A simple type system with two identity types

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Started February 23, 2013. Work in progress.

1 Introduction

We call this system and its further extensions HTS for "homotopy type system". It is an extension of the Martin-Lof type system with some additional constructs which reflect the structures which exist in the target of the canonical univalent model of the Martin-Lof system.

2 Core inference rules

Structural rules

1. for each $i \in \mathbf{N}$

$$\frac{\Gamma, x : T, \Gamma' \rhd \quad \mathit{where}\ l(\Gamma) = i}{\Gamma, x : T, \Gamma' \vdash x : T}$$

2.

$$\frac{\Gamma, x : T \rhd}{\Gamma \vdash T \stackrel{d}{=} T}$$

3.

$$\frac{\Gamma \vdash T_1 \stackrel{d}{=} T_2}{\Gamma \vdash T_2 \stackrel{d}{=} T_1}$$

4.

$$\frac{\Gamma \vdash T_1 \stackrel{d}{=} T_2 \quad \Gamma \vdash T_2 \stackrel{d}{=} T_3}{\Gamma \vdash T_1 \stackrel{d}{=} T_3}$$

5. (*)

$$\frac{\Gamma \vdash o : T}{\Gamma \vdash o \stackrel{d}{=} o : T}$$

6. (*)

$$\frac{\Gamma \vdash o_1 \stackrel{d}{=} o_2 : T}{\Gamma \vdash o_2 \stackrel{d}{=} o_1 : T}$$

7. (*)

$$\frac{\Gamma \vdash o_1 \stackrel{d}{=} o_2 : T \quad \Gamma \vdash o_2 \stackrel{d}{=} o_3 : T}{\Gamma \vdash o_1 \stackrel{d}{=} o_3 : T}$$

8.

$$\frac{\Gamma \vdash o : T \quad \Gamma \vdash T \stackrel{d}{=} T'}{\Gamma \vdash o : T'}$$

9. (*)

$$\frac{\Gamma \vdash o \stackrel{d}{=} o' : T \quad \Gamma \vdash T \stackrel{d}{=} T'}{\Gamma \vdash o \stackrel{d}{=} o' : T'}$$

10.

$$\frac{\Gamma \vdash T \quad Fib}{\Gamma, x : T \rhd}$$

Dependent products

1.

$$\frac{\Gamma, x : T_1, y : T_2 \rhd}{\Gamma, y : [\prod](T_1, x.T_2) \rhd}$$

2.

$$\frac{\Gamma \vdash T_1 \quad Fib \quad \Gamma, x : T_1 \vdash T_2 \quad Fib}{\Gamma \vdash [\prod](T_1, x.T_2) \quad Fib}$$

3.

$$\frac{\Gamma \vdash T_1 \stackrel{d}{=} T_1' \quad \Gamma, x : T_1, y : T_2 \rhd}{\Gamma \vdash [\prod](T_1, x.T_2) \stackrel{d}{=} [\prod](T_1', x.T_2)}$$

4.

$$\frac{\Gamma, x : T_1 \vdash T_2 \stackrel{d}{=} T_2'}{\Gamma \vdash [\prod](T_1, x.T_2) \stackrel{d}{=} [\prod](T_1, x.T_2')}$$

5.

$$\frac{\Gamma, x : T_1 \vdash o : T_2}{\Gamma \vdash [\lambda](T_1, x.o) : [\prod](T_1, x.T_2)}$$

6.

$$\frac{\Gamma \vdash T_1 \stackrel{d}{=} T_1' \quad \Gamma, x : T_1 \vdash o : T_2}{\Gamma \vdash [\lambda](T_1, x.o) \stackrel{d}{=} [\lambda](T_1', x.o) : [\prod](T_1, x.T_2)}$$

7.

$$\frac{\Gamma, x : T_1 \vdash o \stackrel{d}{=} o' : T_2}{\Gamma \vdash [\lambda](T_1, x.o) \stackrel{d}{=} [\lambda](T_1, x.o') : [\prod](T_1, x.T_2)}$$

8.

$$\frac{\Gamma \vdash f : [\prod](T_1, x.T_2) \quad \Gamma \vdash o : T_1}{\Gamma \vdash [ev](f, o) : T_2[o/x]}$$

