

# Notes on $T$ -delooping

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Dec. 2000

Dear Fabien,

one can study  $T$ -loop spaces in three different contexts. First one may consider  $T$ -loop spaces in the (pointed) unstable  $\mathbf{A}^1$ -homotopy theory. This is the most interesting but also the most difficult case. Second, one can consider  $T$ -loop spaces in the s-stable  $\mathbf{A}^1$ -homotopy theory. Third, one can consider  $T$ -loop spaces in the  $\mathbf{A}^1$ -homotopy theory of sheaves of abelian groups.

The difference between the second and the third context is not so great, in particular they are equivalent rationally. The difference between the first and the second is more serious but it is still a matter of s-stabilization which is in the realm of the ordinary homotopy theory (modulo the  $\mathbf{A}^1$ -locality problem which we talked about). At least one application of the would-be theory of  $T$ -loop spaces requires only the s-stable context - the new way of constructing the motivic spectral sequence for algebraic K-theory outlined in [?] (a more detailed account is being written).

In what follows I want to concentrate on the context of the homotopy theory of sheaves of abelian groups and consider most basic case of the delooping problem. Namely, I would like to find a  $T$ -analog of the following two theorems in topology:

**Theorem 0.1** [th1top] *Let  $X$  be a discrete space. Then finding a representation of  $X$  as a loop space is equivalent to giving a group structure on  $X$ .*

**Theorem 0.2** [th2top] *Let  $X$  be a group. Then  $\Omega^{-1}(X)$  has a loop space structure if and only if  $X$  is abelian. In that case this loop space structure is unique and is, also, an  $\infty$ -loop space structure.*

For motivic analogs of these results I want to consider, instead of discrete spaces  $X$ ,  $T$ -rigid, strictly homotopy invariant sheaves of abelian groups  $F$ : a sheaf is called  $T$ -rigid if  $F_{-1} = 0$ . We ask the following question: given such a sheaf  $F$ , what structure on  $F$  defines a representation of  $F$  as a  $T$ -loop space? What additional structure or what property of the original one is needed for  $F$  to be a 2-loop space (resp. n-loop space)?

As the zero approximation I suggest to consider the following “conjectures”. It only makes sense in the case of schemes over a field of characteristic zero.

**Conjecture 1 [conj1]** *A  $T$ -loop space structure on  $F$  is determined by giving  $F$  transfer maps for all finite étale coverings which are additive and commute with the base change.*

**Conjecture 2 [conj2]** *Let  $F$  be a  $T$ -rigid, strictly homotopy invariant sheaf with étale transfers. Then  $\Omega_T^{-1}(F)$  is a  $T$ -loop space if and only if for a sequence of finite étale maps  $X \xrightarrow{f} Y \xrightarrow{g} Z$  one has*

$$[\mathbf{com}]tr_{gf} = tr_g tr_f \tag{0.1}$$

*In this case the  $T$ -loop space structure on  $\Omega_T^{-1}(F)$  is unique and is an  $\infty$ -loop space structure.*

Here are some arguments towards these conjectures.

**Lemma 0.3 [l1]** *Let  $G$  be an  $\mathbf{A}^1$ -local complex of sheaves and assume that  $\Omega_T^1(G)$  is quasi-isomorphic to a rigid sheaf  $F$ . Then  $F$  has canonical transfers for finite étale maps which are additive and commute with the base change.*

Note that under the assumptions of the lemma  $F$  is automatically strictly homotopy invariant and that  $F = \underline{H}_0(\Omega_T^1(G))$ .

**Proof:** We knew it for a long time.

**Lemma 0.4 [l2]** *In the notations of Lemma ??, assume that  $G = \Omega_T^1(H)$  for an  $\mathbf{A}^1$ -local  $H$ . Then the transfers on  $F$  satisfy the condition (??)*

**Proof:** Should not be hard.

**Proposition 0.5 [l3]** *Let  $F$  be a strictly homotopy invariant rigid sheaf with étale transfers satisfying (??). Then  $F$  is a sheaf with transfers.*

**Proof:** Same as above.

**Corollary 0.6** *Let  $F$  be a strictly homotopy invariant rigid sheaf with étale transfers satisfying (??). Then  $F$  is an  $\infty$ -loop space.*

