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1 Basics

We will need several simple lemmas about the functorial behavior of pointed simplicial sheaves on the categories of smooth schemes over a base with Nisnevich topology. Many of them have well known analogs for sheaves on "small" sites but we often have to give quite differnet proofs. The main reason for most of the differences is that for sheaves on smooth sites stalks and inverse images do not commute i.e. Theorem 3.2(a) of [**?**] fails to be true.

Everywhere below we work in the context of pointed sheaves of sets on $(Sm/S)_{Nis}$. In [**?**] we called such sheaves "spaces" over S but since the techniques of this paper have much more to do with the usual sheaf theory than with the homotopy theory we do not use this name here. The category of pointed (simplicial) sheaves on $(Sm/S)_{Nis}$ is a pointed category which has all small products and coproducts. Its initial/final object is denoted by pt and the direct sums by $\vee_\alpha F_\alpha$. The smash product $F \wedge G$ of two pointed simplicial sheaves is given by the usual formula ([**?**,]) and satisfies the standard associativity and commutativity conditions. The unit object of the symmetric monoidal structure defined by $\wedge$ is denoted S^0.

For any morphism of schemes $f : S_1 \to S_2$ we have the inverse image functor f^* which is caracterized by the properties that it commutes with colimits and that for a smooth scheme X over S_2 one has $f^*(X_+) = (X \times_{S_2} S_1)_+$. It has the right adjoint called the direct image functor f_*. For smooth morphisms f the functor of inverse image also has the left adjoint $f_\#$ (see []) such that for a smooth scheme X over S_1 one has $f_\#(X_+) = X_+$ where on the right side of the equality X is considered as a smooth scheme over S_2.

Remark 1.1 [**forgetful**]Functors of all three types commute with the functor of free base point $F \mapsto F_+$ from sheaves to pointed sheaves. Functors f^* and f_* also commute with the forgetful functor from pointed sheaves to

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sheaves but functors $f_\#$ do not. If we denote the forgetful functor by ϕ then for a smooth morphism $f: S_1 \to S_2$ and a pointed sheaf F over S_1 one has a push-forward square of sheaves of the form

$$\begin{array}{ccc} S_1 & \to & f_\#(\phi(F)) \\ \downarrow & & \downarrow \\ S_2 & \to & \phi(f_\#(F)) \end{array}$$

The following lemmas can be seen directly from definitions.

Lemma 1.2 *[10] For a composable pair of morphisms $S_1 \stackrel{f}{\to} S_2 \stackrel{g}{\to} S_3$ there is a canonical ismorphism $(g \circ f)^* \to f^* \circ g^*$ and for a composable triple f, g, h the square*

$$\begin{array}{ccc} (fgh)^* & \to & (gf)^* h^* \\ \downarrow & & \downarrow \\ f^*(gh)^* & \to & f^* g^* h^* \end{array}$$

commutes.

By adjunction the isomorphisms of Lemma 1.2 define isomorphisms $g_* \circ f_* \to (g \circ f)_*$ and for smooth f, g isomorphisms $g_\# \circ f_\# \to (g \circ f)_\#$.

Lemma 1.3 *[11] For any $f : S_1 \to S_2$ and any F, G over S_2 there is a canonical isomorphism $f^*(F \wedge G) = f^*(F) \wedge f^*(G)$.*

Lemma 1.4 *[12] For any smooth morphism $f : S_1 \to S_2$ any F over S_1 and G over S_2 the morphism $f_\#(F \wedge f^* G) \to f_\# F \wedge G$ defined by the adjunctions and the isomorphisms of Lemma 1.3 is an isomorphism.*

Lemma 1.5 *[13] For any pull-back square*

$$\begin{array}{ccc} S_1' & \stackrel{f_1}{\to} & S_1 \\ p' \downarrow & & \downarrow p \\ S_2' & \stackrel{f_2}{\to} & S_2 \end{array}$$

such that p is smooth and any F over S_1 the morphism $p'_\# f_1^*(F) \to f_2^* p_\#(F)$ defined by the adjunctions and the isomorphism of Lemma 1.2 is an isomorphism.

In general neither of the three types of functors considered above preserve simplicial or $\mathbf{A}^1$-weak equivalences. In order to define the (left) derived functors for f^* and $f_\#$ we need the following construction.

pre-Definition 1.1 [**ladm**] *An object is called left admissible if it is admissible with respect to all f^*'s and $f_\#$'s.*

Lemma 1.6 [**l6**] *Let $a : F \to G$ be a simplicial (resp. $\mathbf{A}^1$-) weak equivalence of left admissible objects over S. Then for any morphism $f : S' \to S$ the morphism $f^*(a)$ is a simplicial (resp. $\mathbf{A}^1$-) weak equivalence and for any smooth morphism $f : S \to S'$ the morphism $f_\#(a)$ is a simplicial (resp. $\mathbf{A}^1$-) weak equivalence.*

Lemma 1.7 [**l7**] *Let F be a left admissible object. Then for any morphism $f : S' \to S$ the object $f^*(F)$ is left admissible and for any smooth morphism $f : S \to S'$ the object $f_\#(F)$ is left admissible.*

Lemma 1.8 [**admsm**] *Let F and G be left admissible objects. Then $F \wedge G$ is left admissible.*

Lemma 1.9 [**lres**] *For any S there exists a functor $Lres : \Delta^{op}Spc_\bullet \to \Delta^{op}Spc_\bullet$ called the left resolution functor and a natural transformation $Lres \to Id$ such that the following two conditions hold:*

1. *for any F the terms of the simplicial sheaf $Lres(F)$ are direct sums of pointed sheaves of the form U_+ for smooth quasi-projective schemes U over S.*

2. *for any F and any smooth quasi-projective scheme U over S the morphism of simplicial sets $Lres(F)(U) \to F(U)$ is a trivial Kan fibration.*

Proof: ???

We define the left derived functors of f^* and $f_\#$ setting $\mathbf{L}f^* = f^* \circ Lres$ and $\mathbf{L}f_\# = f_\# \circ Lres$.

Lemma 1.10 [**l5**] *For any morphism $f : S_1 \to S_2$ and a simplicial (resp. $\mathbf{A}^1$-) weak equivalence $a : F \to G$ over S_2 the morphism $\mathbf{L}f^*(a)$ is a simplicial (resp. $\mathbf{A}^1$-) weak equivalence.*

For a smooth morphism $f : S_1 \to S_2$ and a simplicial (resp. $\mathbf{A}^1$-) weak equivalence $a : F \to G$ over S_1 the morphism $\mathbf{L}f_\#(a)$ is a simplicial (resp. $\mathbf{A}^1$-) weak equivalence.

Proof: ???

Lemma 1.11 *[l51] For any morphism $f : S_1 \to S_2$ and a left admissible object F over S_2 the morphism $\mathbf{L}f^*(F) \to f^*(F)$ is a simplicial weak equivalence.*

For a smooth morphism $f : S_1 \to S_2$ and a left admissible object F over S_1 the morphism $\mathbf{L}f_\#(F) \to f_\#(F)$ is a simplicial weak equivalence.

We will need to know how functors $\mathbf{L}f^*$ and $\mathbf{L}f_\#$ behave with respect to homotopy colimits. Let us recall the definition of homotopy colimits first. Let I be a small category and $X : I \to \Delta^{op}Spc_\bullet$ a diagram of pointed (simplicial) sheaves indexed by I. For $i \in I$ one usually denotes $X(i)$ by X_i. Let I/i be the category of objects in I over i (i.e. the category of arrows which end in i) and let $Nerv(I/i)$ be the nerve of I/i i.e. the simplicial set whose n-simplexes are composable sequences of arrows in I/i of length n. For any morphism $\gamma : i \to i'$ in I we have a functor $I/i' \to I/i$ and thus a morphism of simplicial sets $N_\gamma : Nerv(I/i') \to Nerv(I/i)$. Following [, p.328] one defines the homotopy colimit $hocolim_{i\in I}X_i$ as the coequalizer of two morphisms

$$\bigvee_{\gamma:i\to i'} Nerv(I/i')_+ \wedge X_i \rightrightarrows \bigvee_i Nerv(I/i)_+ \wedge X_i$$

where the first arrow is given on $Nerve(I/i')_+ \wedge X_i$ by $Id \wedge X(\gamma)$, the second by $N(\gamma)_+ \wedge Id$ and simplicial sets are considered as constant simplicial sheaves in the usual manner. The following three lemmas describe the main properties of this construction.

