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[1]

1 2-pretheories

1.1 Basic definitions

Definition 1.1.1 *[**2pr**] A 2-functor $H : Sch^{ft}/S \to Cat$ is called a 2-pretheory if the following conditions hold:*

1. *$H(\emptyset)$ is equivalent to the point category and for any X, Y over S the functor*
$$H(i_X) \times H(i_Y) : H(X \coprod Y) \to H(X) \times H(Y)$$
is an equivalence

2. *for any smooth morphism $p : X \to Y$ the functor $H(p)$ has a left adjoint $H_l(p)$ and for any pull-back square*
$$\begin{array}{ccc} X' & \stackrel{f_X}{\to} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \stackrel{f_Y}{\to} & Y \end{array}$$
such that p is smooth the l-exchange morphism $H_l(p')H(f_X) \to H(f_Y)H_l(p)$ is an isomorphism

3. *for a closed embedding $i : Z \to X$ the functor $H(i)$ has a right adjoint $H_r(i)$ and for any pull back square*
$$\begin{array}{ccc} Z' & \stackrel{f_Z}{\to} & Z \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ X' & \stackrel{f_X}{\to} & X \end{array}$$
the r-exchange morphism $H(f_X)H_r(i) \to H_r(i')H(f_Z)$ is an isomorphism.

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Lemma 1.1.2 *[***trivial1***] For any 2-pretheory H and any X the category $H(X)$ has initial and final objects and finite coproducts. The functors $H(f)$ take final objects to final objects, initial objects to initial objects and finite coproducts to finite coproducts.*

Proof: For any X the morphism $e : \emptyset \to X$ is a smooth morphism and a closed embedding. By 1.1.1(1,2,3) it implies that the canonical functor $H(X) \to pt$ has both left and right adjoints i.e. $H(X)$ has initial and final objects.

The morphism $p : X \coprod X \to X$ is smooth and one verifies easily that if (F, G) is an object of $H(X \coprod X)$ corresponding to a pair of objects F, G of $H(X)$ by 1.1.1(1) then $H_l(p)(F, G)$ is a coproduct of F and G in $H(X)$. Functors $H(f)$ preserve coproducts by the second part of 1.1.1(2) applied to the square

$$\begin{array}{ccc} X' \coprod X' & \to & X \coprod X \\ \downarrow & & \downarrow \\ X' & \stackrel{f}{\to} & X \end{array}$$

and initial and final objects by the second parts of 1.1.1(2,1) applies to a similar square for $e : \emptyset \to X$.

Let Sm/X be the category of smooth schemes over X wich we consider as a symmetric monoidal category with respect to the direct products. The for any 2-pretheory H the category $H(X)$ has a module structure

$$S : Sm/X \to Funct(H(X), H(X))$$

over Sm/X defined as follows. For a smooth scheme $p : U \to X$ over X denote by $S(U)$ the functor $H_l(p)H(p) : H(X) \to H(X)$. For a morphism of smooth schemes $f : U_1 \to U_2$ over X let $S(f)$ be the natural transformation l-adjoint to

$$H(p_1) \cong H(f)H(p_2) \stackrel{Id*a*Id}{\to} H(f)H(p_2)H_l(p_2)H(p_2) \cong H(p_1)H_l(p_2)H(p_2)$$

where $a : Id \to H(p_2)H_l(p_2)$ is the adjunction. One can verify that for a composable pair of morphisms of smooth schemes g, f one has $S(g \circ f) = S(g) \circ S(f)$.

For a pair of smooth schemes $p_1 : U_1 \to X$, $p_2 : U_2 \to X$ we have a canonical isomorphism $S(U_1 \times_X U_2) \to S(U_1) \circ S(U_2)$ given by

$$H_l(p_{12})H(p_{12}) \cong H_l(p_1)H_l(pr_1)H(pr_2)H_l(p_2) \to H_l(p_1)H(p_1)H_l(p_2)H(p_2)$$

where $p_{12} = p_2 \circ pr_2 = p_1 \circ pr_1$ is the morphism $U_1 \times_X U_2 \to X$ and the middle arrow is the l-exchange isomorphism of 1.1.1(2). These isomorphisms are compatible with the associativity isomorphisms for direct products and composition of functors and one can check that they are natural in both U_1 and U_2.

