Notes on type systems

Vladimir Voevodsky

Started September 08, 2009, cont. July 23, 2012, November 20, 2012, December 1, 2012

Notes from discarded attempts:

1. It seems that we will have to use some generalization of de Brujin indexes instead of α-equivalence classes since otherwise it is unclear how to "make" $[\prod; x](T_1, T_2)$ from smaller pieces. Indeed in the formulation with alpha-equivalence classes T_2 in this expression has no meaning at all (Nov. 20, 2012).

1a. One can write $[\prod; x](T_1, T_2)$ as $[prod](T_1, [bnd; x](T_2))$ and similarly for all other quantifiers (suggested by D Grayson, around Dec.1, 2012).

1b. The "type" of bnd is $forall(A)(x : A)(T : Exp\,A)(B), Exp(B) \to Exp(A - \{x\} \amalg B)$ (Dec.1, 2012).

Contents

1 C-systems and B-systems

C-systems and B-systems are models of essentially algebraic theories. C-systems are known in type theory as contextual categories. They where introduces by Cartmell in [?] and then described in more detail by Streicher (see [?, Def. 1.2, p.47]). B-systems are seemingly quite different objects which are exemplified by the systems of contexts and typing judgments of a type theory. One of the main ideas of this section is to outline some constructions and results which suggest that the theories of C-systems and B-systems are equivalent thus providing a purely algebraic basis for the connection between type systems and contextual categories. In the present version of the paper we do not give a precise formulation of the equivalence theorem. Work on constructing a formal proof of this theorem using Coq proof assistant is currently being done by Benedikt Ahrens, Chris Kapulkin and the author.

1 C-systems

It will be important for us to distinguish two notions of a category. What is understood by a category by most practicing mathematicians i.e. a category up to an equivalence, will be called, when an explicit distinction is needed, a category of h-level 3. A category as an algebraic object i.e. a category up to an isomorphism will be called a set-level category or category of h-level 2. A set-level category C is a pair of sets $Mor(C)$ and $Ob(C)$ with four maps

$$\partial_0, \partial_1 : Mor(C) \to Ob(C)$$

$$Id : Ob(C) \to Mor(C)$$

and

$$\circ : Mor(C)_{\partial_0} \times_{\partial_1} Mor(C) \to Mor(C)$$

which satisfy the well known conditions (note that we write composition of morphisms in the form $f \circ g$ where $f : Y \to X$ and $g : Z \to Y$).

A C-system is a set-level category CC with additional structure of the form

1. a function $l : Ob(CC) \to \mathbf{N}$,
2. an object pt,
3. a map $ft : Ob(CC) \to Ob(CC)$,
4. for each $X \in Ob(CC)$ a morphism $p_X : X \to ft(X)$,
5. for each $X \in Ob(CC)$ such that $X \neq pt$ and each morphism $f : Y \to ft(X)$ an object f^*X and a morphism $q(f, X) : f^*X \to X$,

which satisfies the following conditions:

1. $l^{-1}(0) = \{pt\}$
2. for X such that $l(X) > 0$ one has $l(ft(X)) = l(X) - 1$
3. $ft(pt) = pt$
4. pt is a final object,
5. for $X \in Ob(CC)$ such that $X \neq pt$ and $f : Y \to ft(X)$ one has $ft(f^*X) = Y$ and the square

   [2009.10.14.eq1]
   $$\begin{array}{ccc} f^*X & \xrightarrow{q(f,X)} & X \\ {\scriptstyle p_{f^*X}}\downarrow & & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow{f} & ft(X) \end{array} \qquad (1)$$

   is a pull-back square,
6. for $X \in Ob(CC)$ such that $X \neq pt$ one has $id^*_{ft(X)}(X) = X$ and $q(id_{ft(X)}, X) = id_X$,
7. for $X \in Ob(CC)$ such that $X \neq pt$, $f : Y \to ft(X)$ and $g : Z \to Y$ one has $(fg)^*(X) = g^*(f^*(X))$ and $q(fg, X) = q(f, X)q(g, f^*X)$.