Lemma 1.12 *[hocolim0] Let $X, Y : I \to Spc_\bullet$ be two diagrams of pointed simplicial sheaves and $a : X \to Y$ a morphism such that for any $i \in I$ the morphism $a_i : X_i \to Y_i$ is a simplicial (resp. $\mathbf{A}^1$-) weak equivalence. Then the morphism $hocolim(a)$ is a simplicial (resp. $\mathbf{A}^1$-weak equivalence).*

Proof: For the simplicial case see [**?**, Cor. 2.1.21]. For the $\mathbf{A}^1$-case see [**?**, Lemma 2.2.12].

The following two lemmas are immediate corollaries of the corresponding results for simplicial sets proven in [, Ch.XII, §3].

Lemma 1.13 *[hocolim1] Let $X : \Delta^{op} \to \Delta^{op}Spc_\bullet$ be a pointed bisimplicial sheaf. Then there is a canonical simplicial weak equivalence $hocolim_{\Delta^{op}}(X) \to diag(X)$ where $diag(X)$ is the diagonal simplicial sheaf of X. In particular* *for any pointed simplicial sheaf considered as a functor $X : \Delta^{op} \to Spc_\bullet \subset \Delta^{op}Spc_\bullet$ there is a canonical simplicial weak equivalence $hocolim_{\Delta^{op}} X_n \to X$ where X_n are the pointed sheaves of n-simplexes of X.*

Lemma 1.14 *[**hocolim2**] For a pushforward square*

$$\begin{array}{ccc} A & \stackrel{i}{\to} & X \\ \downarrow & & \downarrow \\ B & \to & Y \end{array}$$

such that i is a monomorphism, the canonical map $hocolim(\begin{array}{ccc} A & \stackrel{i}{\to} & X \\ \downarrow & & \\ B & & \end{array}) \to Y$ is a simplicial weak equivalence.

Since the functor of inverse image is a left adjoint it commutes with colimits which immediately implies that for any small diagram $(X_i)_{i\in I}$ we have a canonical isomorphism $i^* hocolim_I X_i \to hocolim_I i^*(X_i)$.

Lemma 1.15 *[**hocolim3**] For any small diagram $(X_i)_{i\in I}$ over S such that X_i are left admissible $hocolim_{i\in I} X_i$ is left admissible.*

Proof: ???

Lemma 1.16 *[**Lho**] For any morphism $f : S' \to S$ and any small diagram $(X_i)_{i\in I}$ over S there is a natural (in X) isomorphism in the simplicial homotopy category $H_s(S')$ of the form*

$$\mathbf{L}f^*(hocolim_{i\in I} X_i) \to hocolim_{i\in I} \mathbf{L}f^*(X_i)$$

such that the following square commutes

$$\begin{array}{ccc} \mathbf{L}f^*(hocolim_{i\in I} X_i) & \to & hocolim_{i\in I} \mathbf{L}f^*(X_i) \\ \downarrow & & \downarrow \\ f^*(hocolim_{i\in I} X_i) & \to & hocolim_{i\in I} i^*(X_i) \end{array}$$

Proof: Recall that $\mathbf{L}f^* = f^* \circ Lres$ and consider the diagram

$$\begin{array}{ccccc} f^*Lres(hc_{i\in I}(Lres(X_i))) & \to & f^*hc_{i\in I}(Lres(X_i)) & \to & hc_{i\in I} f^*Lres(X_i) \\ \downarrow & & \downarrow & & \downarrow \\ f^*Lres(hc_{i\in I} X_i) & \to & f^*(hc_{i\in I} X_i) & \to & hc_{i\in I} i^*(X_i) \end{array}$$

where we abbreviated *hocolim* to *hc*. The left vertical arrow is a simplicial weak equivalence by Lemmas 1.12 and 1.10, the first upper horizontal one is a simplicial weak equivalence by Lemmas 1.6 and 1.15 and the second is the canonical isomorphism. We define our isomorphism as the composition of the inverse to the left vertical arrow with the upper horizontal ones. It is clearly natural with respect to morphisms of diagrams. To prove commutativity of the square it is sufficient to show that the two squares of the diagram from above are commutative. The first one is commutative since $Lres \to Id$ is a natural transformation of functors and the second one since $f^*hc_{i\in I} \to hc_{i\in I}f^*$ is a natural transformation of functors.

Lemma 1.17 [**constproj**] *For any morphism $f : S' \to S$ one has:*

1. *For any family $(F_i)_{i\in I}$ of pointed objects over S' the canonical morphism $\vee_i f_*(F_i) \to f_*(\vee_i F_i)$ is an isomorphism. In particular $f_*(pt) = pt$.*
2. *For any object F over S' and any pointed simplicial set K the morphism $K \wedge f_*(F) \to f_*(K \wedge F)$ defined by the adjunction and the isomorphism of Lemma 1.3 is an isomorphism.*

Proof: ???

Let $f : S' \to S$ be any morphism and $(X_i)_{i\in I}$ a diagram over S'. Define the canonical morphism $hocolim_{i\in I} f_*(X_i) \to f_* hocolim_{i\in I} X_i$ by the commutative diagram

$$\begin{array}{ccccc} \bigvee_{\gamma:i\to i'} Nerv(I/i')_+ \wedge p_*(X_i) & \rightrightarrows & \bigvee_i Nerv(I/i)_+ \wedge p_*(X_i) & \to & hc_i p_*(X_i) \\ \downarrow & & \downarrow & & \downarrow \\ p_*(\bigvee_{\gamma:i\to i'} Nerv(I/i')_+ \wedge X_i) & \rightrightarrows & p_*(\bigvee_i Nerv(I/i)_+ \wedge X_i) & \to & p_*(hc_i X_i) \end{array}$$

where the left and the middle vertical arrows are compositions of isomorphisms from Lemma 1.17(1) and 1.17(2) and where we abbreviated *hocolim* to *hc*.

Lemma 1.18 [**adcom**] *For any small diagram $(X_i)_{i\in I}$ over S' the square of canonical morphisms and adjunctions*

$$\begin{array}{ccc} f^* hocolim_{i\in I} f_*(X_i) & \to & hocolim_{i\in I} f^* f_*(X_i) \\ \downarrow & & \downarrow \\ f^* f_* hocolim_{i\in I} X_i & \to & hocolim_{i\in I} X_i \end{array}$$

commutes.

So far we were able to avoid mentioning "stalks" of sheaves on $(Sm/S)_{Nis}$ but some of the proofs below require their use. Let X be a smooth scheme over S and x be a point of the Zariski topological space of X. To any such pair we can assign a point $F \mapsto F_{(X,x)}$ of the site $(Sm/S)_{Nis}$ setting

$$F_{(X,x)} = colim_{(U,u)\to(X,x)} F(U)$$

where the colimit is taken over the category of all diagrams of the form

$$\begin{array}{ccc} & & U \\ {\scriptstyle u} & \nearrow & \downarrow \\ Spec(k_x) & \stackrel{x}{\to} & X \end{array}$$

with $U \to X$ being etale. One verifies easily that this is indeed a point i.e. that the functor $(-)_{(X,x)} : Shv((Sm/S)_{Nis}) \to Sets$ commutes with both limits nd colimits. One can also verify that the set of points corresponding to all the pairs (X, x) where X runs through smooth quasi-projective (or affine) schemes over S is a "sufficient" set of points i.e. that the following lemma holds.

