$H(n; k_1, \ldots, k_n)_{st} = \begin{cases} pt & \text{if } k_1 = \ldots = k_n = 1 \\ (s,t] & \text{otherwise} \end{cases}$$

Clearly, the subsets $H(n; k_1, \ldots, k_n)_{st}$ are measurable. For a process μ on $\mathcal{H}$ the measures $\mu(k \to *)_{st}$ give us a collection of measures $\mu(n; k_1, \ldots, k_n)_{st}$ on $(s,t]$ and (for $k_1 = \ldots = k_n = 1$) a collection of numbers $x(n)_{st}$ (since measures on a point are non-negative real numbers).

Lemma 2.5 *[**elprop**] The collection $(\mu(n; k_1, \ldots, k_n)_{st}, x(n)_{st})_{s<t}$ corresponding to a process on $\mathcal{H}$ satisfies the following condition. For any $s < t < u$ in T one has*

$$[\mathbf{cond1}] x(n)_{su} = x(n)_{st} x(n)_{tu} \tag{6}$$

and if $k_i \neq 1$ for some i then

$$[\mathbf{cond2}] \mu(n; k_1, \ldots, k_n)_{su} = \mu(n; k_1, \ldots, k_n)_{st} x(n)_{tu} + x(n)_{st} \mu(n; k_1, \ldots, k_n)_{tu}. \tag{7}$$

Proof: ???

Theorem 2.6 *[**th1**] There is a bijection between the set of processes on $\mathcal{H}$ over T and the set of collections $(\mu(n; k_1, \ldots, k_n)_{st}, x(n)_{st})_{s<t}$ satisfying conditions (6),(7) of Lemma 2.5.*

Proof: ???

A pre-process over $T = [a,b]$ is called *normal* if for any $a \leq s < b$ and any k one has

$$\mu(k \to *)_{sb}(H(k \to *)_{sb}) \neq 0, \infty.$$

Lemma 2.7 *[**deter**] A normal pre-process on $\mathcal{H}$ over $T=[a,b]$ is uniquely determined by the measures $\mu(k\rightarrow *)_{sb}$ for $a\le s<b$.*

Proof: ???

3 Reducible processes

Recall from Section 1 that we let $D(*\stackrel{m}{\rightarrow} n)$ denote the space of ordered genealogies with n final individuals and m deaths events over $T=[a,b]$. Our construction of these spaces gave as a side effect a construction of continuous maps

$$D(*\stackrel{m}{\rightarrow} n)\rightarrow D(*\stackrel{m-1}{\rightarrow} n)$$

which correspond to the removal of the edge ending in the m-th death event. The composition of these maps give us a map

$$[\mathbf{r0}]D(*\stackrel{m}{\rightarrow} n)\rightarrow D(*\stackrel{0}{\rightarrow} n) \tag{8}$$

which sends a genealogy h to the genealogy obtained from h by removing all the lines of descend which do not reach the final population. One can easily see that the map (8) is equivariant under the action of $\Sigma_m\times\Sigma_n$ and therefore gives us a continuous maps

$$r_s: H_{sb}\rightarrow H_{sb}$$

for all $a\le s<b$. For a genealogy $h\in H_{sb}$ we call $r_s(h)$ the *ansestral genealogy* defined by h. Intuitively, h describes genealogy of all descendants of the initial population and $r(h)$ describes the genealogy of the direct ansestors of the final population.

The inference problems which we consider below concern the reconstructions of the parametrs of a process μ_{st} on $\mathcal{H}$ from an observation of the ansestral genealogy of a genealogy generated by this process. An important intermediate object is given by the projections of the measures $\mu_{sb}(n\rightarrow *)$ with respect to r_s i.e. by the compositions $r_s\circ\mu_{sb}$. These compositions almost never correspond to a process themselves since they do not give sections of the morphism $i: H_{sb}\rightarrow \mathbf{N}$ unless μ is death free. However, in many interesting cases one can define the *reduced process* $\tilde{\mu}$ of a given process μ whose measures are closely related to $r_s\mu(k\rightarrow *)$.

We start by considering the situation when we are given for each $a\le s<b$ a section $\mu_s:\mathbf{N}\rightarrow H_{sb}$ of i without assuming that these sections correspond to a process.