Definition 1.1.3 [**additive**] *A 2-pretheory is called additive if all the categories H(X) are additive.*

Lemma 1.1.4 [**add0**] *If H is an additive pretheory then the functors $H(f)$, $H_l(p)$ and $H_r(i)$ are additive.*

Proof: Follows from Lemma 1.1.2

Consider a pull-back square

$$\begin{array}{ccc} Z_X & \stackrel{i_X}{\to} & X \\ {\scriptstyle p_Z}\downarrow & & \downarrow{\scriptstyle p} \\ Z_Y & \stackrel{i_Y}{\to} & Y \end{array} \qquad (1)$$

such that p is smooth and i is a closed embedding. Define the lr-exchange morphism $H_l(p)H_r(i_Y) \to H_r(i_X)H_l(p_Z)$ as the r-adjoint to the composition

$$H(i_X)H_l(p)H_r(i_Y) \to H_l(p_Z)H(i_Y)H_r(i_Y) \to H_l(p_Z)$$

where the first arrow is the inverse to the l-base change morphism which exists by 1.1.1(2) and the second is given by the adjunction.

Definition 1.1.5 [**exc**] *A 2-pretheory H is said to satisfy open excision (resp. etale excision, smooth excision) if for any pull-back square of the form (1) such that p is an open embedding (resp. an etale morphisms, a smooth morphism) the lr-exchange morphism is an isomorphism.*

Recall that a functor ϕ is called *conservative* if any morphism f such that $\phi(f)$ is an isomorphism is itself an isomorphism.

Definition 1.1.6 [**wd**] *A 2-pretheory (or more generally a 2-functor) is said to have weak Zariski descent (resp. weak Nisnevich descent, weak etale descent etc.) if for any Zariski (resp. Nisnevich, etale etc.) covering $\{j_i : U_i \to X\}_{i\in I}$ the functor $\prod H(j_i) : H(X) \to \prod H(U_i)$ is conservative.*

Let $p : \mathbf{P}^1_X \to X$ be the canonical morphism and $i : X \to \mathbf{P}^1_X$ a section of p.

Definition 1.1.7 *[**stable**] A 2-pretheory is called T-stable if for any X and i as above the functor $H_l(p)H_r(i) : H(X) \to H(X)$ is an equivalence.*

Let $j : U \to \mathbf{P}^1_X$ be the open embedding complimentary to i and $p_U = p \circ j$.

Definition 1.1.8 *[**hi**] A 2-pretheory is called homotopy invariant if for any X and i as above the adjunction $H_l(p_U)H(p_U) \to Id_{H(X)}$ is an isomorphism.*

Consider the sequence

$$H_l(p_U)H(p_U) \stackrel{\eta_i}{\to} H_l(p)H(p) \stackrel{\sigma_i}{\to} H_l(p)H_r(i)$$

obtained from the sequence of adjunctions

$$H_l(j)H(j) \to Id \to H_r(i)H(i)$$

by applying $H(p)$ on the right and $H_l(p)$ on the left and using composition isomorphisms.

Definition 1.1.9 *[**A**] A 2-pretheory is said to satisfy axiom A if it is additive and for any X and i as above the sequence*

$$0 \to H_l(p_U)H(p_U) \stackrel{\eta_i}{\to} H_l(p)H(p) \stackrel{\sigma_i}{\to} H_l(p)H_r(i) \to 0$$

is split-exact.