Lemma 1.19 *[***points0***] A morphism $f : F \to G$ of sheaves on $(Sm/S)_{Nis}$ is an isomorphism (rep. a monomorphism, an epimorphism) if and only if for any smooth quasi-projective X over S and any point x of X the corresponding map of pointed sets $F_{(X,x)} \to G_{(X,x)}$ is an isomorphism (resp. a monomorphism, an epimorphism).*

The main difference between smooth sites and small sites is that for a closed embedding $i : Z \to S$ and a sheaf F on S one has $(i^*F)_{(Z,z)} \neq F_{(S,i(z))}$. Indeed if $F = X_+$ for a smooth scheme X over S we have

$$(i^*F)_{(Z,z)} = Hom_S(Spec(\mathcal{O}^h_{Z,z}), X)_+$$

and

$$F_{(S,i(z))} = Hom_S(Spec(\mathcal{O}^h_{S,i(z)}), X)_+$$

Note that if X is etale over S these two sets are the same which is the reason for the equality $(i^*F)_{(Z,z)} = F_{(S,i(z))}$ on small sites.

Lemma 1.20 *[***stfin***] Let $f : S' \to S$ be a finite morphism and F a sheaf on S'. Then for any (X, x) over S there is a canonical isomorphism*

$$f_*(F)_{(X,x)} = \prod_{x'\in X'_{Zar},\, pr(x')=x} F_{(X',x')}$$

where $X' = X \times_S S'$ and $pr : X' \to X$ is the projection.

Proof: ???

Lemma 1.21 *[**stet**] Let $f : S' \to S$ be an etale morphism and F a sheaf on S'. Then for any (X, x) over S there is a canonical isomorphism*

$$f_\#(F)_{(X,x)} = \bigvee_{x'} F_{(X',x')}$$

where $X' = X \times_S S'$ and the sum is taken over the points x' such that $pr(x') = x$ and the morphism $Spec(k_{x'}) \to Spec(k_x)$ is an isomorphism.

Proof: ???

2 The gluing theorem and its corollaries

Let $i : Z \to S$ be a closed embedding and $j : U \to S$ the complimentary open one. In this section we consider the $\mathbf{A}^1$-homotopy theoretical analogs of the classical results relating sheaves on S, Z and U. For pointed sheaves of sets on *small* sites the standard picture can be summarized as follows:

1. for any sheaf F on Z the adjunction $i^*i_*(F) \to F$ is an isomorphism
2. for any sheaf F on U the adjunction $F \to j^*j_\#(F)$ is an isomorphism
3. for any sheaf F on S the adjunctions $j_\#j^*(F) \to F$ and $F \to i_*i^*(F)$ fit into a pushforward square

$$\begin{array}{ccc} j_\#j^*(F) & \to & F \\ \downarrow & & \downarrow \\ pt & \to & i_*i^*(F) \end{array}$$

These facts have two important corollaries:

1. *Projection formula*: for a sheaf F on Z and sheaf G on S the morphism $F \wedge i_*(G) \to i_*(i^*F \wedge G)$ is an isomorphism
2. *Base change:* for a pull-back square

$$\begin{array}{ccc} Z' & \stackrel{f_Z}{\to} & Z \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ S' & \stackrel{f_S}{\to} & S \end{array}$$

and a sheaf F on Z the morphism $f_S^*i_*(F) \to i'_*f_Z^*$ is an isomorphism

Proposition 2.1 *[***nd1***] Let $p : Z \to S$ be a finite morphism. Then the functor of direct image p_* is right exact i.e. for any diagram $(X_i)_{i\in I}$ over Z the canonical morphism $colim_{i\in I} p_*(X_i) \to p_*(colim_{i\in I} X_i)$ is an isomorphism.*

Proof: Follows from Lemmas 1.19 and 1.20.

Corollary 2.2 *[***dirhc***] Let $p : Z \to S$ be a finite morphism. Then for any diagram $(X_i)_{i\in I}$ over Z the canonical morphism $hocolim_{i\in I} p_*(X_i) \to p_* hocolim_{i\in I} X_i$ is an isomorphism.*

Proof: Follows from Proposition 2.1 and the definition of the canonical morphism $hocolim_{i\in I} p_*(X_i) \to p_* hocolim_{i\in I} X_i$.

Lemma 2.3 *[***closed2***] Let $i : Z \to S$ be a closed embedding and $X \to Z$ a smooth scheme over Z. Then there exist a finite Zariski covering $X = \cup V_i$, smooth schemes W_i over S and isomorphisms $V_i \cong W_i \times_S Z$ over Z.*

Proof: We may assume that $S = Spec(R)$ and $Z = Spec(Q)$ are affine. By [**?**, Prop. 3.24(b)] we can find a covering $X = \cup V_i$ such that V_i are etale over $\mathbf{A}^n_Z$. By [**?**, Th. 3.4] we can further choose V_i's such that

$$V_i = Spec((A_i[T]/P_i)[1/b_i]),\ A = Q[x_1, \dots, x_n][1/f_i]$$

and P_i' is a unit in $(A_i[T]/P_i)[1/b_i]$. Let $\tilde{f}_i$ be a lifting of f_i to an element in R, $\tilde{P}_i$ a lifting of P_i to an element in $R[x_1, \dots, x_n][T]$ and $\tilde{b}_i$ a lifting of b_i to an element of $R[x_1, \dots, x_n][T]$. Set $W_i = Spec(\tilde{A}_i[T]/\tilde{P}_i[1/\tilde{b}_i, 1/\tilde{P}_i'])$ where $\tilde{A}_i = R[x_1, \dots, x_n](1/\tilde{f}_i)$. Then W_i is etale over $Spec(\tilde{A}_i)$ (by [**?**, Example 3.4]) and thus smooth over S and $W_i \times_S Z \cong V_i$ by construction.

Lemma 2.4 *[***nd2***] Let $i : Z \to S$ be a closed embedding. Then for any G over S the adjunction $G \to i_* i^*(G)$ is an epimorphism.*

Proof: Any pointed sheaf G over S is a colimit of a digram of sheaves of the form $(X_i)_+$ where X_i are smooth schemes over S. The functor i^* commutes with colimits because it is a left adjoint and i_* commutes with colimits by Lemma 2.1. Thus it is sufficient to show that $X_+ \to i_* i^* X_+$ is an epimorphism for a smooth scheme X over S. For any smooth U over S sections of $i_* i^*(X_+)$ over U are just sections of $X \times_S U \to U$ over the closed subscheme $Z \times_S U \to U$. Since X is smooth over S for any such section locally (in the Nisnevich topology) extends to a section over U.

Proposition 2.5 *[**p1**] Let $i : Z \to S$ be a closed embedding and $j : U \to S$ the complimentary open embedding. Then one has:*

1. *for any object F over Z the adjunction $i^*i_*(F) \to F$ is an isomorphism*
2. *for any object F over U the adjunction $F \to j^*j_\#(F)$ is an isomorphism*

Proof: By Lemma 2.3 any smooth scheme over Z has a Zariski covering by smooth schemes which come from S. Thus any pointed sheaf over Z is a colimit of pointed sheaves of the form $i^*((W_\alpha)_+)$ where W_α are smooth schemes over S. The functor i^* commutes with colimits because it is a left adjoint and i_* commutes with colimits by Lemma 2.1. Thus it is sufficient to prove that $i^*i_*i^*G \to i^*G$ is an isomorphism for any G over S. Since i^* and i_* are adjoint functors the composition $i^*G \to i^*i_*i^*G \to i^*G$ where the first arrow is $i^*(G \to i_*i^*G)$ is identity. On the other hand the first arrow is an epimorphism by Lemma 2.4. Therefore both arrows are isomorphisms.

To prove the second claim represent F as a colimit of a diagram of representable sheaves. Both j^* and $j_\#$ are left adjoints and therefore commute with colimits. Thus it is sufficient to verify the case $F = X_+$ where X is a smooth scheme over U. By construction of $j_\#$ we have $j_\#(X_+) = X_+$ where on the right hand side X is considered as a smooth scheme over S and $j^*j_\#(X_+) = X \times_S U$. Our claim follows now from the fact that the projection $X \times_S U \to X$ is an isomorphism.

The second part of this proposition togther with Lemma 1.4 imply:

Corollary 2.6 *[**c0**] For any F, G over U the morphism $j_\#(F \wedge G) \to j_\#F \wedge j_\#G$ given by the adjunction and isomorphism of Lemma 1.3 is an isomorphism.*

Let $j^*j_\# \to Id$ be the natural transformation inverse to the isomorphism of Proposition 2.5(2). By adjunction it defines a natural transformation $j_\# \to j_*$. One can immediately verify the following fact.