Let $B_n(CC) = \{X \in Ob(CC) \,|\, l(X) = n\}$ and let $Mor_{n,m}(CC) = \{f : Mor(CC) | \partial_0(f) \in B_n \, and \, \partial_1(f) \in B_m\}$. One can reformulate the definition of a C-system using $B_n(CC)$ and $Mor_{n,m}(CC)$ as the underlying sets together with the obvious analogs of maps and conditions the definition given above. In this reformulation there will be no use of $\neq$ and the only use of the existential qualifier will be as a part of "there exists a unique" condition. This shows that C-systems can be considered as models of an essentially algebraic theory with sorts B_n, and $Mor_{n,m}$ and in particular all the results of [**?**] are applicable to C-systems.

We will also use the following notations:

1. $B(X) = \{Y \in Ob(CC) \,|\, ft(Y) = X \, and \, Y \neq pt\}$,

2. $\widetilde{Ob}(CC)$ is the set of pairs of the form (X, s) where $X \in Ob(CC)$, $X \neq pt$ and s is a section of the canonical morphism $p_X : X \to ft(X)$ i.e. a morphism $s : ft(X) \to X$ such that $p_X \circ s = Id_{ft(X)}$,

3. $\widetilde{B}_n = \{(X, s) \in \widetilde{Ob}(CC) \,|\, X \in B_n\}$ (note that $\widetilde{B}_0 = \emptyset$),

4. $\partial : \widetilde{B}_n \to B_n$ is the function defined by $\partial(X, s) = X$,

5. $\widetilde{B}(X) = \partial^{-1}(X)$ (note that $\widetilde{B}(pt) = \emptyset$).

2 C-subsystems.

A C-subsystem CC' of a C-system CC is a subcategory of the underlying set-level category which is closed, in the obvious sense under the operations which define the C-system on CC and such that the canonical squares which belong to CC' are pull-back squares in CC'. A C-subsystem is called non-trivial if it contains at least one element other than pt. A C-subsystem is itself a C-system with respect to the induced structure. The following elementary result plays a key role in many constructions of type theory:

Proposition 2.1 [**2009.10.15.prop1**] *Let CC be a C-system. Then for any family CC_α of C-subsystems of CC, the intersection $CC' = \cap_\alpha CC_\alpha$ is a C-subsystem.*

Proof: The only condition to check is that a canonical square which belongs to CC' is a pull-back square in CC'. This follows from the definition of pull-back squares and the fact that fiber products of sets commute with intersections of sets.

Corollary 2.2 [**2009.10.15.cor1**] *Let CC be a C-system, C_0 a set of objects of CC and C_1 a set of morphisms of CC. Then there exists the smallest C-subsystem $[C_1, C_0]$ which contains C_0 and C_1. It is called the C-subsystem generated by C_0 and C_1.*

Lemma 2.3 [**2009.10.15.l1**] *Let CC be a C-system and CC', CC'' be two C-subsystems such that $Ob(CC') = Ob(CC'')$ (as subsets of $Ob(CC)$) and $\widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$ (as subsets of $\widetilde{Ob}(CC)$). Then $CC' = CC''$.*

Proof: Let $f : Y \to X$ be a morphism in CC'. We want to show that it belongs to CC''. Proceed by induction on m where $X \in B_m$. For $m = 0$ the assertion is obvious. Suppose that $m > 0$. Since CC is a C-system we have a commutative diagram

[**2009.11.07.oldeq1**]
$$\begin{array}{ccccc} Y & \xrightarrow{s_f} & (p_X f)^* X & \xrightarrow{q(p_X f, X)} & X \\ \downarrow & & \downarrow p' & & \downarrow p \\ Y & \xrightarrow{=} & Y & \xrightarrow{p_X f} & ft(X) \end{array} \qquad (2)$$

such that $f = q(p_X f, X)\, s_f$. Since the right hand side square is a canonical one, $((p_X f)^* \Gamma', s_f) \in \widetilde{Ob}(CC)$ and $ft(X) \in B_{m-1}$, the inductive assumption implies that $f \in CC''$.