For any 2-pretheory H the diagram

$$\begin{array}{ccc} H_l(p_U)H(p_U) & \stackrel{\eta_i}{\to} & H_l(p)H(p) \\ \downarrow & & \downarrow \\ Id & = & Id \end{array}$$

where the vertical arrows are the adjunctions commutes. Thus if H is a homotopy invariant 2-pretheory then the composition of the right vertical arrow with the inverse to the left one gives a canonical projection π_i for η_i. If in addition H satisfies Axiom A there is a unique section $\lambda_i : H_l(p)H_r(i) \to H_l(p)H(p)$ of σ_i such that $\pi_i \circ \lambda_i = 0$. One can easily see that for any other point i' of $\mathbf{P}^1_X$ over X the composition

$$\phi_{ii'} : H_l(p)H_r(i) \stackrel{\lambda_i}{\to} H_l(p)H(p) \stackrel{\sigma_{i'}}{\to} H_l(p)H_r(i')$$

is an isomorphism. These isomorphisms satisfy the conditions $\phi_{ii'} = \phi_{i'i}^{-1}$, $\phi_{i'i''}\phi_{ii'} = \phi_{ii''}$. We will use them to identify the functors $H_l(p)H_r(i)$ for different i's and will denote them all by Σ and call the *suspension functor*. Similarly we will omit the index from the notations for π and λ. Using this terminology one can say that for any homotopy invariant 2-pretheory satisfying Axiom A there is a canonical split-exact sequence

$$0 \to \Sigma \xrightarrow{\lambda} H_l(p)H(p) \xrightarrow{\pi} Id \to 0$$

and a point $i : X \to \mathbf{P}^1_X$ of $\mathbf{P}^1$ over X defines a section η_i for π and a projection σ_i for λ. It is easy to construct examples which show that σ_i and η_i actually depend on i.

Lemma 1.1.10 *[**Sbch**] Let H be a homotopy invariant 2-pretheory which satisfies Axiom A, $f : X' \to X$ a morphism and $p : \mathbf{P}^1_X \to X$, $p' : \mathbf{P}^1_{X'} \to X'$ the canonical projections. There is a unique isomorphism $\Sigma H(f) \to H(f)\Sigma$ such that the diagram*

$$\begin{array}{ccc} \Sigma H(f) & \to & H(f)\Sigma \\ \downarrow & & \downarrow \\ H_l(p')H(p')H(f) & \to & H(f)H_l(p)H(p) \end{array}$$

where the lower arrow is obtained from the l-exchange isomorphism of **??***(2), commutes.*
If $i : X \to \mathbf{P}^1_X$ is a point of $\mathbf{P}^1$ over X and $i' : X' \to \mathbf{P}^1_{X'}$ the corresponding point of $\mathbf{P}^1$ over X' then the diagram

$$\begin{array}{ccc} H_l(p')H(p')H(f) & \to & H(f)H_l(p)H(p) \\ \downarrow & & \downarrow \\ \Sigma H(f) & \to & H(f)\Sigma \end{array}$$

where the vertical arrows are given by σ_i and $\sigma_{i'}$ respectively commutes.

Lemma 1.1.11 *[**Slbch**] Let H be a homotopy invariant 2-pretheory which satisfies Axiom A, $q : X \to Y$ a smooth morphism and $p_X : \mathbf{P}^1_X \to X$, $p_Y : \mathbf{P}^1_Y \to Y$ the canonical projections. Then there is a unique isomorphism $H_l(q)\Sigma \to \Sigma H_l(q)$ such that the diagram*

$$\begin{array}{ccc} H_l(q)\Sigma & \to & \Sigma H_l(q) \\ \downarrow & & \downarrow \\ H_l(q)H_l(p_X)H(p_X) & \to & H_l(p_Y)H(p_Y)H_l(q) \end{array}$$

where the lower arrow is obtained from 1.1.1(2) commutes.

Proof: Follows from the fact that $H_l(q)H_l(p_X)H(p_X) \to H_l(p_Y)H(p_Y)H_l(q)$ fits into a commutative diagram

$$\begin{array}{ccc} H_l(q)H_l(p_X)H(p_X) & \to & H_l(p_Y)H(p_Y)H_l(q) \\ \downarrow & & \downarrow \\ H_l(q) & = & H_l(q) \end{array}$$

where the vertical arrows are morphisms π i.e. the adjunctions.