Lemma 2.7 *[**ves**] Let $j : U \to S$ be an open embedding such that $j(U)$ is a connected component of S. Then $j_\# \to j_*$ is an isomorphism.*

Let now $p : U \to S$ be an etale morphism. Define a natural transformation $p_\# \to p_*$ as the adjoint to the natural transformation $p^*p_\# \to Id$ given by the composition

$$p^*p_\#(F) = (pr_2)_\# pr_1^*(F) \to (pr_2)_\#\Delta_*(F) \cong (pr_2)_\#\Delta_\#(F) \cong F$$

where the first arrow is the isomorphism of Lemma 1.5 for the square

$$\begin{array}{ccc} U\times_S U & \stackrel{pr_1}{\to} & U \\ {\scriptstyle pr_2}\downarrow & & \downarrow \\ U & \to & S \end{array}$$

the second is obtained from the composition $pr_1^*(F)\to\Delta_*\Delta^*pr_1^*(F)\cong\Delta_*(F)$ where $\Delta: U\to U\times_S U$ is the diagonal and the third is the isomorphism of Lemma 2.7.

Proposition 2.8 *[***l4***] Let $i_U : Z\to U$ be a closed embedding and $p: U\to S$ an etale morphism such that the composition $i_S = p\circ i_U$ is again a closed embedding. Then for any F over Z the composition $p_\#(i_U)_*(F)\to p_*(i_U)_*(F) = (i_S)_*(F)$ is an isomorphism.*

Proof: Follows from Lemmas 1.19, 1.20 and 1.21.

Proposition 2.9 *[***clconst***] Let $i : Z\to S$ be a closed embedding and $j : U\to S$ the complimentary open embedding. For any pointed simplicial set K considered as an object over S the canonical square*

$$\begin{array}{ccc} j_\#j^*(K) & \to & K \\ \downarrow & & \downarrow \\ pt & \to & i_*i^*(K) \end{array}$$

is a push-forward square.

Proof: It follows from Lemmas 1.19, 1.20 and 1.21.

Example 2.10 The statements of Propositions 2.9 and 2.5(1) would be false if we considered the category Sm/S with Zariski topology instead of the Nisnevich one. Let S be the spectrum of a local non-henselian ring and $i : Spec(k)\to S$ the embedding of the closed point. Let further $U\to S$ be a local scheme etale over S such that $U\times_S Spec(k) = \coprod_{i=1}^n U_i$ where U_i are connected and $n>1$. Then $S^0(U) = S^0$ and $i_*i^*(S^0)(U) = \vee_{i=1}^n S^0$ and thus the morphism $S^0\to i_*i^*(S^0)$ is not an epimorphism.

In the following examples $S = Spec(A)$ is the spectrum of a henselian local ring A and $i : Z\to S$ is the embedding of the closed point $Z = Spec(A/m)$.

Example 2.11 [**noncocart**]Consider the pointed sheaf of sets $(\mathbf{A}^1, 0)$ on $(Sm/S)_{Nis}$. Then the square

$$\begin{array}{ccc} j_\# j^*(F) & \to & F \\ \downarrow & & \downarrow \\ pt & \to & i_* i^*(F) \end{array}$$

is not a pushforward square. Indeed $j_\# j^*(F)(S) = pt$, $(F/j_\# j^*(F))(S) = A$ and $i_* i^*(F)(S) = A/m$.

Example 2.12 [**nonbf**]An explicit computation shows that for S and Z as above one has $\mathbf{L}i^* i_*(\mathbf{A}^1, 0) \cong (\mathbf{A}^1, 0) \times B_{simpl}\mathbf{G}_a$ i.e. the canonical morphism $\mathbf{L}i^* i_*(F) \to F$ is not a simplicial weak equivalence for general F.

Theorem 2.13 [**gluing**] *Let $i : Z \to S$ be a closed embedding, $j : U \to S$ the complimentary open embedding and F a left admissible object over S. Then the adjunction $j_\# j^*(F) \to F$ is a monomorphism and the square*

$$\begin{array}{ccc} j_\# j^*(F) & \to & F \\ \downarrow & & \downarrow \\ pt & \to & i_* i^*(F) \end{array}$$

is $\mathbf{A}^1$-homotopy cocartesian i.e. the canonical morphism $F/j_\# j^*(F) \to i_* i^*(F)$ is an $\mathbf{A}^1$-weak equivalence.

Proof: This is the pointed version of [**?**, Th. 3.2.21]. One can verify it using Remark 1.1.

Proposition 2.14 [**p2**] *Let F be an object over Z. Then the composition $\mathbf{L}i^* i_*(F) \to i^* i_*(F) \to F$ is an $\mathbf{A}^1$-weak equivalence.*

Proof: Let us consider first the case when $F = i^* G$ for a left admissible object G over S. Consider the commutative diagram

$$\begin{array}{ccc} i^* Lres(G/j_\# j^* G) & \to & i^* Lres(i_* i^* G) \\ \downarrow & & \downarrow \\ i^*(G/j_\# j^* G) & \to & i^* i_* i^* G \\ & \searrow & \downarrow \\ & & i^* G \end{array}$$

We have to show that the composition of the right vertical arrows is an $\mathbf{A}^1$-weak equivalence. By Theorem 2.13 the canonical morphism $G/j_\# j^*(G) \to i_* i^*(G)$ is an $\mathbf{A}^1$-weak equivalence. Thus by Lemma 1.10 the upper horizontal arrow is an $\mathbf{A}^1$-weak equivalence. By Lemma 1.7 and our definition of left admissible objects the left hand side is left admissible. Thus by Lemma 1.6 the left vertical arrow is a simplicial weak equivalence. The composition of the slanted arrow with the canonical morphism $i^*G \to i^*(G/j_\# j^*G)$ is identity by the definition of adjoint functors. This morphism is an isomorphism since i^* commutes with coproducts and $i^*j_\# = pt$ by Lemma 1.5 and thus the slanted arrow is an isomorphism which finishes the proof for F of the form i^*G.
To prove the proposition for all F we need the following two lemmas.

Lemma 2.15 *[**closed3**] Let $i : Z \to S$ be a closed embedding. Then for any object F over S there exists a diagram of the form $(i^*((W_i)_+))_{i\in\Delta^{op}}$ where W_i are smooth schemes over S and a simplicial weak equivalence $hocolim_{i\in\Delta^{op}} i^*((W_i)_+) \to F$.*

Proof: Define a functor $Lres_S : \Delta^{op}Spc_\bullet(Z) \to \Delta^{op}Spc_\bullet(Z)$ and a natural transformation $Lres_S \to Id$ in the same way as we did with $Lres$ in the proof of Lemma **??** but starting with smooth schemes of the form $W \times_S Z$ for smooth quasi-projective schemes over S. Consider the composition

$$hocolim_{i\in\Delta^{op}} Lres_S(F)_i \to Lres_S(F) \to F$$

The first arrow is a simplicial weak equivalence by Lemma 1.13. To show that the second one is a simplicial weak equivalence one uses the same argument as in the proof of Lemma **??** together with Lemma 2.3.

Lemma 2.16 *[**clhoco**] Let $p : Z \to S$ be a finite morphism and $(X_i)_{i\in I}$ a small diagram over Z. There exists a natural (in X) isomorphism*

$$\mathbf{L}p^*p_* hocolim_{i\in I} X_i \to hocolim_{i\in I} \mathbf{L}p^*p_* X_i$$

in the simplicial homotopy category over Z such that the diagram

$$\begin{array}{ccc} \mathbf{L}p^*p_* hocolim_{i\in I} X_i & \to & hocolim_{i\in I} \mathbf{L}p^*p_* X_i \\ \downarrow & \swarrow & \\ hocolim_{i\in I} X_i & & \end{array}$$

commutes.

