Notes on T-delooping

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Dear Fabien,

I probably reached some progress on the question of the recognition for T-loop spaces and T-delooping which we discussed recently. Here is how things are.

Observe first that one can study T-loop spaces in three different contexts. First one may consider T-loop spaces in the (pointed) unstable $\mathbf{A}^1$-homotopy theory. This is the most interesting but also the most difficult case. Second, one can consider T-loop spaces in the s-stable $\mathbf{A}^1$-homotopy theory. Third, one can consider T-loop spaces in the $\mathbf{A}^1$-homotopy theory of sheaves of abelian groups.

The diffrence between the second and the third context is not so great, in particular they are equivalent rationally. The diffrence between the fisrt and the second is more serious but it is still a matter of s-stabilization which is in the realm of the ordinary homotopy theory (modulo the $\mathbf{A}^1$-locality problem which we talked about). At least one application of the would-be theory of T-loop spaces requires only the s-stable context - the new way of constructing the motivic spectral sequence for algebraic K-theory outlined in [**?**] (a more detailed account is being written).

In what follows I want to concentrate on the context of the homotopy theory of sheaves of abelian groups and consider most basic case of the delooping problem. Namely, I would like to find a T-analog of the folowing two theorems in topology:

Theorem 0.1 *[**th1top**] Let X be a discrete space. Then finding a representation of X as a loop space is equivalent to giving a group structure on X.*

Theorem 0.2 *[**th2top**] Let X be a group. Then $\Omega^{-1}(X)$ has a loop space structure if and only if X is abelian. In that case this loop space structure is unique and is, also, an ∞-loop space structure.*

For motivic analogs of these results I want to consider, instead of discrete spaces X, T-rigid, strictly homotopy invariant sheaves of abelian groups F: a sheaf is called T-rigid if $F_{-1} = 0$. We ask the following question: given such a sheaf F, what structure on F defines a representation of F as a T-loop space? What additional structure or what property of the original one is needed for F to be a 2-loop space (resp. n-loop space).

Here is my conjectural answer. I only know how to formulate it in the case of schemes over a field of characteristic zero.

Conjecture 1 *[**conj1**] A T-loop space structure on F is determined by giving F transfer maps for all finite etale coverings which are additive and commute with the base change.*

Conjecture 2 *[**conj2**] Let F be a T-rigid, strictly homotopy invariant sheaf with etale transfers. Then $\Omega_T^{-1}(F)$ is a T-loop space if and only if for a sequence of finite etale maps $X \stackrel{f}{\to} Y \stackrel{g}{\to} Z$ one has*

$$[\mathbf{com}] tr_{gf} = tr_g tr_f \tag{0.1}$$

In this case the T-loop space structure on $\Omega_T^{-1}(F)$ is unique and is an ∞-loop space structure.

Here are some arguments towards these conjectures.

Lemma 0.3 *[**l1**] Let G be a an $\mathbf{A}^1$-local complex of sheaves and assume that $\Omega_T^1(G)$ is quasi-isomorphic to a rigid sheaf F. Then F has canonical transfers for finite etale maps which are additive and commute with the base change.*

Note that under the assumptions of the lemma F is automatically strictly homotopy invariant and that $F = \underline{H}_0(\Omega_T^1(G))$.
Proof: We knew it fo a long time.

Lemma 0.4 *[**l2**] In the notations of Lemma **??**, assume that $G = \Omega_T^1(H)$ for an $\mathbf{A}^1$-local H. Then the transfers on F satisfy the condition (**??**)*

Proof: Should not be hard.

Proposition 0.5 *[**l3**] Let F be a strictly homotopy invariant rigid sheaf with etale transfers satisfying (**??**). Then F is a sheaf with transfers.*

Proof: Same as above.

Corollary 0.6 *Let F be a strictly homotopy invariant rigid sheaf with etale transfers satisfying (**??**). Then F is an ∞-loop space.*

These results were, kind of, known. To prove Conjectures **??**, **??** in general I suggest to construct "a theory of cycles with coefficients in F". Let us start with the following definition. For F as above, a smooth scheme S over k and a smooth scheme X over S denote by $z(X/S,d,F)$ the group $colim_Z \oplus_i F(Spec(k_{Z_i}))$ where Z runs through all closed subsets in X equidimensional of relative dimension d over S, Z_i are the irreducible components of Z and k_{Z_i} is the function field of Z_i. Similarly, we can define $c(X/S,d,F)$ as the subgroup in $z(X/S,d,F)$ obtained by taking only Z's which are proper over S.

The groups $z(X/S,d,F)$ and $c(X/S,d,F)$ are well defined for all F and they are contravariantly functorial with respect to dominant morphisms. My feeling is that they are not functorial with respect to all morphisms in general. Namely, I expect that transfers on F satisfying the conditions of Conjecture **??** are required to make these groups functorial for $dim(X/S) \geq 1$ and transfers satisfying (**??**) are needed to make them functorial for all X. I further expect that $z(\mathbf{A}^n/S,0,F)$ provides an n-fold T-delooping of F.

Example 0.7 [**ex1**]Let $F = \mathbf{Z}$, then $z(X/S,F,d)$ described above is just $z_{equi}(X/S,d)$. For $X = \mathbf{A}^n$ it is known to be a model for $K(\mathbf{Z}(n),2n)$.

To get the general functoriality of the groups $z(X/S,F,d)$ it is clearly sufficient to define, for a point s of S, a specialization morphism $z(X/S,d,F)/srz(X_s/Spec(k_s),d,F)$ and to check that these specialization morphisms are "transitive".

