Direct image with compact supports in the stable $\mathbf{A}^1$-homotopy theory [1]

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1 Introduction

In this paper we study functoriality of the stable $\mathbf{A}^1$-homotopy categories $SH(S)$ with respect to the base scheme S. There are two obvious types of functors associated to morphisms which can be constructed directly from the definition of SH. First, categories SH are "contravariantly functorial" in S. For any morphism $f : S' \to S$ of base schemes there is a functor $f^* : SH(S) \to SH(S')$ (called the *inverse image functor*) and for any composable pair of morphisms f, g a natural isomorphism $(f \circ g)^* \cong g^* \circ f^*$ which satisfy the usual coherence condition for any composable triple of morphisms f, g, h. The functors f^* are essentially characterized by the property that for a smooth scheme X over S one has a canonical isomorphism $f^*(\Sigma_T X_+) \cong \Sigma_T(X \times_S S')_+$ togther with the fact that they commute with arbitrary direct sums and respect the triangulated structure.

For any *smooth* morphism $p_X : X \to S$ the functor p_X^* has a left adjoint $(p_X)_\#$ and for any smooth scheme U over X there is a canonical isomorphism $(p_X)_\#(\Sigma_T U_+) \cong \Sigma_T U^+$ where on the right hand side U is considered as a smooth scheme over S. In particular $\Sigma_T X_+ = (p_X)_\#(\mathbf{1})$ where $\mathbf{1}$ is the unit of the smash product in $SH(X)$. Functors $f_\#$ also commute with arbitrary direct sums and respect the triangulated structure. For a commutative square

$$\begin{array}{ccccc} & X' & \stackrel{g_X}{\to} & X & \\ p' & \downarrow & & \downarrow & p \\ & S' & \stackrel{g_S}{\to} & S & \end{array}$$

such that p is smooth the adjunctions give a canonical morphism of functors $p'_\# g_X^* \to g_S^* p_\#$. Using the explicit description of these functors on suspension spectra one verifies that this morphism is an isomorphism if the square is a pull-back square. Thus we have the *smooth base change theorem* for the pair $(g_S^*, p_\#)$.

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Inverse image functors respect the tensor structure on SH given by the smash product. For any two specta $\mathbf{E}, \mathbf{F}$ we have a canonical isomorphism $f^*(\mathbf{E}\wedge\mathbf{F}) = f^*(\mathbf{E})\wedge f^*(\mathbf{F})$ which satisfies the usual coherence conditions. By adjunction this implies that for a smooth morphism $p : X \to S$ and spectra $\mathbf{E}$, $\mathbf{F}$ over X and S respectively we have a canonical morphism $p_\#(\mathbf{E}) \wedge \mathbf{F} \to p_\#(\mathbf{E} \wedge p^*(\mathbf{F}))$. Using the explicit description of functors p^* and $p_\#$ on suspension spectra one verifies that this morphism is always an isomorphism which gives the *projection formula* for the pair $(p^*, p_\#)$.

Finally one can define in the usual way the right adjoint f_* to the inverse image functor which is called the direct image functor. Unlike functors f^* and $f_\#$ the direct image functor does not have a geometrical description in terms of its values on suspension spectra. Moreover it is not known to take compact objects to compact objects even for smooth morphisms (in fact even for open embeddings). This makes working with this functor difficult and we try to avoid it except in the particular case of closed embeddings where a partial geometrical description is available due to the gluing theorem.

The main results of this paper can be summarized as follows. We construct for any morphism of finite type $f : X \to S$ of Noetherian schemes a functor $f_! : SH(X) \to SH(S)$ (called the *direct image with compact supports*) such that one has:

1. for a composable pair of morphisms f, g there is a canonical isomorphism $(f \circ g)_! \cong f_! \circ g_!$ and for a composable triple the usual coherence condition holds

2. functors $f_!$ respect triangulated structure, commute with arbitrary direct sums and take compact objects to compact objects

3. for a *proper* morphism f the functor $f_!$ is right adjoint to f^* i.e. $f_! = f_*$

4. for a *smooth* morphism f the functor $f_!(- \wedge Th(T_{X/S}))$ is left adjoint to f^* i.e. $f_!(- \wedge Th(T_{X/S})) = f_\#$