Proof: Consider the diagram

$$\begin{array}{ccccc}
\mathbf{L}p^*p_*hocolim_{i\in I}X_i & \leftarrow & \mathbf{L}p^*hocolim_{i\in I}p_*(X_i) & \rightarrow & hocolim_{i\in I}\mathbf{L}p^*p_*(X_i) \\
\downarrow & & \downarrow & & \downarrow \\
p^*p_*hocolim_{i\in I}X_i & \leftarrow & p^*hocolim_{i\in I}p_*(X_i) & \rightarrow & hocolim_{i\in I}p^*p_*(X_i) \\
& \searrow & & \swarrow & \\
& & hocolim_{i\in I}X_i & &
\end{array}$$

The first upper horizontal arrow is the isomorphism of Corollary 2.2. The second one is the isomorphism of Lemma 1.16. We define our isomorphism as the compostion of the inverse to the first one with the second. To prove commutativity of the triangle claimed in the lemma it is sufficient to prove commutativity of three squares in the diagram above. The upper left one is commutative since $\mathbf{L}p^* \rightarrow p^*$ is a natural transformation. The upper right one by Lemma 1.16 and the lower one by Lemma 1.18.

To finish the proof of Proposition 2.14 consider the simplicial weak equivalence $hocolim_{i\in\Delta^{op}}i^*((W_i)_+) \rightarrow F$ constructed in Lemma 2.15. We have a commutative square

$$\begin{array}{ccc}
\mathbf{L}i^*i_*hocolim_{i\in\Delta^{op}}i^*((W_i)_+) & \rightarrow & \mathbf{L}i^*i_*F \\
\downarrow & & \downarrow \\
hocolim_{i\in\Delta^{op}}i^*((W_i)_+) & \rightarrow & F
\end{array}$$

The upper horizontal arrow is a simplicial weak equivalence by [**?**, Prop. 3.1.27] and Lemma 1.10 and the lower horizontal one by construction. Thus it is sufficient to show that the left vertical arrow is an $\mathbf{A}^1$-weak equivalence. This follows from the first part of the proof, Lemma 2.16 and Lemma 1.12.

Corollary 2.17 *[***p3***] Let F be an object over Z. Then the canonical morphism $\mathbf{L}i^*i_*(F) \rightarrow i^*i_*(F)$ is an $\mathbf{A}^1$-weak equivalence.*

Proposition 2.18 *[***projform***] Let $i : Z \rightarrow S$ be a closed embedding, F a left admissible object over S and G a pointed simplicial sheaf over Z. Then the morphism $F \wedge i_*G \rightarrow i_*(i^*F \wedge G)$ defined by the adjunction and the isomorphism of Lemma 1.3 is an $\mathbf{A}^1$-weak equivalence.*

Proof: Consider first the case when $G = i^*F'$ for a left admissible object F' over S. We have the following commutative diagram of morphisms of sheaves

$$\begin{array}{ccc} F\wedge (F'/j_{\#}j^*F') & \to & F\wedge i_*i^*F' \\ \downarrow & & \downarrow \\ (F\wedge F')/(F\wedge j_{\#}j^*F') & & i_*(i^*F\wedge i^*F') \\ \downarrow & & \downarrow \\ (F\wedge F')/j_{\#}j^*(F\wedge F') & \to & i_*i^*(F\wedge F') \end{array}$$

where all the vertical arrows except for the upper right one are isomorphisms for obvious reasons (Lemmas 1.3 and 1.4). The upper horizontal arrow is an $\mathbf{A}^1$-weak equivalence by Theorem 2.13 and the lower one by Lemma 1.8 and Theorem 2.13. Thus $F\wedge i_*i^*F' \to i_*(i^*F\wedge i^*F')$ is an $\mathbf{A}^1$-weak equivalence. To prove the case of an arbitrary G one uses Lemma 2.15, Corollary 2.2, Lemma 1.12 and [**?**, Prop. 3.1.27].

Corollary 2.19 *[**p0**] Let $i: Z\to S$ be a closed embedding, $j: U\to S$ the complimentary open embedding and F a left admissible object over S. Then the morphism $F\wedge i_*(S^0)\to i_*i^*(F)$ defined by the adjunction and isomorphism of Lemma 1.3 is an $\mathbf{A}^1$-weak equivalence.*

Proposition 2.20 *[**clbasechange**] For a pull-back square*

$$\begin{array}{ccc} Z' & \stackrel{f_Z}{\to} & Z \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ S' & \stackrel{f_S}{\to} & S \end{array}$$

such that i is a closed embedding and a left admissible F on Z the composition $\mathbf{L}f_S^*i_*(F)\to f_S^*i_*(F)\to i'_*f_Z^*(F)$ is an $\mathbf{A}^1$-weak equivalence.

Proof: Consider first the case when $F = i^*G$ for a left admissible object G over S. Then we have a diagram

$$\begin{array}{ccccccc} \mathbf{L}f_S^*(G/j_{\#}j^*G) & \to & f_S^*(G/j_{\#}j^*G) & \to & f_S^*(G)/f_S^*j_{\#}j^*G & \to & f_S^*(G)/j'_{\#}(j')^*f_S^*(G) \\ \downarrow & & \downarrow & & & & \downarrow \\ \mathbf{L}f_S^*i_*i^*(G) & \to & f_S^*i_*i^*(G) & \to & i'_*f_Z^*i^*G & \to & i'_*(i')^*f_S^*(G) \end{array}$$

where the left square is commutative because $\mathbf{L}f_S^*\to f_S^*$ is a natural transformation and the commutativity of the right hexagon can be easily verified from definitions. The left vertical arrow is an $\mathbf{A}^1$-weak equivalence by Theorem 2.13 and Lemma 1.10. The first upper horizontal arrow is a simplicial weak equivalence by Lemma 1.7 and Lemma 1.11. Two other upper horizontal arrows and the right lower horizontal one are isomorphisms for obvious reasons. The right vertical arrow is an $\mathbf{A}^1$-weak equivalence by Theorem 2.13 and Lemma 1.7. Thus the composition of the first two lower horizontal arrows is an $\mathbf{A}^1$-weak equivalence.

To prove the case of a general F one uses Lemma 2.15 in a way similar to how it is used in the proof of Proposition 2.14.

Remark 2.21 It can be shown that in the notations of Proposition 2.20 the the morphism $f_S^* i_*(F) \to i'_* f_Z^*$ is an isomorphism for any F but since $\mathbf{L} f_S^* i_*(F) \to f_S^* i_*(F)$ is not generally a simplicial weak equivalence this has little use.

3 Duality for smooth quasi-projective morphisms

3.1 Formulation of the main theorem

Definition 3.1 *Let $p : X \to S$ be a smooth morphism. The dualizing object of X over S is the pointed sheaf $D_{X/S} = (X \times_S X)/(X \times_S X - \Delta(X))$ considered over X with respect to the projection to the second component.*

Note that by Lemma 2.9 the dualizing object can be written as $D_{X/S} = (pr_2)_\# \Delta_*(S^0)$ where $pr_2 : X \times_S X \to X$ is the projection to the second component and $\Delta : X \to X \times_S X$ is the diagonal.

Let $p : X \to S$ be a smooth morphism and $p = \bar{p} \circ j$ be a decomposition of p such that $j : X \to \bar{X}$ is an open embedding and $\bar{p} : \bar{X} \to S$ is any morphism. For any left admissible F over X we define a natural (in F) morphism in the $\mathbf{A}^1$-homotopy category

$$\beta_F = \beta_F^{(j,\bar{p})} : \bar{p}^* p_\#(F) \to j_\#(F \wedge D_{X/S})$$

as follows. Consider the following diagram

$$\text{[dmain]} \quad \begin{array}{ccc} X \times_S X & \stackrel{p''}{\to} & X \\ j' \downarrow & \stackrel{\Gamma}{\swarrow} & \downarrow j \\ X \times_S \bar{X} & \stackrel{p'}{\to} & \bar{X} \\ \bar{p}' \downarrow & & \downarrow \bar{p} \\ X & \stackrel{p}{\to} & S \end{array} \qquad (1)$$

where both squares are Cartesian and Γ is the closed embedding of the graph of j. We define β_F as the morphism in the $\mathbf{A}^1$-homotopy category represented by the following diagram of morphisms of pointed simplicial sheaves

$$\begin{array}{ccccc} \mathbf{L}p'_{\#}(\bar{p}')^*F & & \mathbf{L}p'_{\#}j'_{\#}((p'')^*F \wedge \Delta_* S^0) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \bar{p}^* p_{\#}(F) & & \mathbf{L}p'_{\#}\Gamma_* F & & j_{\#}(F \wedge D_{X/S}) \end{array}$$

where $\Delta : X \to X \times_S X$ is the diagonal embedding and the morphisms are given by:

1. the left vertical arrow is the composition $\mathbf{L}p'_{\#}(\bar{p}')^*F \to p'_{\#}(\bar{p}')^*F \to \bar{p}^* p_{\#}$ where the second arrow is the isomorphism of Lemma 1.5 and the first one is a simplicial weak equivalence by Lemmas 1.7 and 1.11;

2. the left slanted arrow is obtained by applying $\mathbf{L}p'_{\#}$ to the morphism adjoint to the ismorphism $\Gamma^*(\bar{p}')^* \cong Id$ of Lemma 1.2;

3. the middle vertical arrow is obtained by applying $\mathbf{L}p'_{\#}$ to the composition

$$j'_{\#}((p'')^*F \wedge \Delta_* S^0) \to j'_{\#}(\Delta_* \Delta^*((p'')^*(F))) \to j'_{\#}\Delta_* F \to \Gamma_* F$$

   where the first arrow is an $\mathbf{A}^1$-weak equivalence by Corollary 2.19 and Lemma **??**, the second is the isomorphism of Lemma 1.2 and the third is the isomorphism of Proposition 2.8;

4. the right slanted arrow is the composition

$$\begin{array}{ccl} \mathbf{L}p'_{\#}j'_{\#}((p'')^*F \wedge \Delta_* S^0) & \to & p'_{\#}j'_{\#}((p'')^*F \wedge \Delta_* S^0) \\ & & \downarrow \\ & & j_{\#}p''_{\#}((p'')^*F \wedge \Delta_* S^0) \quad \to j_{\#}(F \wedge D_{X/S}) \end{array}$$

   where the first arrow is the canonical morphism, the second is an isomorphism of Lemma 1.2 and the third is the isomorphism of Lemma 1.5.

The following *duality theorem* is the main result of this paper.

Theorem 3.2 *[***main***] For any smooth morphism $p : X \to S$ and any decomposition $p = \bar{p} \circ j$ such that $j : X \to \bar{X}$ is an open embedding and $\bar{p} : \bar{X} \to S$*

is a projective morphism there exists $n \geq 0$ such that for any left admissible F on X the n-th T-suspension of the morphism

$$\delta_F : p_\#(F) \to \mathbf{R}\bar{p}_* j_\#(F \wedge D_{X/S})$$

adjoint to β_F is an isomorphism in $H_{\mathbf{A}^1}(S)$.

Lemma 3.3 *[**bc**] Consider a pull back diagram*

$$\begin{array}{ccc} X_2 & \stackrel{f''}{\to} & X_1 \\ {\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ \bar{X}_2 & \stackrel{f'}{\to} & \bar{X}_1 \\ {\scriptstyle \bar{p}_2}\downarrow & & \downarrow{\scriptstyle \bar{p}_1} \\ S_2 & \stackrel{f}{\to} & S_1 \end{array}$$

where j_1, j_2 are open embeddings and $p_1 = \bar{p}_1 \circ j_1$, $p_2 = \bar{p}_2 \circ j_2$ are smooth morphisms. Then for any left admissible F over X_1 the diagram of morphisms in $H_{\mathbf{A}^1}(\bar{X}_2)$

$$\begin{array}{ccc} (f')^*\bar{p}_1^*(p_1)_\#(F) & \to & (f')^*(j_1)_\#(F \wedge D_{X_1/S_1}) \\ \downarrow & & \downarrow \\ \bar{p}_2^* f^*(p_1)_\#(F) & & (j_2)_\#(f'')^*(F \wedge D_{X_1/S_1}) \\ \downarrow & & \downarrow \\ \bar{p}_2^*(p_2)_\#(f'')^*(F) & \to & (j_2)_\#((f'')^*(F) \wedge D_{X_2/S_2}) \end{array}$$

commutes.

Proof: ???

Consider a diagram of the form

$$\begin{array}{ccc} & & \bar{X}' \\ & \stackrel{k}{\nearrow} & \downarrow{\scriptstyle \bar{q}} \\ X & \stackrel{j}{\to} & \bar{X} \\ & \stackrel{p}{\searrow} & \downarrow{\scriptstyle \bar{p}} \\ & & S \end{array}$$

and denote the composition $\bar{p}\bar{q}$ by $\bar{r}$. Then for any left admissible F over X we have a square of morphisms in $H_{\mathbf{A}^1}(S)$

$$\begin{array}{ccc} \bar{q}^*\bar{p}^*p_\#(F) & \longrightarrow & \bar{r}^*p_\#(F) \\ \downarrow & & \downarrow \\ \bar{q}^*j_\#(F\wedge D_{X/S}) & \longrightarrow & k_\#(F\wedge D_{X/S}) \end{array} \qquad (2)$$

where the upper horizontal arrow is the isomorphism of Lemma 1.2, the left vertical arrow is $\bar{q}^*(\beta_F^{(j,\bar{p})})$, the right vertical arrow is $\beta_F^{(j',\bar{r})}$ and the lower horizontal arrow is $\beta^{(k,\bar{q})}_{(F\wedge D_{X/S})}$.

Proposition 3.4 *[**long**] The square 2 commutes.*

Proof: Consider the diagrams of the form 1 for $(j,\bar{p})$ and $(k,\bar{r})$

$$\begin{array}{cccc} X\times_S X & \stackrel{p''}{\to} & X \\ & \Gamma_j & \\ j'\downarrow & \swarrow & \downarrow j \\ X\times_S \bar{X} & \stackrel{p_1'}{\to} & \bar{X} \\ \bar{p}'\downarrow & & \downarrow\bar{p} \\ X & \stackrel{p}{\to} & S \end{array} \qquad \begin{array}{ccc} X\times_S X & \stackrel{p''}{\to} & X \\ & \Gamma_k & \\ k'\downarrow & \swarrow & \downarrow k \\ X\times_S \bar{X}' & \stackrel{p_2'}{\to} & \bar{X}' \\ \bar{r}'\downarrow & & \downarrow\bar{r} \\ X & \stackrel{p}{\to} & S \end{array} \qquad (3)$$

and the pull-back square

$$\begin{array}{ccc} X\times_S \bar{X}' & \stackrel{p_2'}{\to} & \bar{X}' \\ \bar{q}'\downarrow & & \downarrow \bar{q}' \\ X\times_S \bar{X} & \stackrel{p_1'}{\to} & \bar{X} \end{array}$$

which connects them. We have:

$$\begin{array}{ccccc} \bar{q}^*\bar{p}^*p_\# & & \longrightarrow & & \bar{r}^*p_\# \\ \uparrow & & 1 & & \uparrow \\ \bar{q}^*(p_1')_\#(\bar{p}')^* & \leftarrow & (p_2')_\#(\bar{q}')^*(\bar{p}_1')^* & \rightarrow & (p_2')_\#(\bar{r}')^* \\ \uparrow & 2 & \uparrow & 3 & \uparrow \\ \bar{q}^*\mathbf{L}(\bar{p}_1')_\#(\bar{p}')^* & \leftarrow & \mathbf{L}(p_2')_\#\mathbf{L}(\bar{q}')^*(\bar{p}')^* & \rightarrow & \mathbf{L}(p_2')_\#(\bar{r}')^* \\ \downarrow & 4 & \downarrow & 5 & \downarrow \\ \bar{q}^*\mathbf{L}(\bar{p}_1')_\#(\Gamma_j)_* & \leftarrow & \mathbf{L}(p_2')_\#\mathbf{L}(\bar{q}')^*(\Gamma_j)_* & \rightarrow & \mathbf{L}(p_2')_\#(\Gamma_k)_* \\ \uparrow & 6 & \uparrow & 7 & \uparrow \\ \bar{q}^*\mathbf{L}(\bar{p}_1')_\#j'_\#((p'')^*\wedge\Delta_*S^0) & \leftarrow & \mathbf{L}(p_2')_\#\mathbf{L}(\bar{q}')^*j'_\#((p'')^*\wedge\Delta_*S^0) & \rightarrow & \mathbf{L}(p_2')_\#k'_\#((p'')^*\wedge\Delta_*S^0) \\ \downarrow & & 8 & & \downarrow \\ \bar{q}^*j_\#p''_\#((p'')^*\wedge\Delta_*S^0) & & \longrightarrow & & k_\#p''_\#((p'')^*\wedge\Delta_*S^0) \\ \downarrow & & 9 & & \downarrow \\ \bar{q}^*j_\#(F\wedge D_{X/S}) & & \longrightarrow & & k_\#(F\wedge D_{X/S}) \end{array}$$

