# A simplest type system with annotation language

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Hi Dan,

in continuation of our conversation in Ocean Grove on Sunday here is an attempt to make the discussion of annotation languages which allow the proof checker to reconstruct the derivation tree in a deterministic manner more precise.

The simplest type system which I can think of which has non-trivial definitional equalities between types and therefore non-trivial conversions is the system generated by one universe $U$ and dependent products, abstractions, applications and $\beta$-reduction. Here is the list of its derivation rules:

1. Structural Rules.

   (a) for each $i \in \mathbf{N}$

   $$\frac{\Gamma, x : T, \Gamma' \rhd \quad \textit{where } l(\Gamma) = i}{\Gamma, x : T, \Gamma' \vdash x : T}$$

   (b)

   $$\frac{\Gamma, x : T \rhd}{\Gamma \vdash T \stackrel{d}{=} T}$$

   (c)

   $$\frac{\Gamma \vdash T_1 \stackrel{d}{=} T_2}{\Gamma \vdash T_2 \stackrel{d}{=} T_1}$$

   (d)

   $$\frac{\Gamma \vdash T_1 \stackrel{d}{=} T_2 \quad \Gamma \vdash T_2 \stackrel{d}{=} T_3}{\Gamma \vdash T_1 \stackrel{d}{=} T_3}$$

   (e)

   $$\frac{\Gamma \vdash o : T}{\Gamma \vdash o \stackrel{d}{=} o : T}$$

   (f)

   $$\frac{\Gamma \vdash o_1 \stackrel{d}{=} o_2 : T}{\Gamma \vdash o_2 \stackrel{d}{=} o_1 : T}$$

   (g)

   $$\frac{\Gamma \vdash o_1 \stackrel{d}{=} o_2 : T \quad \Gamma \vdash o_2 \stackrel{d}{=} o_3 : T}{\Gamma \vdash o_1 \stackrel{d}{=} o_3 : T}$$

   (h)

   $$\frac{\Gamma \vdash o : T \quad \Gamma \vdash T \stackrel{d}{=} T'}{\Gamma \vdash o : T'}$$

(i)

$$\frac{\Gamma \vdash o \stackrel{d}{=} o' : T \quad \Gamma \vdash T \stackrel{d}{=} T'}{\Gamma \vdash o \stackrel{d}{=} o' : T'}$$

2. Dependent Products.

(a)

$$\frac{\Gamma, x : T_1, y : T_2 \rhd}{\Gamma, y : [\prod](T_1, x.T_2) \rhd}$$

(b)

$$\frac{\Gamma \vdash T_1 \stackrel{d}{=} T_1' \quad \Gamma, x : T_1, y : T_2 \rhd}{\Gamma \vdash [\prod](T_1, x.T_2) \stackrel{d}{=} [\prod](T_1', x.T_2)}$$

(c)

$$\frac{\Gamma, x : T_1 \vdash T_2 \stackrel{d}{=} T_2'}{\Gamma \vdash [\prod](T_1, x.T_2) \stackrel{d}{=} [\prod](T_1, x.T_2')}$$

(d)

$$\frac{\Gamma, x : T_1 \vdash o : T_2}{\Gamma \vdash [\lambda](T_1, x.o) : [\prod](T_1, x.T_2)}$$

(e)

$$\frac{\Gamma \vdash T_1 \stackrel{d}{=} T_1' \quad \Gamma, x : T_1 \vdash o : T_2}{\Gamma \vdash [\lambda](T_1, x.o) \stackrel{d}{=} [\lambda](T_1', x.o) : [\prod](T_1, x.T_2)}$$

(f)

$$\frac{\Gamma, x : T_1 \vdash o \stackrel{d}{=} o' : T_2}{\Gamma \vdash [\lambda](T_1, x.o) \stackrel{d}{=} [\lambda](T_1, x.o') : [\prod](T_1, x.T_2)}$$

(g)

$$\frac{\Gamma \vdash f : [\prod](T_1, x.T_2) \quad \Gamma \vdash o : T_1}{\Gamma \vdash [ev](f, o) : T_2[o/x]}$$

(h)

$$\frac{\Gamma \vdash f \stackrel{d}{=} f' : [\prod](T_1, x.T_2) \quad \Gamma \vdash o : T_1}{\Gamma \vdash [ev](f, o) \stackrel{d}{=} [ev](f', o) : T_2[o/x]}$$