Definition 3.1 *[**anses0**] A family of morphisms μ_s as above is called reducible if for any $a\le s<b$ there exist a (stochastic) morphism $\sigma_s:\mathbf{N}\rightarrow\mathbf{N}$ and a section $\tilde{\mu}_s$ of $i: H_{sb}\rightarrow\!\!\!\rightarrow \mathbf{N}$ such that the square*

$$[\mathbf{needsq}][\mathbf{sq1}]\quad \begin{array}{ccc} \mathbf{N} & \xrightarrow{\sigma_s} & \mathbf{N}\\ {\scriptstyle\mu_s}\downarrow & & \downarrow{\scriptstyle\tilde{\mu}_s}\\ H_{sb} & \xrightarrow{r_s} & H_{sb}\end{array} \tag{9}$$

commutes.

A family μ_s is called uniquely reducible if it is reducible and the morphisms σ_s and $\tilde{\mu}_s$ which make (9) commutative are unique. If μ_s is a uniquely reducible family then the family $\tilde{\mu}_s$ is called the reduction of μ_s.

Let μ_s and $\tilde{\mu}_s$ be as in Definition 3.1. The condition that $\tilde{\mu}_s$ is a section of i together with the commutativity of (9) shows that

$$[\mathbf{int2}]\sigma_s = ir_s\mu_s. \tag{10}$$

In particular we conclude that σ_s which makes (9) commutative is always unique. Let us write σ_s in the form

$$< m > \mapsto \sum_k \sigma_s^{mk} < k >$$

where we used the notation $< i >$ for points of $\mathbf{N}$ to distinguish them from the coefficients. Then (10) shows that

$$\sigma_s^{mk} = \mu_s(m \to *)(H_{sb}((m,k) \to *))$$

where

$$H((m,k) \to *)_{sb} = H(m \to *)_{sb} \cap r_s^{-1}(H(k \to *)_{sb}).$$

The subset $H((m,k) \to *)_{sb}$ consists of genealogies with the inital population of size m and such that exactly k members of the initial population have descendants in the final population. Hence, σ_s^{mk} is the probability that in a population with m individuals at time s exactly k will have descendants in the final population.

Let $\mu_s((m,k) \to *)$ be the measure equal to $\mu_s(m \to *)$ on $H_{sb}((m,k) \to *)$ and to zero on the rest of H_{sb}.

Lemma 3.2 *[**redcr**] A family μ_s is reducible if and only if for all m, n, k one has*

$$[\mathbf{redcond}]\sigma_s^{mk} r_s(\mu((n,k) \to *)) = \sigma_s^{nk} r_s(\mu((m,k) \to *)) \tag{11}$$

i.e. the projections of the measures $\mu_s((i,k) \to *)$ to $H_{sb}(k \to *)$ agree for different i up to multiplication by a constant. It is uniquely reducible if and only if it is reducible and for each k and s there exists m such that $\sigma_s^{mk} \neq 0$.

Proof: The commutativity of (9) is equivalent to the condition that for all m, k one has

$$\sigma_s^{mk}\tilde{\mu}_s(k \to *) = r_s(\mu_s((m,k) \to *))$$

This implies "only if" in the first part of the lemma and "if" in the second part. Assume now that (11) is satisfied. If for a given k there exists m such that $\sigma_s^{mk} \neq 0$ we set

$$\tilde{\mu}_s(k \to *) = (1/\sigma_s^{mk}) r_s(\mu_s((m,k) \to *)).$$

Otherwise take $\tilde{\mu}_s(k \to *)$ to be any probability measure on $H_{sb}(k \to *)$. One verifies immediately that this defines $\tilde{\mu}_s$ which are sections of $i : H_{sb} \to \mathbf{N}$ such that (9) commutes. This proves "if" in the first part of the lemma "only if" in the second.