These results were, kind of, known. To prove Conjectures ??, ?? in general I suggest to construct “a theory of cycles with coefficients in  $F$ ”. Let us start with the following definition. For  $F$  as above, a smooth scheme  $S$  over  $k$  and a smooth scheme  $X$  over  $S$  denote by  $z(X/S, d, F)$  the group  $\text{colim}_Z \oplus_i F(\text{Spec}(k_{Z_i}))$  where  $Z$  runs through all closed subsets in  $X$  equidimensional of relative dimension  $d$  over  $S$ ,  $Z_i$  are the irreducible components of  $Z$  and  $k_{Z_i}$  is the function field of  $Z_i$ . Similarly, we can define  $c(X/S, d, F)$  as the subgroup in  $z(X/S, d, F)$  obtained by taking only  $Z$ 's which are proper over  $S$ .

The groups  $z(X/S, d, F)$  and  $c(X/S, d, F)$  are well defined for all  $F$  and they are contravariantly functorial with respect to dominant morphisms. My feeling is that they are not functorial with respect to all morphisms in general. Namely, I expect that transfers on  $F$  satisfying the conditions of Conjecture ?? are required to make these groups functorial for  $\dim(X/S) \geq 1$  and transfers satisfying (??) are needed to make them functorial for all  $X$ . I further expect that  $z(\mathbf{A}^n/S, 0, F)$  provides an  $n$ -fold  $T$ -delooping of  $F$ .

**Example 0.7** [ex1] Let  $F = \mathbf{Z}$ , then  $z(X/S, F, d)$  described above is just  $z_{\text{equi}}(X/S, d)$ . For  $X = \mathbf{A}^n$  it is known to be a model for  $K(\mathbf{Z}(n), 2n)$ .

To get the general functoriality of the groups  $z(X/S, F, d)$  it is clearly sufficient to define, for a point  $s$  of  $S$ , a specialization morphism  $z(X/S, d, F) \rightarrow z(X_s/\text{Spec}(k_s), d, F)$  and to check that these specialization morphisms are “transitive”.

We first define specialization to a point  $s \in S$  of codimension 1. Localizing  $S$  in  $s$  we may assume that  $S$  is the spectrum of a discrete valuation ring. Let  $Z$  be our closed subset which we consider as a reduced scheme  $Z_0$  the generic fiber of  $Z$  and  $f \in F(Z_0) \subset z(X/S, d, F)$  be the element which we want to specialize. Let  $Z'$  be the semilocal scheme of the set generic points of the closed fiber of  $Z$  and let  $\tilde{Z}$  be the normalization of  $Z'$ . Since  $\dim(Z') = 1$ ,  $\tilde{Z}$  is a normal scheme of dimension  $\leq 1$  and, therefore, it is smooth. Since  $F$  is rigid the section of  $F$  over  $Z_0$  will extend to a unique section of  $F$  over  $\tilde{Z}$ .

Let  $Z_{cl}$  be the closed fiber of  $Z$  and  $Z_{cl,0}$  be set of the generic points of  $Z_{cl}$  (i.e. the set of closed points of  $Z'$ ) and let  $\tilde{Z}_{cl,0} = (\tilde{Z} \times_Z Z_{cl})_{\text{red}}$ . We have an étale map  $p : \tilde{Z}_{cl,0} \rightarrow Z_{cl,0}$ . Set

$$(Z, f)_s = (Z_{cl}, p_*(\tilde{f}|_{\tilde{Z}_{cl}}))$$

Let now  $s$  be a point of arbitrary codimension. Let  $S_s$  be the blow-up of  $S$  in  $s$  and  $s'$  be the generic point of the special fiber of  $p : S_s \rightarrow S$ . Then, using

the construction of the previous paragraph we can define the specialization of  $p^*(Z, f)$  to any point in an open neighborhood of  $s'$ . Let  $\tilde{s}$  be any  $k$ -point in this open neighborhood lying over  $s$ . Define  $(Z, f)_s$  as  $(p^*(Z, f))_{\tilde{s}}$ .

**Conjecture 3** [A]/[conj3] *For any  $F$  satisfying the conditions of Conjecture ?? and any smooth  $X \rightarrow S$  of relative dimension  $d$  the construction described above define a structure of a presheaf on  $z(X/S, F, d - 1)$ .*

This ‘‘conjecture’’ is what I am most interested in at the moment. If it is false then the conditions on transfers should be added in Conjecture ?? to make it right. Note that if we start with the transfer structure defined by a delooping then it holds since for  $F = \Omega_T^1 G$  we can write

$$z(X/S, F, d - 1) = \operatorname{colim}_{Z \subset X} \operatorname{Hom}_H(X/(X - Z), G)$$

**Conjecture 4** [Aprime] *For any  $F$  satisfying the conditions of Conjecture ?? and any smooth  $X \rightarrow S$ , the construction described above define a structure of a presheaf on  $z(X/S, F, d)$ .*

So far I can not see this either.