5. for a Cartesian square

$$\begin{array}{ccc} X' & \stackrel{g_X}{\to} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \stackrel{g_S}{\to} & S \end{array}$$

one has a canonical isomorphism $g_S^* \circ f_! \cong f'_! \circ g_X^*$

We want to mention the following two applications of these results. The first one is the $\mathbf{A}^1$-analog of the Spanier-Whitehead duality for manifolds. For a morphism $f : X \to S$ which is both proper and smooth the combination of items (3) and (4) implies that the Thom spectrum $Th(N_{X/S})$ of the normal bundle to X over S represents the dual to the suspension spectrum of X_+ in $SH(S)$. In particular for any spectrum $\mathbf{E}$ over S one has canonical isomorphisms

$$E^{p,q}(X_+) = E_{-p,-q}(Th(N_{X/S}))$$
$$E_{p,q}(X_+) = E^{-p,-q}(Th(N_{X/S})).$$

For spectra $\mathbf{E}$ representing theories with Thom isomorphism such as motivic cohomology or algebraic cobordism this leads to the Poincare duality theorem for smooth proper morphisms over arbitrary Noetherain base. In particular we prove the existence of homological fundamental class in algebraic cobordisms which was used in [**?**].

The second application is the blow-up long exact sequence for any generalized cohomology theory. This is a corollary of the proper base change theorem (item (5) above). The only previously known construction of this sequence for motivic cohomology was based on the use of resolution of singularities. It has important consequences for the comparison between cdh- and Nisnevich cohomology and localization for algebraic cycles homology over arbitrary base which are considered in the second part of the paper.

We do not consider here the inverse image with compact supports $f^!$. It is easy to define it formally as the right adjoint to $f_!$ using the general existence theorem of A. Neeman. However at the moment we can not prove any nontrivial theorems about this functors. In particular the problem of proving that $f^!$ for a morphism of finite type (in fact already for a closed embedding) takes compact objects to compact objects seems to be a hard one and may require blow-up techniques.

Although we work with the stable homotopy category SH throughout the paper all our proofs work without change for the stable homotopy categories of modules over symmetric ring spectra. They also work in topological setting for apropriately defined relative stable homotopy categories of spectra. They should also work in the equivariant setting but this is a separate story.

In the first part of the paper we deal only with quasi-projective morphisms. In the second we use the results of the first together with the fact that sheaves (unlike schemes) admit one-point compactifications to define canonically the direct image with compact supports for all morphisms of finite type.

One of our main technical tools is the stable version of the gluing theorem ([**?**, Th. 3.2.21]) which says that for any spectrum $\mathbf{F}$ over S and a closed embedding $i : Z \to S$ one has a canonical distinguished triangle

$$j_\# j^* \mathbf{F} \to F \to i_* i^* \mathbf{F} \to j_\# j^* \mathbf{F}[1]$$

where $j : S - i(Z) \to S$ is the complimentary open embedding to i. We show that the category $SH(Z)$ is generated by objects of the form $i^*\mathbf{F}$ which gives us two corollaries. One is that the proper base change (see (5) above) holds for closed embeddings and another that i_* takes compact objects to compact objects.

For any smooth morphism $\bar{p} : \bar{X} \to S$ we define the *dualizing object* of X over S as $D_{X/S} = (pr_1)_\# \Delta_*(\mathbf{1})$ where $pr_1 : X \times_S X \to X$ is the first projection and $\Delta : X \to X \times_S X$ is the diagonal closed embedding. For a proper morphism $\bar{p} : \bar{X} \to S$ and an open embedding $j : X \to \bar{X}$ such that $p = \bar{p} \circ j$ is smooth we define a natural transformation of functors $\beta : \bar{p}^* p_\# \to j_\#(- \wedge D_{X/S})$. The main result of the first part of the paper is that the morphism $p_\# \to \bar{p}_* j_\#(- \wedge D_{X/S})$ adjoint to β is an isomorphism for any smooth quais-projective p.

We first prove that for pairs $(\bar{p}, j)$ such that both $\bar{p}$ and $\bar{q} : \bar{X} - j(X) \to S$ are smooth the composition $j_\# p^*$ has a left adjoint $\phi_{\bar{p},j}$ which has properties very similar to the properties of $\bar{p}_\#$ including the smooth base change theorem and the projection formula.