We proceed now to the lemmas relating r_s with the composition morphisms. Define for $k \leq m$ and $s < t$ in T a morphism

$$q_{st}^{mk} : H_{st}(* \to m) \to H_{st}(* \to k)$$

as follows. For a genealogy h and a subset A in $F(h)$ let $r(h,A)$ be the genealogy obtained from h by removing all lines of descend which end outside A (in particular all of those which do not reach $F(h)$). Set:

$$q_{st}^{mk}(h) = C(m,k)^{-1} \sum_{A\subset F(h),\ \#(A)=k} r(h,A).$$

Let further

$$q_{stb} : H_{st} \times_{\mathbf{N}} H_{tb} \to H_{st} \times_{\mathbf{N}} H_{tb}$$

be given on $H_{st}(* \to m) \times H_{tb}((m,k) \to *)$ by

$$q_{stb}^{mk}(h,h') = (q_{st}^{mk}(h), r_t(h')).$$

Lemma 3.3 *[**lowersq**] For any $s < t < b$ in T the square*

$$[\mathbf{lower}] \quad \begin{array}{ccc} H_{st} \times_{\mathbf{N}} H_{tb} & \xrightarrow{q_{stb}} & H_{st} \times_{\mathbf{N}} H_{tb} \\ \phi \downarrow & & \downarrow \phi \\ H_{sb} & \xrightarrow{r_s} & H_{sb} \end{array} \tag{12}$$

commutes (here ϕ is the composition morphism).

Proof: Let $h \in H_{st}(* \to m)$ and $h' \in H_{tb}((m,k) \to *)$. Going through the upper right corner of the square we get

$$\phi(q_{stb}(h,h')) = C(m,k)^{-1} \sum_{A\subset F(h),\ \#(A)=k} \phi(r(h,A), r_s(h')) =$$

$$= C(m,k)^{-1}(k!)^{-1} \sum_{A\subset F(h),\ \#(A)=k} \ \sum_{\alpha:A\cong I(r_s(h'))} r(h,A) \cup_\alpha r_{sb}(h').$$

Going through the lower left corner we get

$$r_s(\phi(h,h')) = (m!)^{-1} \sum_{\beta:F(h)\cong I(h')} r_s(h \cup_\beta h')$$

Let $B = I_{sb}(h') \subset I(h')$. Then each β defines $A_\beta = \beta^{-1}(B)$ and $\alpha_\beta : A_\beta \cong I(r_{sb}(I'))$. Clearly

$$r_s(h \cup_\beta h') \cong r(h, A_\beta) \cup_{\alpha_\beta} r_t(h')$$

In addition for each (A,α) there is exactly $(m-k)!$ different β's such that $(A,\alpha) = (A_\beta, \alpha_\beta)$. Since $C(m,k) = m!/(k!(m-k)!)$ we get the required equality.

Lemma 3.4 *[**uppersq**] Let $\mu, \tilde{\mu}$ be as above and assume that the squares (9) commute. Let*

$$p_{st} : H_{st} \to H_{st}$$

be the morphism given on $H_{st}(* \to m)$ by

$$p_{st}^m(h) = \sum_{k\leq m} \sigma_t^{mk} q_{st}^{mk}(h).$$

Let

$$pr^*\mu_{tb} : H_{st} \to H_{st} \times_{\mathbf{N}} H_{tb}$$

and

$$pr^*\tilde{\mu}_{tb} : H_{st} \to H_{st} \times_{\mathbf{N}} H_{tb}$$

be the morphisms defined by μ_t *and* $\tilde{\mu}_t$ *respectively. Then the square*

$$\begin{array}{ccc} H_{st} & \xrightarrow{p_{st}} & H_{st} \\ {\scriptstyle pr^*\mu_t}\downarrow & & \downarrow{\scriptstyle pr^*\tilde{\mu}_t} \\ H_{st} \times_{\mathbf{N}} H_{tb} & \xrightarrow{q_{stb}} & H_{st} \times_{\mathbf{N}} H_{tb} \end{array} \quad [\textbf{upper}] \tag{13}$$

commutes.

