2-theories

As in your list of main properties of four functors in the theory of derived categories of sheaves one can informally summarize the known functorial properties of the stable homotopy categories over schemes as follows:

for any Noetherian scheme S of finite dimension there is a category $SH(S)$

for any morphism $f : S_1 \to S_2$ there are functors

$$f_* : SH(S_1) \to SH(S_2) \quad f^* : SH(S_2) \to SH(S_1)$$

for any quasi-projective morphism $f : S_1 \to S_2$ there are functors

$$f_! : SH(S_1) \to SH(S_2) \quad f^! : SH(S_2) \to SH(S_1)$$

Adjunctions $Hom(f_!A, B) = Hom(A, f^!B)$
$Hom(f^*A, B) = Hom(f_*A, B)$

Base change for any pull-back square $\begin{array}{ccc} X' & \stackrel{g}{\to} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \stackrel{g}{\to} & Y \end{array}$ one has

$$g^* f_! = f_! g^* \quad \text{if f is quasi-projective}$$

$$g^! f_* = f_* g^! \quad \text{if g is quasi-projective}$$

2-functoriality

$$(fg)_* = f_* g_* \quad (fg)^* = g^* f^*$$
$$(fg)_! = f_! g_! \quad (fg)^! = g^! f^!$$

Identities 1. For a projective morphism $f_* = f_!$

2. For a smooth quasi-projective morphism $f : X \to S$ one has $f^! = \Sigma_\Delta f^*$ where $\Sigma_\Delta = \Delta^* pr^!$, $X \stackrel{\Delta}{\to} X \times_S X \stackrel{pr}{\to} X$.

In addition one has:

Gluing axioms For a complimentary pair $Z \stackrel{i}{\to} X \stackrel{j}{\leftarrow} U$ both pairs of functors (j^*, i^*) and $(j^*, i^!)$ reflect isomorphisms

$\mathbf{A}^1$-homotopy invariance For $p : \mathbf{A}^1_S \to S$ the adjunctions

$$Id \to p_* p^* \quad p_! p^! \to Id$$

are isomorphisms

The categories $SH(S)$ are additive (in fact triangulated by this palys no role below), have small direct sums and sets of compact generators. If $SH^c(S)$ the subcategory of compact objects in $SH(S)$ then for any morphism f the functor f^* takes SH^c to SH^c and for any quasi-projective morphism the functor $f_!$ takes SH^c to SH^c. The following is known for varieties over a field with resolution of singularities and I believe should be true always:

Conjecture 1 *For any quasi-projective morphism f the functors f_* and $f^!$ take SH^c to SH^c.*

The categories SH and functors $f_*, f^*, f_!.f^!$ give an example of an $\mathbf{A}^1$-homotopy invariant "2-theory" with gluing. The rest of this text contains a formal definition of 2-category of 2-theories $2-Th$. Assuming Conjecture 1 the categories SH^c also form a 2-theory and I believe the following:

Conjecture 2 *For any Noetherian scheme of finite dimension S, SH^c is a universal $\mathbf{A}^1$-homotopy invariant 2-theory with gluing in the sense that the functor $\Phi \to Hom(SH^c, \Phi)$ from $2-Th_{\mathbf{A}^1,gluing}/S$ to $\mathcal{C}at$ is equivalent to the functor $\Phi \mapsto \Phi(S)$.*

E-functors Let C be a category and $\mathcal{D}$ a 2-category. Recall that a 2-functor H from C to $\mathcal{D}$ is given by

an object $\mathsf{H}(X)$ of $\mathcal{D}$ for any object X of C

a 1-morphism $H(p)$ of $\mathcal{D}$ for any morphism p of C

a 2-morphism $h(p,q) : H(pq) \to H(p)H(q)$ of $\mathcal{D}$ for any composable pair of morphisms p, q of C

a 2-morphism $h_X : H(id_X) \to Id_{\mathsf{H}}(X)$ for any object X of C

A 1-morphism F from a 2-functor $\mathsf{H} : \mathsf{C} \to \mathcal{D}$ to a 2-functor $\mathsf{H}' : \mathsf{C} \to \mathcal{D}$ is given by:

a 1-morphism $F_X : \mathsf{H}(X) \to \mathsf{H}'(X)$ for any object X of C

a 2-morphism $F_p : H'(p)F_X \to F_{X'}H(p)$ for any morphism $p : X \to X'$ of C

A 2-morphism $\phi : F \to F'$ between 1-morphisms of 2-functors is given by

a 2-morphism $\phi_X : F_X \to F'_X$ for any object X of C.

These data should satisfy the usual axioms. 2-functors from C to $\mathcal{D}$, their 1-morphisms and 2-morphisms form a 2-category $Funct(\mathsf{C}, \mathcal{D})$ which is strict if $\mathcal{D}$ is. A contravariant 2-functor from C to $\mathcal{D}$ is a 2-functor from C^{op} to $\mathcal{D}$.