We use this construction to show that the adjoint to β is an isomorphism for $\bar{p} = (\mathbf{P}^n_S \to S)$ and $j = (\mathbf{A}^n \to \mathbf{P}^n)$. The key ingredient of the proof is the construction of a morphism $T^n \to \phi_{\bar{p},j}(\mathbf{1})$ which plays the role of fundamental class.

The we deduce that the adjoint to β is an isomorphism for $\bar{p} = (\mathbf{P}^n_S \to S)$ and $j = Id$. As a corollary of this fact and the proper base change for closed embeddings we get the proper base change theorem (5) for projective morphisms f. This theorem in the usual way implies that for a quais-projective morphism p any two decompositions $p = \bar{p} \circ j$, $p' = \bar{p}' \circ j'$ such that j, j' are open embeddings and $\bar{p}, \bar{p}'$ are projective morphisms there is a canonical isomorphism $\bar{p}_* j_\# \cong \bar{p}'_* j'_\#$. Thus we can define $p_!$ for quasi-projective morphisms such that $(p \circ q)_! = p_! \circ q_!$ and the base change theorem holds in the form given in (5).

We can now rewrite the adjoint to β as a morphism $p_{\#} \to p_!(- \wedge D_{X/S})$ and show that if it is an isomorphism for two smooth morphisms then it is an isomorphism for their composition. Then we show that the adjoint to β is an isomorphism for an etale morphism p using again an explicit construction of the "fundamental class" $\mathbf{1} \to \bar{p}_{\#}(\mathbf{1})$. Since any smooth morphism can be represented locally as a composition of an etale morphism and the projection $\mathbf{A}^n_S \to S$ this implies that the adjoint to β is an isomorphism for any smooth quais-projective morphism.

Finally we use a version of the deformation to the normal cone construction to get an isomorphism $D_{X/S} \to Th_{X/S}$ thus finishing the proof of all the properties of direct image with compact supports for quasi-projective morphisms.

2 The stable homotopy category of T-spectra

Everywhere below a *sheaf* over S means a sheaf of sets on $(Sm/S)_{Nis}$. The category $\Delta^{op}Shv_\bullet(S) = \Delta^{op}Shv_\bullet((Sm/S)_{Nis})$ of pointed simplicial sheaves has a proper simplicial closed model structure which is called the $\mathbf{A}^1$-closed model structure and which is described in detail in [**?**]. The localization of $\Delta^{op}Shv_\bullet(S)$ with respect to the class $\mathbf{W}_{\mathbf{A}^1}$ of $\mathbf{A}^1$-weak equivalences is called the pointed $\mathbf{A}^1$-homotopy category over S and denoted by $\mathcal{H}_{\mathbf{A}^1,\bullet}(S)$ or simply $\mathcal{H}_\bullet(S)$.

There is an alternative description of $\mathcal{H}_\bullet(S)$ as a localization of the category of pointed sheaves itself (see [**?**]) but in this paper we will use the simplicial approach of [**?**]. The following two unstable lemmas will be needed.

Lemma 2.1 *Let F be a pointed simplicial sheaf with the following two properties:*

1. *for any quasi-projective smooth scheme X over S and an open subscheme U in X the morphism of simplicial sets $F(X) \to F(U)$ is a Kan fibration*

2. *for any quasi-projective smooth scheme X over S the morphism $F(X \times \mathbf{A}^1) \to F(X)$ corresponding to the point 0 of $\mathbf{A}^1$ is a weak equivalence.*

Then for any pointed simplicial set K and any smooth quasi-projective scheme X over S one has a canonical bijection

$$Hom_{H_\bullet(S)}(X_+ \wedge K, F) = \pi_0 \underline{Hom}_{\Delta^{op}Sets_\bullet}(K, F(X))$$

Proof: Let $F \to Ex_s(F)$ be a simplicial trivial cofibration such that $Ex_s(F)$ is simplicially fibrant. Our first condition on F means that it has the B.G. property in the sense of [**?**, Def.3.1.13] with respect to the class of quasi-projective schemes. By [**?**, Prop. 3.1.16] the maps of simplicial sets $F(X) \to Ex_s(F)(X)$ are weak equivalences for all quasi-projective X. By our second condition on F this implies in particular that for any such X the map $Ex_s(F)(X \times \mathbf{A}^1) \to Ex_s(F)(X)$ is a weak equivalence. Since any scheme has a covering by quasi-projective schemes this implies (by [**?**, Lemma 2.2.8(3)]) that $Ex_s(F)$ is $\mathbf{A}^1$-local object. Therefore

$$Hom_{H_\bullet(S)}(X_+ \wedge K, F) = Hom_{H_{\bullet,s}(S)}(X_+ \wedge K, Ex_s(F)) =$$

$$= \pi_0 \underline{Hom}_{\Delta^{op}Sets_\bullet}(K, Ex_s(F)(X)) = \pi_0 \underline{Hom}_{\Delta^{op}Sets_\bullet}(K, F(X))$$