Remark 2.4 In Lemma 2.3, it is sufficient to assume that $\widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$. The condition $Ob(CC') = Ob(CC'')$ is then also satisfied. Indeed, let $X \in Ob(CC')$. Then $p_X^* X$ is the product $X \times X$ in CC. Consider the diagonal section $\Delta_X : X \to p_X^* X$ of $p_{p_X^*(X)}$. Since CC' is assumed to be a C-subsystem we conclude that $\Delta_X \in \widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$ and therefore $X \in Ob(CC'')$. It is however more convenient to think of C-subsystems in terms of subsets of both Ob and $\widetilde{Ob}$.

Let CC be a C-system. Let us say that a pair of subsets $C \subset Ob(CC)$, $\widetilde{C} \subset \widetilde{Ob}(CC)$ is saturated if there exists a C-subsystem CC' such that $C = Ob(CC')$ and $\widetilde{C} = \widetilde{Ob}(CC')$. By Lemma 2.3 we have a bijection between C-subsystems of CC and saturated pairs $(C, \widetilde{C})$.

Let us introduce the following notations. Let $X \in Ob(CC)$ and $i \geq 0$. Denote by $p_{X,i}$ the composition of the canonical projections $X \to ft(X) \to \ldots \to ft^i(X)$ such that $p_{X,0} = Id_X$ and $p_{X,1} = p_X$. For $f : Y \to ft^i(X)$ denote by $q(f, X, i) : f^*(X, i) \to X$ the morphism defined inductively by the rule

$$f^*(X, 0) = Y \qquad q(f, X, 0) = f,$$

$$f^*(X, i+1) = q(f, ft(X), i)^*(X) \qquad q(f, X, i+1) = q(q(f, ft(X), i), X).$$

In other words, $q(f, X, i)$ is the canonical pull-back of the morphism $f : Y \to ft^i(X)$ with respect to the sequence of canonical projections $X \to ft(X) \to \ldots \to ft^i(X)$.

Let $i \geq 1$, $f : Y \to ft^i(X)$ be a morphism and $s : ft(X) \to X$ an element of $\widetilde{Ob}(CC)$. Denote by $f^*(s, i)$ the element of $\widetilde{Ob}(CC)$ of the form $f^*(ft(X), i-1) \to f^*(X, i)$ which is the pull-back of s with respect to $q(f, ft(X), i-1)$.

Proposition 2.5 *[**2009.10.15.prop2**] A pair $(C, \widetilde{C})$ is saturated if and only if it satisfies the following conditions:*

1. *$pt \in C$,*
2. *if $X \in C$ then $ft(X) \in C$,*
3. *if $(s : ft(X) \to X) \in \widetilde{C}$ then $X \in C$,*
4. *if $(s : ft(X) \to X) \in \widetilde{C}$, $X' \in C$, $i \geq 1$ and $ft^i(X) = ft(X')$ then $q(p_{X'}, ft(X), i-1)^*(s) \in \widetilde{C}$,*
5. *if $(s_1 : ft(X) \to X) \in \widetilde{C}$, $i \geq 1$ and $(s_2 : ft^{i+1}(X) \to ft^i(X)) \in \tilde{C}$ then $q(s_2, ft(X), i-1)^*(s_1) \in \widetilde{C}$,*
6. *if $X \in C$ then the diagonal $s_{id_X} : X \to (p_X)^*(X)$ is in $\widetilde{C}$.*

Conditions (4) and (5) are illustrated by the following diagrams:

$$
\begin{array}{ccc}
p_{X'}^*(ft(X),i-1) & \xrightarrow{q(p_{X'},ft(X),i-1)} & ft(X) \\
\downarrow{\scriptstyle q(p_{X'},ft(X),i-1)^*(s)} & & \downarrow{\scriptstyle s} \\
p_{X'}^*(X,i) & \xrightarrow{q(p_{X'},X,i)} & X \\
\downarrow & & \downarrow{\scriptstyle p_X} \\
p_{X'}^*(ft(X),i-1) & \xrightarrow{q(p_{X'},ft(X),i-1)} & ft(X) \\
\downarrow & & \downarrow \\
\dots & & \dots \\
\downarrow & & \downarrow \\
X' & \xrightarrow{p_{X'}} & ft^i(X)
\end{array}
\qquad
\begin{array}{ccc}
s_2^*(ft(X),i-1) & \xrightarrow{q(s_2,ft(X),i-1)} & ft(X) \\
\downarrow{\scriptstyle q(s_2,ft(X),i-1)^*(s_1)} & & \downarrow{\scriptstyle s_1} \\
s_2^*(X,i) & \xrightarrow{q(s_2,X,i)} & X \\
\downarrow & & \downarrow{\scriptstyle p_X} \\
s_2^*(ft(X),i-1) & \xrightarrow{q(s_2,ft(X),i-1)} & ft(X) \\
\downarrow & & \downarrow \\
\dots & & \dots \\
\downarrow & & \downarrow \\
ft^{i+1}(X) & \xrightarrow{s_2} & ft^i(X)
\end{array}
$$

Proof: The "only if" part of the proposition is straightforward. Let us prove that for any $(C,\widetilde{C})$ satisfying the conditions of the proposition there exists a C-subsystem CC' of CC such that $C = Ob(CC')$ and $\widetilde{C} = \widetilde{Ob}(CC')$.

For a morphism $f : Y \to X$ let $ft(f) = p_X f : Y \to ft(X)$. Any morphism $f : Y \to X$ in CC has a canonical representation of the form $Y \stackrel{s_f}{\to} X_f \stackrel{q_f}{\to} X$ where $X_f = ft(f)^*(X)$, $q_f = q(ft(f),X)$ and $s_f : Y \to X_f$ is the section of the canonical projection $X_f \to Y$ corresponding to f.

Define a candidate subcategory CC' setting $Ob(CC') = C$ and defining the set $Mor(CC')$ of morphisms of CC' inductively by the conditions:

1. $Y \to pt$ is in $Mor(CC')$ if and only if $Y \in C$,
2. $f : Y \to X$ is in $Mor(CC')$ if and only if $X \in Ob(C)$, $ft(f) \in Mor(CC')$ and $s_f \in \widetilde{C}$.

(note that the for $(f : Y \to X) \in Mor(CC')$ one has $Y \in C$ since $s_f : Y \to X_f$).

Let us show that if the condition of the proposition are satisfied then $(Ob(CC'), Mor(CC'))$ form a C-subsystem of CC.

The subset $Ob(CC')$ contains pt and is closed under ft map by the first two conditions. The following lemma shows that $Mor(CC')$ contains identities and the compositions of canonical projections.

Lemma 2.6 *[2009.10.16.l1] Under the assumptions of the proposition, if $X \in C$ and $i \geq 0$ then $p_{X,i} : X \to ft^i(X)$ is in $Mor(CC')$.*

Proof: By definition of C-systems there exists n such that $ft^n(X) = pt$. Then $p_{X,n} \in Mor(CC')$ by the first constructor of $Mor(CC')$. By induction it remains to show that if $X \in C$ and $p_{X,i} \in Mor(CC')$ then $p_{X,i-1} \in Mor(CC')$. We have $ft(p_{X,i-1}) = p_{X,i}$ and $s_{p_{X,i-1}}$ is the pull-back of the diagonal $ft^{i-1}(X) \to (p_{ft^{i-1}(X)})^*(ft^{i-1}(X))$ with respect to the canonical morphism $X \to ft^{i-1}(X)$. The diagonal is in $\widetilde{C}$ by condition (6) and therefore $s_{p_{X,i-1}}$ is in $\widetilde{C}$ by repeated application of condition (4).

Lemma 2.7 *[***2009.10.16.l3***] Under the assumptions of the proposition, let* $X \in C$, $(s : ft(X) \to X) \in \widetilde{C}$, $i \geq 0$, *and* $(f : Y \to ft^i(X)) \in Mor(CC')$. *Then* $q(f, ft(X), i-1)^*(s) : ft(f^*(X,i)) \to f^*(X,i)$ *is in* $Mor(CC')$.