where the left vertical side represents $\bar{q}^*(\beta_F^{(j,\bar{p})})$, the right vertical side represents $\beta_F^{(j',\bar{r})}$, the upper horizontal one is the isomorphism of Lemma 1.2 and the lower horizontal one is $\beta^{(k,\bar{q})}_{(F\wedge D_{X/S})}$.
The horizontal arrows in squares 4 and 6 are given by the composition of natural transformations

$$\mathbf{L}(p_2')_\#\mathbf{L}(\bar{q}')^* \to (p_2')_\#\mathbf{L}(\bar{q}')^* \to \bar{q}^*\mathbf{L}(\bar{p}_1')_\#$$

where the first one is a simplicial weak equivalence by Lemmas 1.7 and 1.11 and the second one is an isomorphism by Lemma 1.5. Since these are natural transformations the squares 4 and 6 commute. The upper horizontal arrow in square 2 is the base change isomorphism of Lemma 1.5 and the right vertical one is the composition

$$\mathbf{L}(p_2')_\#\mathbf{L}(\bar{q}')^* \to (p_2')_\#\mathbf{L}(\bar{q}')^* \to (p_2')_\#(\bar{q}')^*$$

and this square commutes because the base change is a natural transformation. It follows that to prove the proposition it is sufficient to verify commutativity of pentagon 1, squares 3,5,7, pentagon 8 and square 9.

Pentagon 1 is the coherence between the base change isomorphisms of Lemma 1.5 and the composition isomorphisms of Lemma 1.2 (**??**). Square 3 commutes because $\mathbf{L}(p_2')_\# \to (p_2')_\#$ is a natural transformation. Square 5 is obtained by applying $\mathbf{L}(p_2')_\#$ to the diagram

$$\begin{array}{ccccc} \mathbf{L}(\bar{q}')^*(\bar{p}')^* & \to & (\bar{q}')^*(\bar{p}')^* & \to & (\bar{r}')^* \\ \downarrow & 5a & \downarrow & 5b & \downarrow \\ \mathbf{L}(\bar{q}')^*(\Gamma_j)_* & \to & (\bar{q}')^*(\Gamma_j)_* & \to & (\Gamma_k)_* \end{array}$$

where square 5a commutes because $\mathbf{L}(\bar{q}')^* \to (\bar{q}')^*$ is a natural transformation. The upper horizontal arrow of square 5b is the isomorphism of Lemma 1.2. The lower horizontal one is the adjoint to the composition

$$\Gamma_k^*(\bar{q}')^*(\Gamma_j)_* \to \Gamma_j^*(\Gamma_j)_* \to Id$$

where the first arrow is the isomorphism of Lemma 1.2 obtained from the equality $\Gamma_j = \bar{q}'\Gamma_k$ and the second is the adjunction. The left vertical arrow is obtained by applying $(\bar{q}')^*$ to the morphism adjoint to the isomorphism $\Gamma_j^*(\bar{p}')^* = Id$ of Lemma 1.2 and the right vertical one is the adjoint to the isomorphism $\Gamma_k^* \bar{r}^* = Id$ of Lemma 1.2. The commutativity of this square can be seen from the following diagram of adjoint morphisms

$$\begin{array}{ccccc} \Gamma_k^*(\bar{q}')^*(\bar{p}')^* & \to & \Gamma_j^*(\bar{p}')^* & & \\ & & & \stackrel{Id}{\searrow} & \\ \downarrow & & \downarrow & & \\ \Gamma_k^*(\bar{q}')^*(\Gamma_j)_*\Gamma_j^*(\bar{p}')^* & \to & \Gamma_j^*(\Gamma_j)_*\Gamma_j^*(\bar{p}')^* & \to & \Gamma_j^*(\bar{p}')^* \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_k^*(\bar{q}')^*(\Gamma_j)_* & \to & \Gamma_j^*(\Gamma_j)_* & \to & Id \end{array}$$

The left vertical side is by definition Γ_k^* of the left vertical arrow of square 5a and the lower horizontal side is the adjoint to the lower horizontal arrow of square 5a. The composition of the upper horizontal arrow and the right vertical arrow equals by Lemma 1.2 to the composition

$$\Gamma_k^*(\bar{q}')^*(\bar{p}')^* \to \Gamma_k^*(\bar{r}')^* \to Id$$

which is adjoint to the composition of the upper horizontal and right vertical arrows of square 5a. The triangle commutes by definition of adjoints. The right lower square because $\Gamma_j^*(\Gamma_j)_* \to Id$ is a natural transformation. Two left squares because $\Gamma_k^*(\bar{q}')^* \to \Gamma_j^*$ is a natural transformation. Square 7 is obtained by applying $\mathbf{L}(p_2')_\#$ to the diagram

$$\begin{array}{ccccc} \mathbf{L}(\bar{q}')^*(\Gamma_j)_* & \to & (\bar{q}')^*(\Gamma_j)_* & \to & (\Gamma_k)_* \\ \uparrow & 7a & \uparrow & 7b & \uparrow \\ \mathbf{L}(\bar{q}')^* j'_\#((p'')^* \wedge \Delta_* S^0) & \to & (\bar{q}')^* j'_\#((p'')^* \wedge \Delta_* S^0) & \to & k'_\#((p'')^* \wedge \Delta_* S^0) \end{array}$$

where the vertical arrows are taken from the definitions of $\beta^{(j,\bar{p})}$ and $\beta^{(k,\bar{r})}$. The square on the left commute because $\mathbf{L}(\bar{q}')^* \to (\bar{q}')$ is a natural transformation. The right upper horizontal arrow was described in the context of square 5 above and the lower right horizontal arrow is $\beta^{(k',\bar{q}')}_{(p'')^*F\wedge\Delta_*S^0}$. To prove commutativity of square 7b observe first that $(\Gamma_k)_* cong k'_*\Delta_*$ and thus it is sufficient to prove commutativity of the square obtained from 7b by applying $(k')^*$....

Corollary 3.5 *In the notations of the proposition the following square of morphisms in $H_{\mathbf{A}^1}$ commutes*

$$\begin{array}{ccc} p_\#(F) & \stackrel{\delta_F^{(j,\bar{p})}}{\longrightarrow} & \mathbf{R}\bar{p}_* j_\#(F \wedge D_{X/S}) \\ \delta_F^{(k,\bar{r})} \downarrow & & \downarrow \\ \mathbf{R}\bar{r}_* k_\#(F \wedge D_{X/S}) & \to & \mathbf{R}\bar{p}_*\mathbf{R}\bar{q}_* k_\#(F \wedge D_{X/S}) \end{array}$$

(here the right vertical arrow is $\mathbf{R}\bar{p}_*$ of the composition $j_\#(F \wedge D_{X/S}) \to \bar{q}_* k_\#(F \wedge D_{X/S}) \to \mathbf{R}\bar{q}_* k_\#(F \wedge D_{X/S})$ where the first arrow is the morphism adjoint to $\epsilon^{(k,\bar{q})}_{F\wedge D_{X/S}}$).

Proof: ???

For a pull-back square

$$\begin{array}{ccc} X' & \stackrel{q}{\to} & S' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \stackrel{p}{\to} & S \end{array}$$

and any F over S' define a morphism $\mathbf{L}p^*\mathbf{R}f_*(F) \to \mathbf{R}g_*\mathbf{L}q^*(F)$ in the simplicial homotopy category over X by the following diagram

$$p^*Lres(f_*Rres(F)) \leftarrow p^*Lres(f_*Rres(Lres(F))) \to p^*f_*(Rres(Lres(F))) \to$$

$$\to g_*q^*(Rres(Lres(F))) \to g_*Rres(q^*(Rres(Lres(F)))) \leftarrow g_*Rres(q^*Lres(F))$$

where the first left arrow is a simplicial weak equivalence by Lemmas 1.10 and 3.6 and the second left arrow is a simplicial weak equivalence by Lemmas 1.6 and 3.7.