Proof: By definition, for $h \in H_{st}(* \to m)$ one has

$$pr^*\mu_t(h) = h \otimes \mu_t(m \to *)$$

and similarly for $h' \in H_{st}(* \to k)$ one has

$$pr^*\tilde{\mu}_t(h') = h' \otimes \tilde{\mu}_t(k \to *)$$

where we as always identify points and the corresponding δ-measures. Going through the upper right corner of (13) we get for $h \in H_{st}(* \to m)$:

$$pr^*\tilde{\mu}_t(p_{st}(h)) = pr^*\tilde{\mu}_t(\sum_k \sigma_t^{mk} q_{st}^{mk}(h)) =$$

$$= \sum_k \sigma_t^{mk} q_{st}^{mk}(h) \otimes \tilde{\mu}_t(k \to *).$$

Going through the lower left corner we get

$$q_{stb}(pr^*\mu_t(h)) = q_{stb}(h \otimes \mu_t(m \to *)) = \sum_k q_{stb}(h \otimes \mu_t((m,k) \to *)) =$$

$$= \sum_k q_{st}^{mk}(h) \otimes r_t(\mu_t((m,k) \to *))$$

On the other hand the commutativity of (9) implies that one has

$$r_t(\mu_t((m,k) \to *)) = \sigma_t^{mk} \tilde{\mu}_t(k \to *).$$

Definition 3.5 *[***anses***] Let* μ *be a process on* $\mathcal{H}$ *over* $[a,b]$*. It is called reducible (resp. uniquely reducible) if the morphisms* $\mu_s := \mu_{sb}$*,* $a \leq s < b$ *are reducible (resp. uniquely reducible) in the sense of Definition 3.1.*

Theorem 3.6 *[***mainanses***] Let* μ_{st} *be a uniquely reducible process on* $\mathcal{H}$ *over* $T = [a,b]$ *and let* $\tilde{\mu}_s$ *be the reduction of the family* $\mu_s := \mu_{sb}$*. Then there exists a unique process* $\tilde{\mu}_{st}$ *such that* $\tilde{\mu}_s = \tilde{\mu}_{sb}$*.*

Proof: The uniqueness part of the theorem follows from **??**. Consider the projections $\pi_{st} : H_{sb} \to H_{st}$ defined by the cutting map. In order to prove the existence part we have to show that the morphisms

$$\tilde{\mu}_{st} := \pi_{st}\tilde{\mu}_s : \mathbf{N} \to H_{st}$$

satisfy the condition of **??**, i.e. that the composition

$$\mathbf{N} \stackrel{\tilde{\mu}_{st}}{\to} H_{st} \stackrel{pr^*\tilde{\mu}_t}{\to} H_{st} \times_{\mathbf{N}} H_{tb} \stackrel{\phi}{\to} H_{sb}$$

coincides with $\tilde{\mu}_s$. We start with the following lemma.

Lemma 3.7 *[**as1**] For all $a \leq s < t < b$ the square*

$$[\mathbf{s1sq}] \quad \begin{array}{ccc} \mathbf{N} & \stackrel{\sigma_s}{\longrightarrow} & \mathbf{N} \\ {\scriptstyle \mu_{st}}\downarrow & & \downarrow{\scriptstyle \tilde{\mu}_{st}} \\ H_{st} & \stackrel{p_{st}}{\longrightarrow} & H_{st} \end{array} \tag{14}$$

commutes.

Proof:

Consider now the following diagram

$$\begin{array}{ccc} \mathbf{N} & \stackrel{\sigma_s}{\longrightarrow} & \mathbf{N} \\ {\scriptstyle \mu_{st}}\downarrow & & \downarrow{\scriptstyle \tilde{\mu}_{st}} \\ H_{st} & \stackrel{p_{st}}{\longrightarrow} & H_{st} \\ {\scriptstyle pr^*\mu_{tb}}\downarrow & & \downarrow{\scriptstyle pr^*\tilde{\mu}_{tb}} \\ H_{st} \times_{\mathbf{N}} H_{tb} & \stackrel{q_{stb}}{\longrightarrow} & H_{st} \times_{\mathbf{N}} H_{tb} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ H_{sb} & \stackrel{r_s}{\longrightarrow} & H_{sb} \end{array}$$

It commutes by Lemmas 3.7, 3.4, 3.3. Since μ is a process the left vertical side equals μ_{sb}. Let f be the composition of morphisms of the right vertical side. We need to prove that $f = \tilde{\mu}_{sb}$. The commutativity of 9 implies that

$$[\mathbf{almost}] f \circ \sigma_s = \tilde{\mu}_{sb} \circ \sigma_s. \tag{15}$$

Since μ is uniquely reducible, Lemma 3.2 shows that for each k there exists m such that $\sigma_s^{mk} \neq 0$. This guarantees that σ_s is an epimorphism in the expansion category and therefore (15) implies $f = \tilde{\mu}_{sb}$.