Proof: Suppose first that $ft^i(X) = pt$. Then $f = p_{Y,n}$ for some n and the statement of the lemma follows from repeated application of condition (4). Suppose that the lemma is proved for all morphisms to objects of length $j-1$ and let the length of $ft^i(X)$ be j. Consider the canonical decomposition $f = q_f s_f$. The morphism q_f is the canonical pull-back of $ft(f)$ and therefore the pull-back of s relative to q_f coincides with its pull-back relative to $ft(f)$ which is $\widetilde{C}$ by the inductive assumption. The pull-back of an element of $\widetilde{C}$ with respect to s_f is in $\widetilde{C}$ by condition (5).

Lemma 2.8 *[***2009.10.16.l4***] Under the assumptions of the proposition, let* $g : Z \to Y$ *and* $f : Y \to X$ *be in* $Mor(CC')$. *Then* $fg \in Mor(CC')$.

Proof: If $X = pt$ the the statement is obvious. Assume that it is proved for all f whose codomain is of length $< j$ and let X be of length j. We have $ft(fg) = ft(f)g$ and therefore $ft(fg) \in Mor(CC')$ by the inductive assumption. It remains to show that $s_{fg} \in \widetilde{C}$. We have the following diagram whose squares are canonical pull-back squares

$$\begin{array}{ccccc} X_{fg} & \longrightarrow & X_f & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p_X} \\ Z & \xrightarrow{g} & Y & \xrightarrow{ft(f)} & ft(X) \end{array}$$

which shows that $s_{fg} = g^*(s_f)$. Therefore, $s_{fg} \in Mor(CC')$ by Lemma 2.7.

Lemma 2.9 *[***2009.10.16.l5***] Under the assumptions of the proposition, let* $X \in C$ *and let* $f : Y \to ft(X)$ *be in* $Mor(CC')$, *then* $f^*(X) \in C$ *and* $q(f,X) \in Mor(CC')$.

Proof: Consider the diagram

$$\begin{array}{ccccc} f^*(X) & \xrightarrow{q(f,X)} & X & & \\ {\scriptstyle s_{q(f,X)}}\downarrow & & \downarrow{\scriptstyle s_{Id_X}} & & \\ q(f,X)^*(X) & \longrightarrow & p_X^*(X) & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ f^*(X) & \xrightarrow{q(f,X)} & X & \longrightarrow & ft(X) \\ {\scriptstyle p_{f^*(X)}}\downarrow & & \downarrow{\scriptstyle p_X} & & \\ Y & \xrightarrow{f} & ft(X) & & \end{array}$$

where the squares are canonical. By condition (6) we have $s_{Id} \in \widetilde{C}$. Therefore, by Lemma 2.7, we have $s_{q(f,X)} \in \widetilde{C}$. In particular, $q(f,X)^*(X) \in C$ and therefore $f^*(X) = ft(q(f,X)^*(X)) \in C$. The fact that $q(f,X) \in Mor(CC')$ follows from the fact that $s_{q(f,X)} \in \widetilde{C}$ and $ft(q(f,X)) = f \circ p_{f^*(X)}$ is in $Mor(CC')$ by previous lemmas.

Lemma 2.10 *[2009.10.16.l6] Under the assumptions of Lemma 2.9, the square*

$$\begin{array}{ccc} f^*(X) & \xrightarrow{q(f,X)} & X \\ {\scriptstyle p_{f^*(X)}}\downarrow & & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow{f} & ft(X) \end{array}$$

is a pull-back square in CC'.

Proof: We need to show that for a morphism $g : Z \to f^*(X)$ such that $p_{f^*(X)}g$ and $q(f, X)g$ are in $Mor(CC')$ one has $g \in Mor(CC')$. We have $ft(g) = p_{f^*(X)}g$, therefore by definition of $Mor(CC')$ it remains to check that $s_g \in \widetilde{C}$. The diagram

$$\begin{array}{ccccc} (f^*Y)_g & \longrightarrow & f^*Y & \xrightarrow{q(f,X)} & X \\ \downarrow & & \downarrow & & \downarrow \\ Z & \xrightarrow{ft(g)} & Y & \xrightarrow{f} & ft(X) \end{array}$$

shows that $s_g = s_{q(f,X)g}$ and therefore $s_g \in Mor(CC')$.