Definition 2.2 *The subcategory $\Delta^{op}Shv_\bullet^{ft}(S)$ of pointed simplicial sheaves of finite type is the smallest subcategory which contains simplicial sheaves of the form $(X \times \Delta^n)_+$ for all smooth schemes over S and all $n \geq 0$ and has the property that if in a push-forward square*

$$\begin{array}{ccc} A & \stackrel{i}{\to} & X \\ \downarrow & & \downarrow \\ B & \to & Y \end{array}$$

A, X and B belong to $\Delta^{op}Shv^{ft}$ and i is a monomorphism then Y belongs to Shv^{ft}.

Note that since any smooth scheme has a covering by quasi-projective schemes we can equivalently start in this definition with the class of sheaves of the form $X \times \Delta^n$ with quasi-projective X's.

Lemma 2.3 *Let K be a pointed simplicial sheaf of finite type and $(F_\alpha)_{\alpha \in A}$ a filtered system of pointed simplicial sheaves. Then the natural map*

$$colim_\alpha Hom_{H_\bullet(S)}(K, F_\alpha) \to Hom_{H_\bullet(S)}(K, colim_\alpha F_\alpha)$$

is a bijection.

Proof: Replacing $F_\alpha' s$ by their $\mathbf{A}^1$-fibrant models obviously does not change the left hand side and it does not change the right hand side by [**?**, Cor. 2.2.13]. Thus we may assume that the sheaves F_α are $\mathbf{A}^1$-fibrant. For pointed simplicial sheaves X, Y denote by $\mathcal{S}_\bullet(X, Y)$ the pointed simplicial set of the form $\mathcal{S}_\bullet(X, Y)_n = Hom(X \wedge \Delta^n_+, Y)$. For any fibrant Y and any X we have $Hom_{H_\bullet(S)}(X, Y) = \pi_0(\mathcal{S}_\bullet(X, Y))$. Set $F = colim_\alpha F_\alpha$. Since K is compact as a simplicial sheaf we have $colim_\alpha Hom_{H_\bullet(S)}(K, F_\alpha) = \pi_0 \mathcal{S}(K, F)$. If $F \to Ex_{\mathbf{A}^1}(F)$ is an $\mathbf{A}^1$-fibrant replacement of F then $Hom_{H_\bullet(S)}(K, colim_\alpha F_\alpha) = \pi_0(\mathcal{S}_\bullet(K, Ex_{\mathbf{A}^1}(F)))$. Let A_F be the class of pointed simplicial sheaves K such that the map $\mathcal{S}_\bullet(K, F) \to \mathcal{S}_\bullet(K, Ex_{\mathbf{A}^1}(F))$ is a weak equivalence of simplicial sets. Lemma **??** implies easily that A_F contains all the sheaves of the form $(X \times \Delta^n)_+$ for smooth quasi-projective schemes X. Consider a pushforward square

$$\begin{array}{ccc} A & \stackrel{i}{\to} & X \\ \downarrow & & \downarrow \\ B & \to & Y \end{array}$$

where A, X and B are sheaves of finite type which belong to A_F and i is a monomorphism. We have two fibrant squares of simplicial sets

$$\begin{array}{ccc} \mathcal{S}_\bullet(Y, Ex_{\mathbf{A}^1}(F)) & \to & \mathcal{S}_\bullet(X, Ex_{\mathbf{A}^1}(F)) \\ \downarrow & & \downarrow \\ \mathcal{S}_\bullet(B, Ex_{\mathbf{A}^1}(F)) & \to & \mathcal{S}_\bullet(A, Ex_{\mathbf{A}^1}(F)) \end{array} \qquad \begin{array}{ccc} \mathcal{S}_\bullet(Y, F) & \to & \mathcal{S}_\bullet(X, F) \\ \downarrow & & \downarrow \\ \mathcal{S}_\bullet(B, F) & \to & \mathcal{S}_\bullet(A, F) \end{array}$$