Removing the lines of descent of all the members of the initial population which have no descendants in the final population gives us maps

$$p_{m,k} : H((m,k) \to n)_{st} \to H((k,k) \to n)_{st}.$$

The map r_{st} clearly factors through $p_{m,k}$ i.e.

$$(r_{st})_{|H((m,k)\to n)} = (r_{st})_{H((k,k)\to n)} \circ p_{m,k}.$$

4 Additive processes on the singleton paths system

Let us denote by $H^{(k)}$ the subset of H which consists of genealogies h such that $\#I(h) = k$. Let further Z_t be the subset of H which consists of genealogies h such that there exists $v \in \coprod_{i\geq 2} V_i(h)$ satisfying $\tau(v) = t$.

Definition 4.1 [**indmes**] *A measure μ on $H[T]$ is called:*

normalized *if for each I and $k \geq 0$ one has $\mu_I(H^{(k)}[I]) = 1$*

nd-regular *if for each I and $t \in I$ one has $\mu_I(Z_t[I]) = 0$*

branching *if for each I one has $u_*(\mu_I \otimes \mu_I) = \mu_I$*

Definition 4.2 [**mesfam**] *Let $T = [a, b]$ be as above and suppose that for each subinterval $I \subset T$ we are given a measure μ_I on $H[I]$. Such a family (μ_I) is called normalized (resp. nd-regular, branching) if each of the measures μ_I is normalized (resp. nd-regular, branching). In addition, the family (μ_I) is called sequential if for each $s_0 < s_1 < s_2$ one has*

$$c_*(\mu_{[s_0,s_1]} \otimes \mu_{[s_1,s_2]}) = \mu_{[s_0,s_2]}.$$

Let us recall now some basic notions related to the continuous time Markov processes on $\mathbf{N}$. For our purposes, a Markov process on $\mathbf{N}$ over T is a collection of expansion morphisms $\phi_{s_0,s_1} : \mathbf{N} \to \mathbf{N}$ given for all $s_0 \leq s_1$ and satisfying the following conditions:

1. for each pair $s_0 \leq s_1$ and any $i \in \mathbf{N}$, $\phi_{s_0,s_1}(i)$ is a probability measure on $\mathbf{N}$

2. $\phi_{s,s} = Id$

3. for a triple $s_0 \leq s_1 \leq s_2$ one has $\phi_{s_0,s_2} = \phi_{s_1,s_2} \circ \phi_{s_1,s_0}$.

Consider the addition map $+ : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$. A Markov process (ϕ_{s_0,s_1}) is called a Markov *branching* process if for each pair $s_0 \leq s_1$ one has

$$\phi_{s_0,s_1}(0) = \delta_0$$

and the diagram

$$\begin{array}{ccc} \mathbf{N} \times \mathbf{N} & \xrightarrow{\phi_{s_0,s_1} \otimes \phi_{s_0,s_1}} & \mathbf{N} \otimes \mathbf{N} \\ {\scriptstyle +}\big\downarrow & & \big\downarrow{\scriptstyle +} \\ \mathbf{N} & \xrightarrow{\phi_{s_0,s_1}} & \mathbf{N} \end{array}$$

commutes. In other words, (ϕ_{s_0,s_1}) is a branching process if the morphisms ϕ_{s_0,s_1} are homomorphisms of monoids.

For a morphism $\phi : \mathbf{N} \to \mathbf{N}$ we may define a measure $\phi^\vee$ on $\mathbf{N} \times \mathbf{N}$ setting

$$\phi^\vee(p, q) = \phi(p, q)$$

Let further

$$c : (\mathbf{N} \times \mathbf{N}) \times (\mathbf{N} \times \mathbf{N}) \to \mathbf{N} \times \mathbf{N}$$

be an expansion morphism of the form

$$c((n_1, n_2), (m_1, m_2)) = \begin{cases} 0 & \text{if } n_2 \neq m_1 \\ \delta_{n_2} & \text{if } n_2 = m_1 \end{cases}$$

Lemma 4.3 *[**mardes**] There is a bijection between the set of Markov processes on* $\mathbf{N}$ *over* T *and the set of families of measures* μ_I *on* $\mathbf{N}\times\mathbf{N}$ *where* I *runs through all non-degenerate subintervals* I *of* T*, satisfying the following conditions*

1. *for each* I *and each* $i\in\mathbf{N}$ *one has* $\mu_I(\{i\}\times\mathbf{N})=1$
2. *for each* $s_0<s_1<s_2$ *one has*

$$c_*(\mu_{[s_0,s_1]}\otimes\mu_{[s_1,s_2]})=\mu_{[s_0,s_2]}$$

Proof: ???