To finish the proof of the proposition it remains to show that $Ob(CC') = C$ and $\widetilde{Ob}(CC') = \widetilde{C}$. The first assertion is tautological. The second one follows immediately from the fact that for $(s : ft(X) \to X) \in \widetilde{Ob}(CC)$ one has $ft(s) = Id_{ft(X)}$ and $s_s = s$.

3 The sequent axiomatics of C-systems.

Proposition 2.5 suggests that a C-system CC can be reconstructed from the sets $B_n = B_n(CC)$ and $\widetilde{B}_{n+1} = \widetilde{B}_{n+1}(CC)$, $n \geq 0$ together with the structures on these sets which correspond to the conditions of the proposition. Let us show that it is indeed the case.

In addition to the sets B_n and $\widetilde{B}_n$ and maps $ft : B_{n+1} \to B_n$ and $\partial : \widetilde{B}_{n+1} \to B_{n+1}$ let us consider the following maps given for all $m \geq n \geq 0$:

1. $T : (B_{n+1})_{ft} \times_{ft^{m+1-n}} (B_{m+1}) \to B_{m+2}$, which sends (Y, X) such that $ft(Y) = ft^{m+1-n}(X)$ to $p_Y^*(X, m + 1 - n)$,

2. $\widetilde{T} : (B_{n+1})_{ft} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \to \widetilde{B}_{m+2}$, which sends (Y, s) such that $ft(Y) = ft^{m+1-n}\partial(s)$ to $p_Y^*(s, m + 1 - n)$,

3. $S : (\widetilde{B}_{n+1})_\partial \times_{ft^{m+1-n}} (B_{m+2}) \to B_{m+1}$, which sends (r, X) such that $\partial(r) = ft^{m+1-n}(X)$ to $r^*(X, m + 1 - n)$,

4. $\widetilde{S} : (\widetilde{B}_{n+1})_\partial \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+2}) \to \widetilde{B}_{m+1}$, which sends (r, s) such that $\partial(r) = ft^{m+1-n}\partial(s)$ to $r^*(s, m + 1 - n)$.

5. $\delta : B_{n+1} \to \widetilde{B}_{n+2}$ which sends X to the diagonal section of the projection $p_X^* X \to X$.

Note that we have:

1. for $Y \in B_{n+1}$, $X \in B_{m+1}$ such that $ft(Y) = ft^{m+1-n}(X)$ and $m \geq n \geq 0$ one has:
$$ft(T(Y,X)) = \begin{cases} T(Y, ft(X)) & \text{if } m > n \\ Y & \text{if } m = n \end{cases} \tag{3}$$

2. for $Y \in B_{n+1}$, $s \in \widetilde{B}_{m+1}$ such that $ft(Y) = ft^{m+1-n}\partial(s)$ and $m \geq n \geq 0$ one has:
$$\partial(\widetilde{T}(Y,s) = T(Y, \partial(s)) \tag{4}$$

3. for $r \in \widetilde{B}_{n+1}$, $X \in \widetilde{B}_{m+2}$ such that $\partial(r) = ft^{m+1-n}(X)$ and $m \geq n \geq 0$ one has:
$$ft(S(r,X)) = \begin{cases} S(r, ft(X)) & \text{if } m > n \\ ft(Y) & \text{if } m = n \end{cases} \tag{5}$$

4. for $r \in \widetilde{B}_{n+1}$, $s \in \widetilde{B}_{m+2}$ such that $\partial(r) = ft^{m+1-n}\partial(s)$ and $m \geq n \geq 0$ one has:
$$\partial(\widetilde{S}(r,s)) = S(r, \partial(s)) \tag{6}$$