and a morphism bewteen them which is a weak equivalence everywhere except possibly for the left upper corners. The right vertical arrow in the first square is a fibration since i is a monomorphism and $Ex_{\mathbf{A}^1}(F)$ is fibrant. The right vertical arrow in the second square is also a fibration since it is a filtered colimit of fibrations $\mathcal{S}_\bullet(X, F_\alpha) \to \mathcal{S}_\bullet(A, F_\alpha)$. Therefore both squares are homotopy fibrant and thus the left upper corner morphism is also a weak equivalence i.e. Y belongs to A_F. Lemma is proven.

Definition 2.4 *A T-spectrum over a scheme S is a collection $\mathbf{E} = (E_i, e_i : T \wedge E_i \to E_{i+1})_{i \geq 0}$ where E_i are pointed simplicial sheaves over S and $T = (\mathbf{P}^1, \infty)$. A morphism of spectra $f : (E_i, e_i) \to (E'_i, e'_i)$ is a collection of morphisms $f_i : E_i \to E'_i$ such that for all $i \geq 0$ one has $e'_i \circ (Id_T \wedge f_i) = (Id_T \wedge f_{i+1}) \circ e_i$.*

Spectra and their morphisms form a category which we denote by $Spec_T(S)$. This category has all small products and coproducts which are defined termwise. The spectrum $(pt, Id)_{i \geq 0}$ is both the initial and the final object in this category and we denote it by pt. For any $n \in \mathbf{Z}$ we have a functor $\Sigma^\infty_T(-, n)$ from pointed sheaves to spectra such that

$$\Sigma^\infty_T((X, x), n) = \begin{cases} pt & \text{for} \quad i < -n \\ T^{\wedge(i+n)} \wedge (X, x) & \text{for} \quad i \geq -n \end{cases}$$

The functor $\Sigma_T^\infty(-,0)$ is called the suspension spectrum functor and we denote it simply by $\Sigma_T^\infty(-)$ or $\Sigma_T(-)$. The following obvious lemma will be used below.

Lemma 2.5 *For $i \geq 0$ the functor $\Sigma_T^\infty(-,-i)$ is left adjoint to the functor which takes a spectrum (E_i, e_i) to the pointed sheaf E_i i.e. for any pointed sheaf (X,x) one has*

$$Hom_{Spec_T}(\Sigma_T^\infty((X,x),-i),\mathbf{E}) = Hom_{Shv_\bullet}((X,x),E_i)$$

The stable homotopy category $SH(S)$ is the localization of $Spec_T(S)$ with respect to the class of *stable equivalences* which are defined as follows. For any spectrum (E_i, e_i) define functors $E^n : Shv_\bullet^{ft} \to Sets$, $n \in \mathbf{Z}$ setting

$$E^n(X,x) = \mathrm{co}\lim_{i \geq max\{0,-n\}} Hom_{\mathcal{H}_\bullet(S)}(T^{\wedge i} \wedge (X,x), E_{i+n})$$

Definition 2.6 *A morphism of T-spectra $f : \mathbf{E} \to \mathbf{F}$ is called a stable equivalence if for any pointed sheaf of finite type (X,x) and any $n \in \mathbf{Z}$ the corresponding map $E^n(X,x) \to F^n(X,x)$ is a bijection.*

The key result which shows that our definitions are "reasonable" is the following theorem.

Theorem 2.7 *For any sheaf of finite type (X,x) any $n \in \mathbf{Z}$ and any spectrum $\mathbf{E}$ one has $Hom_{SH(S)}(\Sigma_T^\infty((X,x),n),\mathbf{E}) = E^{-n}(X,x)$.*

The proof of this theorem will be given at the end of this section after we define two endo-functors on the category $Spec_T$ which we call the right resolution functor $Rres$ and the left resolution functor $Lres$. The right resolution functor comes together with a natural transformation $i_{res} : Id \to Rres$ such that for a spectrum $\mathbf{E}$ the morphism $\mathbf{E} \to Rres(\mathbf{E})$ plays the role analoguous to the role of injective resolutions for complexes of sheaves. The left resolution functor comes with a natural transformation $p_{res} : Lres \to Id$ such that for a spectrum $\mathbf{E}$ the morphism $Lres(\mathbf{E}) \to \mathbf{E}$ plays the role analoguous to the role of flat (or free) resolutions for complexes of sheaves. In particular the functor $Rres$ is a part of our definition of the direct image functors f_* and $Lres$ is a part of our definition of the functors f^* and $f_\#$.