Let now H be a T-stable homotopy invariant 2-pretheory which satisfies Axiom A. The Σ is an equivalence and we denote by Ω the right adjoint to Σ which then an inverse equivalence. Lemmas 1.1.10 and 1.1.11 formally imply:

Lemma 1.1.12 *[**Obch**] Let H be a T-stable homotopy invariant 2-pretheory which satisfies Axiom A. Then for any morphism $f : X' \to X$ there is a unique isomorphism $H(f)\Omega \to \Omega H(f)$ such that the diagram*

$$\begin{array}{ccc} \Sigma H(f)\Omega & \to & \Sigma\Omega H(f) \\ \downarrow & & \downarrow \\ H(f)\Sigma\Omega & \to & H(f) \end{array}$$

where the lower horizontal arrow is obtained from the isomorphism of Lemma 1.1.10, commutes.
For any smooth morphism $q : X \to Y$ there is a unique isomorphism $H_l(q)\Omega \to \Omega H_l(q)$ such that the diagram

$$\begin{array}{ccc} H_l(q) & \to & H_l(q)\Omega\Sigma \\ \downarrow & & \downarrow \\ \Omega\Sigma H_l(q) & \to & \Omega H_l(q)\Sigma \end{array}$$

where the lower horizontal arrow is obtained from the inverse to the isomorphism of Lemma 1.1.11, commutes.

Lemma 1.1.13 *[**com1**] Let H be a T-stable homotopy invariant 2-pretheory which satisfies Axiom A and $q : X \to Y$ a smooth morphism. Then the following diagram where the upper line consists of isomorphisms of Lemma 1.1.12 and the vertical arrows are obtained from the adjunctions, commutes*

$$\begin{array}{ccccc} H_l(q)H(q)\Omega & \to & H_l(q)\Omega H(q) & \to & \Omega H_l(q)H(q) \\ \downarrow & & & & \downarrow \\ \Omega & & = & & \Omega \end{array}$$

The following lemma is an immediate corollary of Lemma 1.1.13.

Lemma 1.1.14 *[***com2***] Let H be a T-stable homotopy invariant 2-pretheory which satisfies Axiom A. Then there exists a unique isomorphism $\epsilon : \Sigma\Omega \to \Omega\Sigma$ such that the diagram*

$$\begin{array}{ccccc} \Sigma\Omega & & \stackrel{\epsilon}{\to} & & \Omega\Sigma \\ \downarrow & & & & \downarrow \\ H_l(q)H(q)\Omega & \to & H_l(q)\Omega H(q) & \to & \Omega H_l(q)H(q) \end{array}$$

commutes.

1.2 Duality theorem for the projective line

In this section we prove that for an additive T-stable homotopy invariant 2-pretheory H satisfying Axiom A the functor $H_l(p)\Omega$ where p is the canonical morphism $\mathbf{P}^1_X \to X$ is right adjoint to $H(p)$. Consider the square

$$\begin{array}{ccc} \mathbf{P}^1_X \times_X \mathbf{P}^1_X & \stackrel{pr_2}{\to} & \mathbf{P}^1_X \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{P}^1_X & \stackrel{p}{\to} & X \end{array}$$

and let $\Delta : \mathbf{P}^1_X \to \mathbf{P}^1_X \times_X \mathbf{P}^1_X$ be the diagonal. Define two morphisms $\beta : H(p)H_l(p)\Omega \to Id$ and $\phi : Id \to H_l(p)\Omega H(p)$ as follows:

β is the composition

$$H(p)H_l(p)\Omega \to H_l(pr_2)H(pr_1)\Omega \to H_l(pr_2)H_r(\Delta)\Omega = \Sigma\Omega \to Id$$

where the first arrow is the inverse to the l-exchange isomorphism of 1.1.1(2) and the second morphism is obtained from the r-adjoint to $H(\Delta)H(pr_1) \to Id$.

ϕ is the composition

$$Id \to \Sigma\Omega \stackrel{\lambda * Id}{\to} H_l(p)H(p)\Omega \to H_l(p)\Omega H(p)$$

where the first arrow is the inverse to the adjunction, the second one is obtained from λ and the third one is the isomorphism of Lemma 1.1.12.