Let us now outline a proof of the following (assuming Conjecture ??):

**Pretheorem 0.8** [prth1] *Let  $G$  and  $F$  be as in Lemma ?? and assume in addition that  $G$  is  $T$ -connected. Then  $z(\mathbf{A}_S^1/S, F, 0)$  is  $\mathbf{A}^1$ -weakly equivalent to  $G$ .*

Recall that an object is called  $T$ -connected if it belongs to the localizing subcategory generated by  $T$ -suspensions of smooth schemes, or, in our context of sheaves of abelian groups, by objects of the form  $\mathbf{Z}(X) \otimes \mathbf{Z}(T)$ . The proof of Pretheorem ?? consists of two parts.

Consider first the following construction. For a fibrant  $\mathbf{A}^1$ -local complex  $K$  define  $s_0(K)$  as the complex of sheaves associated with the complex of presheaves given by

$$S \mapsto \operatorname{colim}_{Z \subset \mathbf{A}_S^1} K(\mathbf{A}_S^1 - Z)$$

where  $Z$  runs through all closed subsets in  $\mathbf{A}_S^1$  equidimensional of relative dimension 1 over  $S$ . Let further  $\tau_{\geq 1}^T(K)$  be the fiber of the obvious morphism

$\underline{Hom}(\mathbf{Z}(\mathbf{A}^1), K) \rightarrow s_0(K)$  such that we have a distinguished triangle of complexes of sheaves of the form

$$[\mathbf{eq1}] \tau_{\geq 1}^T(K) \rightarrow \underline{Hom}(\mathbf{Z}(\mathbf{A}^1), K) \rightarrow s_0(K) \quad (0.2)$$

Pretheorem ?? is a corollary of the following two results.

**Proposition 0.9** [pr1] *For any  $\mathbf{A}^1$ -fibrant complex  $G$  such that  $\Omega_T^1(G) \cong \underline{H}_0(\Omega_T^1(G))$  and  $\Omega_T^1(G)$  is rigid, one has a canonical quasi-isomorphism of complexes*

$$z(\mathbf{A}_S^1/S, \underline{H}_0(\Omega_T^1(G)), 0) = \tau_{\geq 1}^T(G)$$

**Proof:** ???

**Proposition 0.10** [pr2] *Let  $G$  be a  $T$ -connected fibrant complex. Then  $s_0(G)$  is  $\mathbf{A}^1$ -weakly equivalent to zero.*

**Proof:** This is something I did very long ago in the context of motives. The proof seems to work for all sheaves as well - it is based on an observation that  $s_0(F \wedge T)$  is always  $\mathbf{A}^1$ -weakly equivalent to zero (it is not hard to show directly) and, moreover, this holds for the analog of  $s_0$  defined with respect to any smooth scheme over  $S$  everywhere of relative dimension  $> 0$  instead of  $\mathbf{A}^1$ . Then, one should express the right derived version of  $s_0(-)$  in terms of  $s_0(-)$  with respect to all Nisnevich covers of  $\mathbf{A}^1$  over all schemes over  $S$ .

Proposition ?? together with the following easy lemma, implies that (??) gives a model for the canonical decomposition of any  $G$  into a  $T$ -connected and a  $T$ -rigid part.

**Lemma 0.11** [elemma] *If  $G$  is rigid then  $G \rightarrow s_0(G)$  is a quasi-isomorphism i.e.  $\tau_{\leq 1}^T(G) = 0$ .*

**Proof:** Follows from the homotopy purity.

**Remark 0.12** [iter] It seems that similar reasoning can be used to show that the classifying space build on  $\mathbf{A}^n$  of an  $n$ -connected with respect to  $T$ ,  $n$ -fold loop space is this space again.

Let us consider now the opposite problem - how to show that, for some  $F$ , we have  $\Omega_T^1 z(\mathbf{A}_S^1/S, F, 0) \cong F$ . The following approach, taken from the motivic cohomology theory which is the case  $F = \mathbf{Z}$ , may work.