Let $SmQP(S)$ be the set of all smooth quasi-projective varieties over S i.e. $SmQP(S) = \coprod SmQP(S)_n$ where $SmQP(S)_n$ is the set of open subschemes of closed subschemes of $\mathbf{P}^n_S$ which are smooth over S. We define $Lres$ and

p_{res} first. For a morphism of spectra $f : \mathbf{F} \to \mathbf{E}$ let $l(f)$ be the set of all commutative squares of the form

$$\begin{array}{ccc} \Sigma_T^\infty((X \times \partial\Delta^j)_+, -i) & \to & \mathbf{F} \\ \downarrow & & \downarrow \\ \Sigma_T^\infty((X \times \Delta^j)_+, -i) & \to & \mathbf{E} \end{array}$$

where the left vertical arrow is the obvious monomorphism, the right vertical arrow is f, $j \geq 0$, $i \geq 0$ and $X \in SmQP(S)$. Define $Lres^1(f)$ as the simplicial sheaf given by the pushforward square

$$\begin{array}{ccc} \bigvee_{l(f)} \Sigma_T^\infty((X \times \partial\Delta^j)_+, -i) & \to & \mathbf{F} \\ \downarrow & & \downarrow \\ \bigvee_{l(f)} \Sigma_T^\infty((X \times \Delta^j)_+, -i) & \to & Lres^1(f) \end{array}$$

and let $p^1_{res}(f)$ be the canonical morphism $Lres^1(f) \to \mathbf{E}$. Set inductively $Lres^n(f) = Lres^1(p^{n-1}_{res}(f))$ and $p^n_{res}(f) = p^1_{res}(p^{n-1}_{res}(f))$. We have obvious morphisms $Lres^n(f) \to Lres^{n+1}(f)$ and we set $Lres^\infty(f) = colim_{n\geq 0} Lres^n(f)$ and $p^\infty_{res} = colim_{n\geq 0} p^n_{res}$. Finally we set $Lres(\mathbf{E})$ to be $Lres^\infty(pt \to \mathbf{E})$. Observe that for any $f : \mathbf{F} \to \mathbf{E}$ the sheaf $(Lres^\infty(f)_i)_j$ of j-simplexes of the i-th term of the spectrum $Lres^\infty(f)$ is of the form

$$(Lres^\infty(f)_i)_j = (F_i)_j \vee (\vee_{\alpha \in A_{i,j}} \Sigma_T^{n_\alpha}(X_\alpha)_+)$$

where X_α are smooth quasi-projective schemes over S. In particular for any spectrum $\mathbf{E}$ the sheaves $(Lres(\mathbf{E})_i)_j$ are of the form $\vee_{\alpha \in A_{i,j}} \Sigma_T^{n_\alpha}(X_\alpha)_+$.

Lemma 2.8 *For any morphism $f : \mathbf{F} \to \mathbf{E}$ the morphism $p^\infty_{res}(f) : Lres^\infty(f) \to \mathbf{E}$ is a termwise simplicial weak equivalence.*

Proof: We will show that for any i the morphism of simplicial sheaves $Lres^\infty(f)_i \to E_i$ is a trivial local fibration. Let X be a smooth quasi-projective scheme over S. It is sufficient to show that the morphism of simplicial sets $Lres^\infty(f)_i(X) \to E_i(X)$ is a trivial Kan fibration i.e. that for any commutative square of simplicial sets

$$\begin{array}{ccc} \partial\Delta^j & \to & Lres^\infty(f)_i(X) \\ \downarrow & & \downarrow \\ \Delta^j & \to & E_i(X) \end{array}$$

there exists a morphism $\Delta^j \to Lres^\infty(f)_i(X)$ which makes two triangles in the resulting diagram commutative. Equivalently this means that for any commutative square of pointed simplicial sheaves

$$\begin{array}{ccc} (X \times \partial\Delta^j)_+ & \to & Lres^\infty(f)_i \\ \downarrow & & \downarrow \\ (X \times \Delta^j)_+ & \to & E_i \end{array}$$