5.
$$[\mathbf{2009.12.27.eq1}]\partial(\delta(X)) = T(X,X) \tag{7}$$

Let us denote by
$$T_j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}} (B_{m+1}) \to B_{m+1+j}$$
$$\widetilde{T}_j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \to \widetilde{B}_{m+1+j}$$
the maps which are defined inductively by
$$T_j(Y,X) = \begin{cases} X & \text{if } j = 0 \\ T(Y, T_{j-1}(ft(Y), X)) & \text{if } j > 0 \end{cases} \qquad \widetilde{T}_j(Y,s) = \begin{cases} s & \text{if } j = 0 \\ \widetilde{T}(Y, \widetilde{T}_{j-1}(ft(Y), s)) & \text{if } j > 0 \end{cases} \tag{8}$$
Note that for any $i = 0, \ldots, j$ we have
$$T_j(Y,X) = T_i(Y, T_{j-i}(ft^i(Y), X))$$
and
$$\widetilde{T}_j(Y,s) = \widetilde{T}_i(Y, \widetilde{T}_{j-i}(ft^i(Y), s))$$

Lemma 3.1 *[**Tnft**] One has*
$$T_j(Y, ft(X)) = ft(T_j(Y,X))$$

Proof: For $n = 0$ the statement is obvious. For $n > 0$ we have by induction on j
$$T_j(Y, ft(X)) = T(Y, T_{j-1}(ft(Y), ft(X))) = T(Y, ft(T_{j-1}(ft(Y), X))) =$$
$$= ft(T(Y, T_{j-1}(ft(Y), X))) = ft(T_j(Y,X)).$$

Let $f : Y \to X$ be a morphism such that $Y \in B_n$ and $X \in B_m$. Define a sequence $(s_1(f), \ldots, s_m(f))$ of elements of $\widetilde{B}_{n+1}$ inductively by the rule
$$(s_1(f), \ldots, s_m(f)) = (s_1(ft(f)), \ldots, s_{m-1}(ft(f)), s_f) = (s_{ft^{m-1}(f)}, \ldots, s_{ft(f)}, s_f)$$

where $ft(f) = p_X f$, s_f is defined by the diagram (2) and for $m = 0$ we start with the empty sequence. This construction can be illustrated by the following diagram for $f : Y \to X$ where $X \in B_4$:

$$\begin{array}{ccccccccccc}
Y & \xrightarrow{s_4(f)} & Z_{4,3} & \longrightarrow & Z_{4,2} & \longrightarrow & Z_{4,1} & \longrightarrow & T_n(Y,X) & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & Y & \xrightarrow{s_3(f)} & Z_{3,2} & \longrightarrow & Z_{3,1} & \longrightarrow & T_n(Y,ft(X)) & \longrightarrow & ft(X) \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & Y & \xrightarrow{s_2(f)} & Z_{2,1} & \longrightarrow & T_n(Y,ft^2(X)) & \longrightarrow & ft^2(X) \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & Y & \xrightarrow{s_1(f)} & T_n(Y,ft^3(X)) & \longrightarrow & ft^3(X) \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & Y & \longrightarrow & pt
\end{array} \qquad (9)$$

which is completely determined by the condition that the squares are the canonical ones and the composition of morphisms in the i-th arrow from the top is $ft^i(f)$. For the objects Z_i^j we have:

$$\begin{array}{lll}
Z_{4,1} = S(s_1(f), T_n(Y,X)) & Z_{4,2} = S(s_2(f), Z_{4,1}) & Z_{4,3} = S(s_3(f), Z_{4,2}) \\
Z_{3,1} = S(s_1(f), T_n(Y, ft(X))) & Z_{3,2} = S(s_2(f), Z_{3,1}) & \\
Z_{2,1} = S(s_1(f), T_n(Y, ft^2(X))) & &
\end{array} \qquad (10)$$

A simple inductive argument similar to the one in the proof of Lemma 2.3 show that if $f, f' : Y \to X$ are two morphisms such that $X \in B_m$ and $s_i(f) = s_i(f')$ for $i = 1, \dots, m$ then $f = f'$. Therefore, we may consider the set $Mor(CC)$ of morphisms of CC as a subset in $\amalg_{n,m \geq 0} B_n \times B_m \times \widetilde{B}_{n+1}^m$.