9. (*)

$$\frac{\Gamma \vdash f \stackrel{d}{=} f' : [\prod](T_1, x.T_2) \quad \Gamma \vdash o : T_1}{\Gamma \vdash [ev](f, o) \stackrel{d}{=} [ev](f', o) : T_2[o/x]}$$

10. (*)

$$\frac{\Gamma \vdash f : [\prod](T_1, x.T_2) \quad \Gamma \vdash o = o' : T_1}{\Gamma \vdash [ev](f, o) \stackrel{d}{=} [ev](f, o') : T_2[o/x]}$$

11.

$$\frac{\Gamma \vdash o_1 : T_1 \quad \Gamma, x : T_1 \vdash o_2 : T_2}{\Gamma \vdash [ev]([\lambda](T_1, x.o_2), o_1) \stackrel{d}{=} o_2[o_1/x] : T_2[o_1/x]}$$

12.

$$\frac{\Gamma \vdash f : [\prod](T_1, x.T_2)}{\Gamma \vdash [\lambda](T_1, x.[ev](f, x, T_2[y/x])) \stackrel{d}{=} f : [\prod](T_1, x.T_2)}$$

Exact equality types

1.

$$\frac{\Gamma \vdash o_1 : X \quad \Gamma \vdash o_2 : X}{\Gamma, x : [Id](o_1, o_2) \rhd}$$

2.

$$\frac{\Gamma \vdash o_1 \stackrel{d}{=} o_1' : X \quad \Gamma \vdash o_2 : X}{\Gamma \vdash [Id](o_1, o_2) \stackrel{d}{=} [Id](o_1', o_2)}$$

3.

$$\frac{\Gamma \vdash o_1 : X \quad \Gamma \vdash o_2 \stackrel{d}{=} o_2' : X}{\Gamma \vdash [Id](o_1, o_2) \stackrel{d}{=} [Id](o_1, o_2')}$$

4.

$$\frac{\Gamma \vdash o : X}{\Gamma \vdash [refl](o) : [Id](o, o)}$$

5. (*)

$$\frac{\Gamma \vdash o \stackrel{d}{=} o' : X}{\Gamma \vdash [refl](o) \stackrel{d}{=} [refl](o') : [Id](o, o)}$$

6.

$$\frac{\begin{array}{ll} \Gamma, x : X \rhd & \Gamma \vdash o_1 : X \\ \Gamma, x : X, x' : X, e : [Id](x, x'), y : P \rhd & \Gamma \vdash o_2 : X \\ \Gamma, x0 : X \vdash s0 : P[x0/x, x0/x', [refl](x0)/e] & \Gamma \vdash eo : [Id](o_1, o_2) \end{array}}{\Gamma \vdash [J](X, x.x'.e.P, x0.s0, o_1, o_2, eo) : P}$$

7. (*) Behavior of $[J]$ for $X \stackrel{d}{=} X'$, $P \stackrel{d}{=} P'$, $o \stackrel{d}{=} o'$, $o_1 \stackrel{d}{=} o_1'$, $o_2 \stackrel{d}{=} o_2'$ and $eo \stackrel{d}{=} eo'$.

8. $[J](X, x.x'.e.P, x0.s0, o, o, [refl](o)) \stackrel{d}{=} s0[o/x0]$

9.
$$\frac{\Gamma \vdash o_1 : X \quad \Gamma \vdash o_2 : X \quad \Gamma \vdash o : [Id](o_1, o_2)}{\Gamma \vdash o_1 \stackrel{d}{=} o_2 : X}$$

Path equality types

1.
$$\frac{\Gamma \vdash X \quad Fib \quad \Gamma \vdash o_1 : X \quad \Gamma \vdash o_2 : X}{\Gamma \vdash [Paths](o_1, o_2) \quad Fib}$$

2.
$$\frac{\Gamma \vdash X \quad Fib \quad \Gamma \vdash o_1 = o_1' : X \quad \Gamma \vdash o_2 : X}{\Gamma \vdash [Paths](o_1, o_2) \stackrel{d}{=} [Paths](o_1', o_2)}$$