Definition 0.1 *An exchange structure on a category* C *is given by a pair of subcategories* E_v, E_h *each of them containing all objects and all isomorphisms of* C *and a class of commutative squares* E *with vertical sides in* E_v *and horizontal sides in* E_h *which contains all squares with either vertical or horizontal sides being isomorphisms and is closed under the horizontal and vertical compositions.*

An E-functor $H : (\mathsf{C}, E) \to \mathcal{D}$ from C to $\mathcal{D}$ with respect to a given exchange data $(\mathsf{E}_v, \mathsf{E}_h, \mathsf{E})$ is given by

a covariant 2-functor $\mathsf{H}_c : \mathsf{E}_v \to \mathcal{D}$ and a contravariant 2-functor $\mathsf{H} : \mathsf{E}_h \to \mathcal{D}$ which coincide on objects

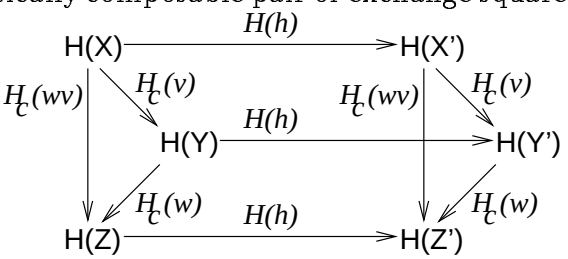
a 2-morphism $H(h_1)H_c(v_1) \to H_c(v_0)H(h_0)$ for any square $Q = \begin{pmatrix} X' & \stackrel{h_0}{\to} & X \\ {\scriptstyle v_0}\downarrow & & \downarrow{\scriptstyle v_1} \\ Y' & \stackrel{h_1}{\to} & Y \end{pmatrix}$ from E

These data should satisfy the following four axioms

Vertical composition for a vertically composable pair of exchange squares

$$\begin{array}{ccc} X' & \xrightarrow{h_0} & X \\ {\scriptstyle v_0}\downarrow & & \downarrow{\scriptstyle v_1} \\ Y' & \xrightarrow{h_1} & Y \\ {\scriptstyle w_0}\downarrow & & \downarrow{\scriptstyle w_1} \\ Z' & \xrightarrow{h_2} & Z \end{array}$$

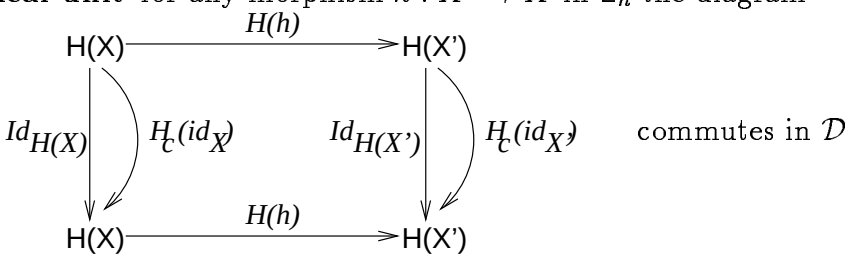
the diagram



commutes in $\mathcal{D}$. If $\mathcal{D}$ is strict this means that the pentagon

$$\begin{array}{ccccc} H(h_2)H_c(w_1v_1) & \rightarrow & H(h_2)H_c(w_1)H_c(v_1) & & \\ & & \downarrow & & \\ \downarrow & & H_c(w_0)H(h_1)H_c(v_1) & \text{of 2-morphisms commutes} & \\ & & \downarrow & & \\ H_c(w_0v_0)H(h_0) & \rightarrow & H_c(w_0)H_c(v_0)H(h_0) & & \end{array}$$

Vertical unit for any morphism $h : X' \to X$ in E_h the diagram



commutes in $\mathcal{D}$

i.e. the pentagon of 2-morphisms

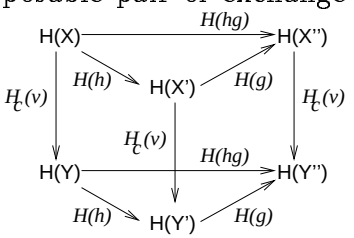
$$\begin{array}{ccc} H(h)H_c(id_{X'}) & \rightarrow & H(h)Id_{\mathsf{H}(X')} \\ & & \downarrow \\ \downarrow & & H(h) \\ & & \uparrow \\ H_c(id_X)H(h) & \rightarrow & Id_{\mathsf{H}(X)}H(h) \end{array}$$

commutes.

Horizontal composition for a horizontally composable pair of exchange squares

$$\begin{array}{ccccc} X'' & \xrightarrow{g_0} & X' & \xrightarrow{h_0} & X \\ {\scriptstyle v_0}\downarrow & & {\scriptstyle v_1}\downarrow & & \downarrow{\scriptstyle v_2} \\ Y'' & \xrightarrow{g_1} & Y' & \xrightarrow{h_1} & Y \end{array}$$

the diagram

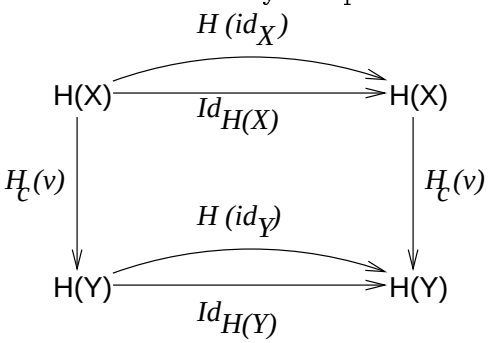


commutes in $\mathcal{D}$. If $\mathcal{D}$ is strict this means that the pentagon of 2-morphisms

$$\begin{array}{ccc} H(h_1 g_1)H_c(v_2) & \to & H(g_1)H(h_1)H_c(v_2) \\ & & \downarrow \\ \downarrow & & H(g_1)H_c(v_1)H(h_0) \\ & & \downarrow \\ H_c(v_0)H(h_0 g_0) & \to & H_c(v_0)H(g_0)H(h_0) \end{array}$$

commutes.

Horizontal unit for any morphism $v : X \to Y$ in E_v the diagram



commutes in $\mathcal{D}$

i.e. the pentagon of 2-morphisms

$$\begin{array}{ccc} H(id_Y)H_c(v) & \to & Id_{\mathsf{H}(Y)}H_c(v) \\ & & \downarrow \\ \downarrow & & H_c(v) \\ & & \uparrow \\ H_c(v)H(id_X) & \to & H_c(v)Id_{\mathsf{H}(X)} \end{array}$$

commutes.