Theorem 1.2.1 *[***Pdual***] The morphisms β and ϕ define on $H_l(p)\Omega$ the structure of a right adjoint to $H(p)$.*

Proof: By [] we have to verify that the compositions

$$H(p) \xrightarrow{Id*\phi} H(p)H_l(p)\Omega H(p) \xrightarrow{\beta*Id} H(p)$$

and

$$H_l(p)\Omega \xrightarrow{\phi*Id} H_l(p)\Omega H(p)H_l(p)\Omega \xrightarrow{Id*\beta} H_l(p)\Omega$$

are identities.

Lemma 1.2.2 *[***key1***] The composition*

$$\begin{array}{ccccccc} H(p)\Sigma & \stackrel{Id*\lambda}{\to} & H(p)H_l(p)H(p) & & & & \\ & & \downarrow & & & & \\ & & H_l(pr_2)H(pr_1)H(p) & \to & H_l(pr_2)H_r(\Delta)H(p) & = & \Sigma H(p) \end{array}$$

is the inverse to the canonical morphism of Lemma 1.1.10.

Proof: Follows from commutativity of the diagram

$$\begin{array}{ccccc} \Sigma H(p) & \to & H_l(pr_2)H(pr_2)H(p) & & \\ & & \downarrow & & \\ \downarrow & & H_l(pr_2)H(pr_1)H(p) & & \\ & & \downarrow & & \\ H(p)\Sigma & \to & H(p)H_l(p)H(p) & & \\ & & \downarrow & & \\ & & H_l(pr_2)H(pr_1)H(p) & \longrightarrow & \Sigma H(p) \\ & & \downarrow & & \downarrow \\ & & H_l(pr_2)H(pr_2)H(p) & \xrightarrow{(\sigma_\Delta)*Id} & \Sigma H(p) \end{array}$$

To prove that the first composition is identity consider the diagram

$$\begin{array}{ccccc} H(p) \to H(p)\Sigma\Omega & \xrightarrow{Id*\lambda*Id} & H(p)H_l(p)H(p)\Omega & \to & H(p)H_l(p)\Omega H(p) \\ & & \downarrow & & \downarrow \\ & & H_l(pr_2)H(pr_1)H(p)\Omega & \to & H_l(pr_2)H(pr_1)\Omega H(p) \\ & & \downarrow & & \downarrow \\ & & \Sigma H(p)\Omega & \to & \Sigma\Omega H(p) \\ & & \downarrow & & \downarrow \\ & & H(p)\Sigma\Omega & \to & H(p) \end{array}$$

where the composition of upper horizontal and right vertical sides is by definition $(\beta{*}Id)\circ(Id{*}\phi)$ Two upper squares are commutative because $H(p)\Omega\to\Omega H(p)$ is a natural transformation and the lower one is commutative by Lemma 1.1.12. By Lemma 1.2.2 the composition of the second upper horizontal arrow with the following three vertical arrows is identity which implies that $(\beta{*}Id)\circ(Id{*}\phi)=Id$

Lemma 1.2.3 [**key2**] *The composition* $((Id*\beta)\circ(\phi*Id))*Id_{H(p)}$

$$H_l(p)\Omega H(p)\xrightarrow{\phi*Id}H_l(p)\Omega H(p)H_l(p)\Omega H(p)\xrightarrow{Id*\beta*Id}H_l(p)\Omega H(p)$$

is an isomorphism.

Proof: Using only Lemma 1.1.12 and Lemma 1.1.14 one can easily see that it is sufficient to check that the following composition

$$\begin{array}{ccccc}
\Sigma H_l(p)H(p) & \stackrel{\lambda*Id}{\to} & H_l(p)H(p)H_l(p)H(p) & \to & H_l(p)H_l(pr_2)H(pr_1)H(p) \\
 & & & & \downarrow \\
 & & & & H_l(p)H_l(pr_2)H(pr_2)H(p) \\
 & & & & \downarrow \\
 & & & & H_l(p)\Sigma H(p)
\end{array}$$