Lemma 4.4 *[**isbran**] The bijection of Lemma 4.3 identifies Markov branching processes with families* (μ_I) *such that for each* I *one has*

$$u_*(\mu_I\otimes\mu_I)=\mu_I$$

where

$$u:(\mathbf{N}\times\mathbf{N})\times(\mathbf{N}\times\mathbf{N})\rightarrow\mathbf{N}\times\mathbf{N}$$

is the map $((n_1,n_2),(m_1,m_2))\mapsto(n_1+m_1,n_2+m_2)$*.*

Proof: ???

5 Sequential measures on $D(*\stackrel{m}{\rightarrow}n)$

Let us start with a general measure-theoretic construction. Recall that a measurable space is a pair (X,A) where X is a set and A is a σ-algebra of subsets of X.

Definition 5.1 *[**mfinite**] Let* $(X,A),(Y,B)$ *be measurable spaces and* $f:X\rightarrow Y$ *a map of sets. Then* f *is called an m-covering if there exists countable families* $(X_i\in A)_{i\in I}$ *and* $(Y_j\in B)_{j\in J}$ *satisfying the following conditions*

1. $X=\coprod X_i$ *and* $Y=\coprod Y_j$ *(in particular different subsets in the families are disjoint)*
2. *for each* j *there exists a finite subset* $I_j\subset I$ *such that* $p^{-1}(Y_j)=\coprod_{i\in I_j}X_i$ *and the restriction of* p *to each* X_i *is an isomorphism of measure spaces* $(X_i,A_{|X_i})\rightarrow(Y_j,B_{|Y_j})$*.*

The following lemma is straightforward.

Lemma 5.2 *[**ism**] Any m-covering is a measurable map in addition if* f *is an m-covering and* $U\in A$ *then* $f(U)\in B$*.*

Definition 5.3 *[**mhom**] let* $f:(X,A)\rightarrow(Y,B)$ *be an m-covering. A measure* μ *on* (X,A) *is called* f*-homogeneous if for any* $U,V\in A$ *such that* $f(U)=f(V)$ *and* $f:U\rightarrow f(U)$*,* $f:V\rightarrow f(V)$ *are bijections one has* $\mu(U)=\mu(V)$*.*

Lemma 5.4 *[**main1**] Let* $f:(X,A)\rightarrow(Y,B)$ *be an m-covering. Then the map* f_* *defines the bijection between the space of m-homogeneous measures on* (X,A) *and the space of measures on* (Y,B)*.*

Proof: ???

Let us go back to the space of genealogies now. Recall that for a space X the set of isomorphisms classes of maps $Y \to X$ where Y is a finite set of n elements is parametrized by the symmetric power $S^n X$ of X. The set of isomorphisms classes of maps from all finite sets to X is then parametrized by

$$S^\bullet X := \coprod_{n\geq 0} S^n X$$

Let h be a genealogy and let $\nu_n = \#(V_n(h))$ is the number of "proper" vertices of h with n outgoing edges. By the previous remark, for each $n = 0, 2, \ldots$ the set V_n together with the function τ defines a point in $S^{\nu_n} T$. Our conditions further imply that for $n = 0$ this point belongs to $S^{\nu_n}((a, b])$ and for $n \geq 2$ it belongs to $S^{\nu_n}((a, b))$. Therefore, we get a map

$$[\textbf{impmap}]\pi : D[T] \to S^\bullet((a, b]) \times \prod_{n=2}^{\infty} S^\bullet((a, b)) \tag{16}$$

Consider both sides of (16) as measurable spaces with respect to the σ-algebras of Borel subsets in the corresponding topologies.

Proposition 5.5 *[**firstprop**] The map (16) is an m-covering.*

Proof: ???