(i)

$$\frac{\Gamma \vdash f : [\prod](T_1, x.T_2) \quad \Gamma \vdash o = o' : T_1}{\Gamma \vdash [ev](f, o) \stackrel{d}{=} [ev](f, o') : T_2[o/x]}$$

(j)

$$\frac{\Gamma \vdash o_1 : T_1 \quad \Gamma, x : T_1 \vdash o_2 : T_2}{\Gamma \vdash [ev]([\lambda](T_1, x.o_2), o_1) \stackrel{d}{=} o_2[o_1/x] : T_2[o_1/x]}$$

3. Universe.

(a)

$$\frac{\Gamma \rhd}{\Gamma, x : \mathcal{U} \rhd}$$

(b)

$$\frac{\Gamma \vdash o : \mathcal{U}}{\Gamma, x : [El](o) \rhd}$$

(c)

$$\frac{\Gamma \vdash o \stackrel{d}{=} o' : \mathcal{U}}{\Gamma \vdash [El](o) \stackrel{d}{=} [El](o')}$$

Let's analyze derivation rules from the point of view of their applicability to sentences of various forms.

1. Sentences of the form $\Gamma, x : T \rhd$ where:
   (a) $T = [\prod](T_1, y.T_2)$:
      i. 2a
   (b) $T = U$:
      i. 3a
   (c) $T = [El](o)$:
      i. 3b
2. Sentences of the form $\Gamma \vdash o : T$ where:
   (a) $o = x$ where $x$ is a variable:
      i. 1a
      ii. 1h
   (b) $o = [\lambda](T, x.o')$:
      i. 2d
      ii. 1h
   (c) $o = [ev](f, o)$ where $x$ is a variable:
      i. 2g
      ii. 1h
3. Sentences of the form $\Gamma \vdash T_1 = T_2$ where:
   (a) $T_1 = [prod](T_3, x.T_4)$ and $T_2 = [prod](T_5, y.T_6)$:
      i. 1b
      ii. 1c
      iii. 1d

iv. 2b
v. 2c

(b) $T_1 = [prod](T_3, x.T_4)$ and $T_2 = [El](o)$:

i. 1c
ii. 1d

(c) $T_1 = [El](o)$ and $T_2 = [prod](T_5, y.T_6)$:

i. 2c
ii. 1d

(d) $T_1 = [El](o_1)$ and $T_2 = [El](o_2)$:

i. 1b
ii. 1c
iii. 1d
iv. 3c

4. Sentences of the form $\Gamma \vdash o_1 = o_2 : T$ where:

(a) $o_1 = x_1$, $o_2 = x_2$ where $x_1$ and $x_2$ are variables:

i. 1e
ii. 1f
iii. 1g
iv. 1i

(b) $o_1 = x_1$, $o_2 = [\lambda](T, x.o_3)$ where $x_1$ is a variable:

i. 1f
ii. 1g
iii. 1i

(c) $o_1 = x_1$, $o_2 = [ev](f, o)$ where $x_1$ is a variable:

i. 1f
ii. 1g
iii. 1i

(d) $o_1 = [\lambda](T, x.o_3)$, $o_2 = x_2$ where $x_2$ is a variable:

i. 1f
ii. 1g
iii. 1i

(e) $o_1 = [\lambda](T_1, x.o_3)$, $o_2 = [\lambda](T_2, x'.o_4)$:

i. 1e
ii. 1f
iii. 1g
iv. 1i

v. 2e
vi. 2f

(f) $o_1 = [\lambda](T, x.o_3)$, $o_2 = [ev](f, o)$:

i. 1f
ii. 1g
iii. 1i

(g) $o_1 = [ev](f, o)$, $o_2 = x_2$ where $x_2$ is a variable:

i. 1f
ii. 1g
iii. 1i
iv. 2j

(h) $o_1 = [ev](f, o)$, $o_2 = [\lambda](T, x.o_3)$:

i. 1f
ii. 1g
iii. 1i
iv. 2j

(i) $o_1 = [ev](f, o)$, $o_2 = [ev](f', o')$:

i. 1e
ii. 1f
iii. 1g
iv. 1i
v. 2h
vi. 2i
vii. 2j