3.
$$\frac{\Gamma \vdash X \quad Fib \quad \Gamma \vdash o_1 : X \quad \Gamma \vdash o_2 \stackrel{d}{=} o_2' : X}{\Gamma \vdash [Paths](o_1, o_2) \stackrel{d}{=} [Paths](o_1, o_2')}$$

4.
$$\frac{\Gamma \vdash X \quad Fib \quad \Gamma \vdash o : X}{\Gamma \vdash [idpath](o) : [Paths](o, o)}$$

5. (*)
$$\frac{\Gamma \vdash X \quad Fib \quad \Gamma \vdash o \stackrel{d}{=} o' : X}{\Gamma \vdash [idpath](o) \stackrel{d}{=} [idpath](o') : [Paths](o, o)}$$

6.
$$\frac{\begin{array}{ll} \Gamma \vdash X \quad Fib & \Gamma \vdash o_1 : X \\ \Gamma, x : X, x' : X, e : [Paths](x, x') \vdash P \quad Fib & \Gamma \vdash o_2 : X \\ \Gamma, x0 : X \vdash s0 : P[x0/x, x0/x', [idpath](x0)/e] & \Gamma \vdash eo : [Paths](o_1, o_2) \end{array}}{\Gamma \vdash [J_F](X, x.x'.e.P, x0.s0, o_1, o_2, eo) : P}$$

7. (*) Behavior of $[J_F]$ for $X \stackrel{d}{=} X'$, $P \stackrel{d}{=} P'$, $o \stackrel{d}{=} o'$, $o_1 \stackrel{d}{=} o_1'$, $o_2 \stackrel{d}{=} o_2'$ and $eo \stackrel{d}{=} eo'$.

8. $[J_F](X, x.x'.e.P, x0.s0, o, o, [idpath](o)) \stackrel{d}{=} s0[o/x0]$

Universe of all types

1.
$$\frac{\Gamma \rhd}{\Gamma \vdash \mathcal{U} \quad Fib}$$

2.
$$\frac{\Gamma \vdash o : \mathcal{U}}{\Gamma, x : [El](o) \rhd}$$

3.

$$\frac{\Gamma \vdash o \overset{d}{=} o' : \mathcal{U}}{\Gamma \vdash [El](o) \overset{d}{=} [El](o')}$$

4. (?)

$$\frac{\Gamma \vdash o : \mathcal{U} \quad \Gamma \vdash o' : \mathcal{U} \quad \Gamma \vdash [El](o) \overset{d}{=} [El](o')}{\Gamma \vdash o \overset{d}{=} o' : \mathcal{U}}$$

5.

$$\frac{\Gamma \vdash o_1 : \mathcal{U} \quad \Gamma, x : [El](o_1) \vdash o_2 : \mathcal{U}}{\Gamma \vdash [forall](o_1, x.o_2) : \mathcal{U}}$$

6.

$$\frac{\Gamma \vdash o_1 : \mathcal{U} \quad \Gamma, x : [El](o_1) \vdash o_2 : \mathcal{U}}{\Gamma \vdash [El][forall](o_1, x.o_2) \overset{d}{=} [\prod]([El](o_1), x.[El](o_2))}$$

7.

$$\frac{\Gamma \vdash o : \mathcal{U} \quad \Gamma \vdash o_1 : [El](o) \quad \Gamma \vdash o_2 : [El](o)}{\Gamma \vdash [id](o_1, o_2) : \mathcal{U}}$$

8.

$$\frac{\Gamma \vdash o : \mathcal{U} \quad \Gamma \vdash o_1 : [El](o) \quad \Gamma \vdash o_2 : [El](o)}{\Gamma \vdash [El][id](o_1, o_2) \overset{d}{=} [Id](o_1, o_2)}$$

Universe of fibrant types

1.

$$\frac{\Gamma \rhd}{\Gamma \vdash \mathcal{U}_F \quad Fib}$$

2.

$$\frac{\Gamma \vdash o : \mathcal{U}_F}{\Gamma \vdash [El_F](o) \quad Fib}$$

3.

$$\frac{\Gamma \vdash o \overset{d}{=} o' : \mathcal{U}_F}{\Gamma \vdash [El_F](o) \overset{d}{=} [El_F](o')}$$

4. (?)