Lemma 3.6 *[**l5d**] For any morphism $f : S_1 \to S_2$ and a simplicial (resp. $\mathbf{A}^1$-) weak equivalence $a : F \to G$ over S_2 the morphism $\mathbf{R}f_*(a)$ is a simplicial (resp. $\mathbf{A}^1$-) weak equivalence.*

Lemma 3.7 *[**mix**] For a left admissible object F over S the object $Rres(F)$ is left admissible.*

Lemma 3.8 *[**smbc**] Consider a pull-back square*

$$\begin{array}{ccc} X' & \stackrel{q}{\to} & S' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \stackrel{p}{\to} & S \end{array}$$

such that p is a smooth morphism. Then for any F over S' the morphism $p^*f_*(F) \to g_*q^*(F)$ defined by the adjunctions and Lemma 1.2 is an isomorphism.

Proof: ???

Lemma 3.9 *[n1] In the notations of Diagram 1 assume that p is an open embedding. The for any F over X the morphism $F \to p^*\bar{p}_* j_\#(F)$ adjoint to $\epsilon_F^{(j,\bar{p})}$ is an isomorphism.*

Proof: ???

Lemma 3.10 *[n11] In the notations of Lemma 3.9 the canonical morphism $p^*\bar{p}_* j_\#(F) \to p^*\mathbf{R}\bar{p}_* j_\#(F)$ is a simplicial weak equivalence.*

Proof: Consider the diagram

$$\begin{array}{ccc} p^*\bar{p}_* j_\#(F) & \to & \bar{p}'_*(p')^* j_\#(F) \\ \downarrow & & \downarrow \\ p^*\bar{p}_* Rres(j_\#(F)) & \to & \bar{p}'_*(p')^* Rres(j_\#(F)) \end{array}$$

The horizontal arrows are ismorphisms by Lemma 3.8 and therefore it is sufficient to check that the right vertical arrow is a simplicial weak equivalence. Assume first that F is right admissible. Then $(p')^* j_\#(F) = j'_\#(p'')^*(F) = j''_*(p'')^*(F)$ is right admissible by Lemma 3.13, $(p')^* Rres(j_\#(F))$ is right admissible by Lemma 3.12 and the morphism $(p')^* j_\#(F) \to (p')^* Rres(j_\#(F))$ is a simplicial weak equivalence. Thus the right vertical arrow is a simplicial weak equivalence in this case by Lemma 3.11. On the other hand the morphism $\bar{p}'_*(p')^* j_\#(F) \to \bar{p}'_*(p')^* j_\#(Rres(F))$ is a simplicial weak equivalence since $\bar{p}'_*(p')^* j_\# \cong Id$ and the morphism $\bar{p}'_*(p')^* Rres(j_\#(F)) \to \bar{p}'_*(p')^* Rres(j_\#(Rres(F)))$ is a simplicial weak equivalence because $j_\#$ preserves simplicial weak equivalences by [**?**, Prop. 3.1.26], $(p')^*$ takes right admissible objects to right admissible objects by Lemma 3.12 and $\bar{p}'_*$ preserves simplicial weak equivalences bewteen right admissible objects by Lemma 3.11.

Lemma 3.11 *[l6d] Let $a : F \to G$ be a simplicial (resp. $\mathbf{A}^1$-) weak equivalence of right admissible objects over S. Then for any morphism $f : S' \to S$ the morphism $f_*(a)$ is a simplicial (resp. $\mathbf{A}^1$-) weak equivalence.*

Lemma 3.12 *[rtsm] Let F be a right admissible object and $f : S' \to S$ be a smooth morphism. Then $f^*(F)$ is right admissible.*

Lemma 3.13 *[l7d] Let F be a right admissible object. Then for any morphism $f : S' \to S$ the object $f_*(F)$ is right admissible.*

Lemma 3.14 *[n2] In the notations of Lemma 3.9 let $i : Z \to S$ be the reduced closed subscheme $S - p(X)$ of S. Assume that for any F over $\bar{X}$ the base change morphism $\mathbf{L}i^*\mathbf{R}\bar{p}_*(F) \to \mathbf{R}(\bar{p}_Z)_*\mathbf{L}(i')^*(F)$ associated with the pull-back square*

$$\begin{array}{ccc} Z' & \stackrel{i'}{\to} & \bar{X} \\ {\scriptstyle \bar{p}_Z}\downarrow & & \downarrow{\scriptstyle \bar{p}} \\ Z & \stackrel{i}{\to} & S \end{array}$$

is an $\mathbf{A}^1$-weak equivalence. Then for any left admissible F the first simplicial suspension of the composition $p_\#(F) \to \bar{p}_*j_\#(F) \to \mathbf{R}\bar{p}_*j_\#(F)$ where the first arrow is the morphism adjoint to $\epsilon_F^{(j,\bar{p})}$ is an $\mathbf{A}^1$-weak equivalence.

Proof: Consider the diagram

$$\begin{array}{ccccc} p_\#p^*p_\#(F) & \to & p_\#p^*\bar{p}_*j_\#(F) & \to & p_\#p^*\mathbf{R}\bar{p}_*j_\#(F) \\ \downarrow & & \downarrow & & \downarrow \\ p_\#(F) & \to & \bar{p}_*j_\#(F) & \to & \mathbf{R}\bar{p}_*j_\#(F) \end{array}$$

where the upper line is obtained from the lower one by applying functor $p_\#p^*$ and the vertical arrows come from the natural transformation $p_\#p^* \to Id$. The left vertical arrow is an isomorphism by Proposition 2.5(2) and the composition of the inverse with the left upper horizontal arrow is $p_\#$ of the isomorphism of Lemma 3.9. The right upper horizontal arrow is a simplicial weak equivalence by Lemma 3.10 and [**?**, Prop. 3.1.26]. It remains to show that the first simplicial suspension of the right vertical arrow is an $\mathbf{A}^1$-weak equivalence.

By [**?**, Prop. 3.1.26] and [**?**, Cor. 3.1.24] the functor $p_\#p^*$ preserves simplicial weak equivalences. Thus it is sufficient to prove that the first simplicial suspension of the morphism $p_\#p^*Lres\mathbf{R}\bar{p}_*j_\#(F) \to Lres\mathbf{R}\bar{p}_*j_\#(F)$ is an $\mathbf{A}^1$-weak equivalence. By Theorem 2.13 the canonical morphism

$$Lres\mathbf{R}\bar{p}_*j_\#(F)/p_\#p^*Lres\mathbf{R}\bar{p}_*j_\#(F) \to i_*i^*Lres\mathbf{R}\bar{p}_*j_\#(F)$$

is an $\mathbf{A}^1$-weak equivalence. This by Lemma 3.15 it is sufficient to show that the right hand side is $\mathbf{A}^1$-weakly equivalent to pt.

By the assumption of the Lemma there exists an isomorphism in the $\mathbf{A}^1$-homotopy category of the form

$$i_*i^*Lres\mathbf{R}\bar{p}_*j_\#(F) \to i_*\mathbf{R}(\bar{p}_Z)_*\mathbf{L}(i')^*j_\#(F)$$

By Lemma 1.7 $j_{\#}(F)$ is left admissible and thus the canonical morphism $\mathbf{L}(i')^* j_{\#}(F) \to (i')^* j_{\#}(F)$ is a simplicial weak equivalence. By Lemma 1.5 $(i')^* j_{\#}(F) = pt$ since $X \times_{\bar{X}'} Z' = \emptyset$. Lemma is proven.

Lemma 3.15 *[***cone***] Let $i : F \to G$ be a monomorphism such that the canonical morphism $G/F \to pt$ is an $\mathbf{A}^1$-weak equivalence. Then the first simplicial suspension $Id_{S^1_s} \wedge i$ is an $\mathbf{A}^1$-weak equivalence.*

Proof: ???