where the second vertical arrow is $Id*\sigma_\Delta*Id$, is an isomorphism. Denote the composition of the second horizontal arrow with the two vertical ones by γ. Let $\sigma: H_l(p)H(p)H_l(p)H(p)\to H_l(p)H(p)H_l(p)H(p)$ be the automorphism corresponding to the permutation of factors on $\mathbf{P}^1\times\mathbf{P}^1$ under the isomorphism

$$H_l(p)H(p)H_l(p)H(p)=S(\mathbf{P}^1)S(\mathbf{P}^1)\cong S(\mathbf{P}^1\times\mathbf{P}^1)$$

let $i: S(\mathbf{P}^1\times\mathbf{P}^1-\Delta(\mathbf{P}^1))\to H_l(p)H(p)H_l(p)H(p)$ be the morphism corresponding under the same isomorphism to the embedding $\mathbf{P}^1\times\mathbf{P}^1-\Delta(\mathbf{P}^1)\to\mathbf{P}^1\times\mathbf{P}^1$ and let

$$\sigma_0: S(\mathbf{P}^1\times\mathbf{P}^1-\Delta(\mathbf{P}^1))\to S(\mathbf{P}^1\times\mathbf{P}^1-\Delta(\mathbf{P}^1))$$

be the isomorphism corresponding to the permutation of factors on $\mathbf{P}^1\times\mathbf{P}^1-\Delta(\mathbf{P}^1)$. One can easily verify the following facts:

1. the sequence

   $$0 \to S(\mathbf{P}^1 \times \mathbf{P}^1 - \Delta(\mathbf{P}^1)) \xrightarrow{i} H_l(p)H(p)H_l(p)H(p) \xrightarrow{\gamma} H_l(p)\Sigma H(p) \to 0$$

   is split-exact

2. the sequences

   $$0 \to \Sigma H_l(p)H(p) \stackrel{\lambda * Id}{\to} H_l(p)H(p)H_l(p)H(p) \stackrel{a * Id}{\to} H_l(p)H(p) \to 0$$

   $$0 \to H_l(p)H(p)\Sigma \stackrel{Id * \lambda}{\to} H_l(p)H(p)H_l(p)H(p) \stackrel{Id * a}{\to} H_l(p)H(p) \to 0$$

   are split exact.

3. the diagram

   $$\begin{array}{ccc} S(\mathbf{P}^1 \times \mathbf{P}^1 - \Delta(\mathbf{P}^1)) & \to & H_l(p)H(p)H_l(p)H(p) \\ \sigma_0 \downarrow & & \downarrow \sigma \\ S(\mathbf{P}^1 \times \mathbf{P}^1 - \Delta(\mathbf{P}^1)) & \to & H_l(p)H(p)H_l(p)H(p) \end{array}$$

   commutes

4. the diagram

   $$\begin{array}{ccc} H_l(p)H(p)H_l(p)H(p) & \stackrel{a * Id}{\to} & H_l(p)H(p) \\ \sigma \downarrow & & \downarrow Id \\ H_l(p)H(p)H_l(p)H(p) & \stackrel{a * Id}{\to} & H_l(p)H(p) \end{array}$$

   commutes.

Therefore there exist isomorphisms

$$\sigma' : \Sigma H_l(p)H(p) \to H_l(p)H(p)\Sigma$$

$$\sigma'' : H_l(p)\Sigma H(p) \to H_l(p)\Sigma H(p)$$

such that the diagram

$$\begin{array}{ccccc} \Sigma H_l(p)H(p) & \stackrel{\lambda * Id}{\to} & H_l(p)H(p)H_l(p)H(p) & \stackrel{\gamma}{\to} & H_l(p)\Sigma H(p) \\ \sigma' \downarrow & & \sigma \downarrow & & \sigma'' \downarrow \\ H_l(p)H(p)\Sigma & \stackrel{Id * \lambda}{\to} & H_l(p)H(p)H_l(p)H(p) & \stackrel{\gamma}{\to} & H_l(p)\Sigma H(p) \end{array}$$

commutes. Observe now that the composition of lower horizontal arrows is $Id * \psi$ where ψ is the isomorphism considered in Lemma 1.2.2 and therefore the composition of upper horizontal arrows is an isomorphism.