We first define specialization to a point $s \in S$ of codimenison 1. Localizing S in s we may assume that S is the spectrum of a disceret valuation ring. Let Z be our closed subset which we consider as a reduced scheme Z_0 the generic fiber of Z and $f \in F(Z_0) \subset z(X/S,d,F)$ be the element which we want to specialize. Let Z' be the semilocal scheme of the set generic points of the closed fiber of Z and let $\tilde{Z}$ be the normalization of Z'. Since $dim(Z') = 1$, $\tilde{Z}$ is a normal scheme of dimension ≤ 1 and, therefore, it is smooth. Since F is rigid the section of F over Z_0 will extend to a unique section of F over $\tilde{Z}$.

Let Z_{cl} be the closed fiber of Z and $Z_{cl,0}$ be set of the generic points of Z_{cl} (i.e. the set of closed points of Z') and let $\tilde{Z}_{cl,0} = (\tilde{Z} \times_Z Z_{cl})_{red}$. We have an etale map $p : \tilde{Z}_{cl,0} \to Z_{cl,0}$. Set

$$(Z,f)_s = (Z_{cl}, p_*(\tilde{f}_{|\tilde{Z}_{cl}}))$$

Let now s be a point of arbitrary codimension. Let S_s be the blow-up of S in s and s' be the genric point of the special fiber of $p : S_s \to S$. Then, using the construction of the previous paragraf we can define the specialization of $p^*(Z, f)$ to any point in an open neighborhood of s'. Let $\tilde{s}$ be any k-point in this open neighborhood lying over s. Define $(Z, f)_s$ as $(p^*(Z, f))_{\tilde{s}}$.

Lemma 0.8 *[**l4**] For any F satisfying the conditions of Conjecture **??** and any smooth $X \to S$ such that $dim(X/S) \leq 1$, the construction described above define a structure of a presheaf on $z(X/S, F, 0)$.*

Proof: I do not know how to prove this "lemma".

Lemma 0.9 *[**l4prime**] For any F satisfying the conditions of Conjecture **??** and any smooth $X \to S$, the construction described above define a structure of a presheaf on $z(X/S, F, d)$.*

Proof: I do not know how to prove this "lemma" either.

Let us now outline a proof of the following:

Pretheorem 0.10 *[**prth1**] Let G and F be as in Lemma **??** and assume in addition that G is T-connected. Then $B_T F$ is $\mathbf{A}^1$-weakly equivalent to G.*

Recall that an object is called T-connected if it belongs to the localizing subcategory generated by T-suspensions of smooth schemes, or, in our context of sheaves of abelian groups, by objects of the form $\mathbf{Z}(X) \otimes \mathbf{Z}(T)$. The proof of Pretheorem **??** consists of two parts.

Consider first the following construction. For a fibrant $\mathbf{A}^1$-local complex K define $s_0(K)$ as the complex of sheaves associated with the complex of presheaves given by

$$S \mapsto colim_{Z \subset \mathbf{A}^1_S} K(\mathbf{A}^1_S - Z)$$

where Z runs through all closed subsets in $\mathbf{A}^1_S$ equidimensional of relative dimension 1 over S. Let further $\tau^T_{\geq 1}(K)$ be the fiber of the obvious morphism $\underline{Hom}(\mathbf{Z}(\mathbf{A}^1), K) \to s_0(K)$ such that we have a distinguiished triangle of complexes of sheaves of the form

$$[\mathbf{eq1}]\tau^T_{\geq 1}(K) \to \underline{Hom}(\mathbf{Z}(\mathbf{A}^1), K) \to s_0(K) \tag{0.2}$$

Pretheorem **??** is a corollary of the following two results.

Proposition 0.11 *[***pr1***] For any $\mathbf{A}^1$-fibrant complex G such that $\Omega^1_T(G) \cong \underline{H}_0(\Omega^1_T(G))$ and $\Omega^1_T(G)$ is rigid, one has a canonical quasi-isomorphism of complexes*

$$B_T\underline{H}_0(\Omega^1_T(G)) = \tau^T_{\geq 1}(G)$$

Proof: ???

Proposition 0.12 *[***pr2***] Let G be a T-connected fibrant complex. Then $s_0(G)$ is $\mathbf{A}^1$-weakly equivalent to zero.*

Proof: This is something I did very long ago in the context of motives. The proof seems to work for all sheaves as well - it is based on an observation that $s_0(F \wedge T)$ is always $\mathbf{A}^1$-weakly equivalent to zero (it is not hard to show directly) and, moreover, this holds for the analog of s_0 defined with respect to any smooth scheme over S everywhere of relative dimension > 0 instead of $\mathbf{A}^1$. Then, one should express the right derived version of $s_0(-)$ in terms of $s_0(-)$ with respect to all Nisnevich covers of $\mathbf{A}^1$ over all schemes over S.

Proposition **??** togther with the following easy lemma, implies that (**??**) gives a model for the canonical decomposition of any G into a T-connected and a T-rigid part.

Lemma 0.13 *[***elemma***] If G is rigid then $G \to s_0(G)$ is a quasi-isomorphism i.e. $\tau^T_{\leq 1}(G) = 0$.*

Proof: Follows from the homotopy purity.

Remark 0.14 [**iter**] It seems that similar reasoning can be used to show that the classifying space build on $\mathbf{A}^n$ of an n-connected with respect to T, n-fold loop space is this space again.

Let us consider now the opposite problem - how to show that, for some F, we have $\Omega^1_T B_T F \cong F$. The following approach, taken from the motivic cohomology theory which is the case $F = \mathbf{Z}$, may work. First let us extend the B_T construction in another direction - instead of closed subsets of codimension 1 in $\mathbf{A}^1$ we may consider closed subsets of codimesnion 1 in any other smooth scheme over S (as opposed to Remark **??** where one consideres closed subsets of relative dimension 1).