Let us show how to describe this subset in terms of the operations introduced above.

Lemma 3.2 *[2009.11.07.l1] An element $(Y, X, s_1, \dots, s_m)$ of $B_n \times B_m \times \widetilde{B}_{n+1}^m$ corresponds to a morphism if and only if the element $(Y, ft(X), s_1, \dots, s_{m-1})$ corresponds to a morphism and $\partial(s_m) = Z_{m,m-1}$ where $Z_{m,i}$ is defined inductively by the rule:*

$$Z_{m,0} = T_n(Y,X) \qquad Z_{m,i+1} = S(s_{i+1}, Z_{m,i})$$

Proof: Straightforward from the example considered above.

Let us show now how to identify the canonical morphisms $p_{X,i} : X \to ft^i(X)$ and in particular the identity morphisms.

Lemma 3.3 *[2009.11.10.l1] Let $X \in B_m$ and $0 \leq i \leq m$. Let $p_{X,i} : X \to ft^i(X)$ be the canonical morphism. Then one has:*

$$s_j(p_{X,i}) = \widetilde{T}_{m-j}(X, \delta_{ft^{m-j}(X)}) \qquad j = 1, \dots, m-i$$

Proof: Let us proceed by induction on $m-i$. For $i=m$ the assertion is trivial. Assume the lemma proved for $i+1$. Since $ft(p_{X,i}) = p_{X,i+1}$ we have $s_j(p_{X,i}) = s_j(p_{X,i+1})$ for $j = 1, \ldots, m-i-1$. It remains to show that

$$[\mathbf{2009.11.10.eq1}] s_{m-i}(p_{X,i}) = \widetilde{T}_i(X, \delta_{ft^i(X)}) \tag{11}$$

By definition $s_{m-i}(p_{X,i}) = s_{p_{X,i}}$ and (11) follows from the commutative diagram:

$$\begin{array}{ccccc}
X & \longrightarrow & ft^i(X) & & \\
{\scriptstyle s_p}\downarrow & & \downarrow{\scriptstyle \delta_{ft^i(X)}} & & \\
p^*_{X,i+1}(ft^i(X)) & \longrightarrow & p^*_{ft^i(X)}(ft^i(X)) & \longrightarrow & ft^i(X) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{ft^i(X)}} \\
X & \longrightarrow & ft^i(X) & \longrightarrow & ft^{i+1}(X)
\end{array}$$

where $p = p_{X,i}$.

Lemma 3.4 *[***2009.11.10.l2***] Let $(X, s) \in \widetilde{B}_{m+1}$, $Y \in B_n$ and $f : Y \to ft(X)$. Define inductively $(f, i)^*(s) \in \widetilde{B}_{n+m+1-i}$ by the rule*

$$(f, 0)^*(s) = \widetilde{T}_n(Y, s)$$

$$(f, i+1)^*(s) = \widetilde{S}(s_{i+1}(f), (f, i)^*(s))$$

Then $f^*(s) = (f, m)^*(s)$.

Proof: It follows from the diagram:

$$\begin{array}{ccccccccccc}
Y & \xrightarrow{s_m(f)} & * & \longrightarrow & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(X) \\
{\scriptstyle f^*(s)}\downarrow & & \downarrow{\scriptstyle (f,m-1)^*(s)} & & & & \downarrow{\scriptstyle (f,1)^*(s)} & & \downarrow{\scriptstyle (f,0)^*(s)} & & \downarrow{\scriptstyle s} \\
* & \longrightarrow & * & \longrightarrow & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & X \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
Y & \xrightarrow{s_m(f)} & * & \longrightarrow & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(X) \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
 & & Y & \xrightarrow{s_{m-1}(f)} & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft^2(X) \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & \ldots & & \ldots & & \ldots \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & Y & \xrightarrow{s_1(f)} & * & \longrightarrow & ft^{m-1}(X) \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & Y & \longrightarrow & pt
\end{array}$$