there exists a morphism $(X \times \Delta^j)_+ \to Lres^\infty(f)_i$ which makes two triangles in the resulting diagram commutative. By Lemma **??** this in turn means that for any commutative square of spectra

$$\begin{array}{ccc} \Sigma^\infty_T((X \times \partial\Delta^j)_+, -i) & \to & Lres^\infty(f) \\ \downarrow & & \downarrow \\ \Sigma^\infty_T((X \times \Delta^j)_+, -i) & \to & \mathbf{E} \end{array}$$

there exists a morphism $\Sigma^\infty_T((X \times \Delta^j)_+, -i) \to Lres^\infty(f)$ which makes two triangles in the resulting diagram commutative. Objects of the form $\Sigma^\infty_T((X \times \partial\Delta^j)_+, -i)$ are compact in $Spec_T$ and therefore any morphism $\Sigma^\infty_T((X \times \partial\Delta^j)_+, -i) \to Lres^\infty(f)$ factors through the canonical morphism $Lres^n(f) \to Lres^\infty(f)$ for some n. By definition of $Lres(f)$ such factorization defines a morphism $\Sigma^\infty_T((X \times \Delta^j)_+, -i) \to Lres^{n+1}(f)$ whose composition with the canonical morphism $Lres^{n+1}(f) \to Lref^\infty(f)$ makes two triangles commutative.

The construction of the right resolution functor is based on a similar principle. For any morphism $f : \mathbf{F} \to \mathbf{E}$ we construct a functorial decomposition $\mathbf{F} \stackrel{i^\infty(f)}{\to} Rres^\infty(f) \to \mathbf{E}$ and then define $Rres(\mathbf{F})$ as $Rres^\infty(\mathbf{F} \to pt)$. The main difference is in the set of commutative squares with which the construction starts. For the right resolution functor we set $r(f)$ to be the union of three sets of commutative squares $r_{B.G.}(f)$, $r_{\mathbf{A}^1}(f)$ and $r_\Omega(f)$ defined as follows.

For a smooth quasi-projective scheme X over S, an open subscheme U in X and integers $i \geq 0$, $j \geq 0$, $k = 0, \ldots, j$ denote by $C(X, U, i, j, k)$ the spectrum given by the pushforward square

$$\begin{array}{ccc} \Sigma^\infty_T((U \times \Lambda^j_k)_+, -i) & \to & \Sigma^\infty_T((U \times \Delta^j)_+, -i) \\ \downarrow & & \downarrow \\ \Sigma^\infty_T((X \times \Lambda^j_k)_+, -i) & \to & C(X, U, i, j, k) \end{array}$$

Define $r_{B.G.}(f)$ as the set of all commutative squares of the form

$$\begin{array}{ccc} C(X,U,i,j,k) & \to & \mathbf{F} \\ \downarrow & & \downarrow \\ \Sigma^\infty_T((X\times\Delta^j)_+,-i) & \to & \mathbf{E} \end{array}$$

where the left vertical arrow is the morphism obvious from the definition of $C(X,U,i,j,k)$ and the right vertical arrow is f.

For a quasi-projective smooth scheme X over S and integers $j\geq 0$, $i\geq 0$ denote by $C_{\mathbf{A}^1}(X,i,j)$ the spectrum given by the pushforward square

$$\begin{array}{ccc} \Sigma^\infty_T((X\times\partial\Delta^j)_+,-i) & \to & \Sigma^\infty_T((X\times\Delta^j)_+,-i) \\ \downarrow & & \downarrow \\ \Sigma^\infty_T((X\times\mathbf{A}^1\times\partial\Delta^j)_+,-i) & \to & C_{\mathbf{A}^1}(X,i,j) \end{array}$$

where the left vertical arrow is given by the point 0 of $\mathbf{A}^1$. Define $r_{\mathbf{A}^1}(f)$ as the set of all commutative squares of the form

$$\begin{array}{ccc} C_{\mathbf{A}^1}(X,i,j) & \to & \mathbf{F} \\ \downarrow & & \downarrow \\ \Sigma^\infty_T((X\times\mathbf{A}^1\times\Delta^j)_+,-i) & \to & \mathbf{E} \end{array}$$

where the left vertical arrow is the morphism obvious from the definition of $C_{\mathbf{A}^1}(X,i,j)$ and the right vertical arrow is f.

For a quasi-projective smooth scheme X over S and integers $j\geq 0$, $i\geq 0$ denote by $C_\Omega(X,i,j)$ the spectrum given by the pushforward square

$$\begin{array}{ccc} \Sigma^\infty_T(T\wedge(X\times\partial\Delta^j)_+,-i-1) & \to & \Sigma^\infty_T((X\times\partial\Delta^j)_+,-i) \\ \downarrow & & \downarrow \\ \Sigma^\infty_T(T\wedge(X\times\Delta^j)_+,-i-1) & \to & C_\Omega(X,i,j) \end{array}$$

where the upper horizontal arrow is the obvious monomorphism. Define $r_\Omega(f)$ as the set of all commutative squares of the form

$$\begin{array}{ccc} C_\Omega(X,i,j) & \to & \mathbf{F} \\ \downarrow & & \downarrow \\ \Sigma^\infty_T((X\times\Delta^j)_+,-i) & \to & \mathbf{E} \end{array}$$

where the left vertical arrow is the morphism obvious from the definition of $C_\Omega(X,i,j)$ and the right vertical arrow is f.

We now proceed exactly as in the construction of the left resolutions. The following lemma is an immediate consequence of our choice of classes $r_{B.G.}$, $r_{\mathbf{A}^1}$ and r_Ω and of the fact that spectra of the form $C(X,U,i,j,k)$, $C_{\mathbf{A}^1}(X,i,j)$, $C_\Omega(X,i,j)$ are compact.

Lemma 2.9 *For any spectrum* $\mathbf{F}$ *the spectrum* $Rres(\mathbf{F})$ *has the following properties:*

1. *For a smoooth quasi-projective scheme* X *over* S*, an open subscheme* U *of* X *and an integer* $i \geq 0$ *the morphism of simplicial sets* $Rres(F)_i(X) \to Rres(F)_i(U)$ *is a Kan fibration. In particular* $Rres(F)_i(X)$ *is fibrant.*

2. *For a smoooth quasi-projective scheme* X *over* S *and an integer* $i \geq 0$ *the morphism of simplicial sets* $Rres(F)_i(X \times \mathbf{A}^1) \to Rres(F)_i(X)$ *corresponding to the point* 0 *of* $\mathbf{A}^1$ *is a trivial Kan fibration*

3. *For a smoooth quasi-projective scheme* X *over* S *and an integer* $i \geq 0$ *the morphism of simplicial sets* $Rres(F)_i(X) \to (\Omega_T Rres(F)_{i+1})(X)$ *corresponding to the adjoint to the assembly morphism* $\Sigma_T Rres(F)_i \to Rres(F)_{i+1}$ *is a trivial Kan fibration.*

Lemma 2.10 *For any spectrum* $\mathbf{F}$ *the morphism* $i_{res} : \mathbf{F} \to Rres(\mathbf{F})$ *is a stable equivalence.*

Proof: It follows from the construction of $Rres(\mathbf{F})$ that there exists a filtration $\mathbf{F} = \mathbf{E}^0 \subset \mathbf{E}^1 \subset \ldots \subset Rres(\mathbf{F})$ such that the components $E_i^m \subset E_i^{m+1}$ of the monomorphisms $\mathbf{E}^m \subset \mathbf{E}^{m+1}$ are $\mathbf{A}^1$-weak equivalences of simplicial sheaves for $i \geq m$. Thus for any pointed sheaf of finite type K we have

$$Rres(F)^n(K) = colim_{i \geq \{-n,0\}} Hom_{H_\bullet(S)}(T^{\wedge i} \wedge K, Rres(F)_{i+n}) =$$

$$= colim_i colim_m Hom_{H_\bullet(S)}(T^{\wedge i} \wedge K, E^m_{i+n}) =$$

$$= colim_m colim_i Hom_{H_\bullet(S)}(T^{\wedge i} \wedge K, E^m_{i+n}) = F^n(K)$$

where the second equality follows from Lemma **??**.

Lemma 2.11 *Let* $f : \mathbf{E} \to \mathbf{F}$ *be a stable equivalence. Then for any* $i \geq 0$ *the morphism of sheaves* $Rres(\mathbf{E})_i \to Rres(\mathbf{F})_i$ *is a simplicial weak equivalence.*