$$\frac{\Gamma \vdash o : \mathcal{U}_F \quad \Gamma \vdash o' : \mathcal{U}_F \quad \Gamma \vdash [El_F](o) \overset{d}{=} [El_F](o')}{\Gamma \vdash o \overset{d}{=} o' : \mathcal{U}_F}$$

5.

$$\frac{\Gamma \vdash o_1 : \mathcal{U}_F \quad \Gamma, x : [El_F](o_1) \vdash o_2 : \mathcal{U}_F}{\Gamma \vdash [forall_F](o_1, x.o_2) : \mathcal{U}_F}$$

6.
$$\frac{\Gamma \vdash o_1 : \mathcal{U}_F \quad \Gamma, x : [El_F](o_1) \vdash o_2 : \mathcal{U}_F}{\Gamma \vdash [El_F][forall_F](o_1, x.o_2) \stackrel{d}{=} [\prod]([El_F](o_1), x.[El_F](o_2))}$$

7.
$$\frac{\Gamma \vdash o : \mathcal{U}_F \quad \Gamma \vdash o_1 : [El_F](o) \quad \Gamma \vdash o_2 : [El_F](o)}{\Gamma \vdash [paths](o_1, o_2) : \mathcal{U}_F}$$

8.
$$\frac{\Gamma \vdash o : \mathcal{U}_F \quad \Gamma \vdash o_1 : [El_F](o) \quad \Gamma \vdash o_2 : [El_F](o)}{\Gamma \vdash [El_F][paths](o_1, o_2) \stackrel{d}{=} [Paths](o_1, o_2)}$$

Universe inclusion

1.
$$\frac{\Gamma \rhd}{\Gamma \vdash [j] : [\prod](\mathcal{U}_F, x.\mathcal{U})}$$

2.
$$\frac{\Gamma \vdash o : \mathcal{U}_F}{\Gamma \vdash [El_F](o) \stackrel{d}{=} [El][ev]([j], o)}$$

Resizing rules

1.
$$\frac{\Gamma \vdash T \quad Fib \quad \Gamma \vdash p : Weq(T, [El_F](t'))}{\Gamma \vdash [rr0](T, t', p) : \mathcal{U}_F}$$

2.
$$\frac{\Gamma \vdash T \quad Fib \quad \Gamma \vdash p : Weq(T, [El_F](t'))}{\Gamma \vdash [El_F][rr0](T, t', p) \stackrel{d}{=} T}$$

3.
$$\frac{\Gamma \vdash T \quad Fib \quad \Gamma \vdash p : Ishprop(T)}{\Gamma \vdash [rr1](T, p) : \mathcal{U}_F}$$

4.
$$\frac{\Gamma \vdash T \quad Fib \quad \Gamma \vdash p : Ishprop(T)}{\Gamma \vdash [El_F][rr1](T, p) \stackrel{d}{=} T}$$

The system described above is the smallest system where some interesting mathematics can be developed. One can also consider the following additional constructions:

The unit type and dependent sums

The inference rules are the usual ones. The η-rules saying that $x \stackrel{d}{=} pair\,(pr1\,x)\,(pr2\,x)$ for x in a dependent sum and $x \stackrel{d}{=} x'$ for x, x' in the unit type are derivable.

The unit type is fibrant and the dependent sum where both arguments are fibrant is fibrant.

There are "unit" as object of $\mathcal{U}_F$ and dependent sum operation on families of objects of $\mathcal{U}$ (reps. $\mathcal{U}_F$) parametrized by types given by object of $\mathcal{U}$ (reps. $\mathcal{U}_F$).

The empty type and disjoint union

The inference rules are the usual ones. The η-rules are derivable.

The empty type is fibrant and the disjoint union of two fibrant types is fibrant.

There are "empty" as object of $\mathcal{U}_F$ and obvious disjoint union operation on objects of $\mathcal{U}$ (reps. $\mathcal{U}_F$).

Recursive types

As far as I understand the material of the last section of the "Notes on Type systems" can be made precise in this type systems. Namely, there is a way to express any strictly positive inductive definition as a combination of the constructions already discussed with *parametrized W-types*. The later correspond to the inductive definitions of Coq of the form

Inductive IC(A:Type)(a:A)(B:A->Type)(D:forall x:A, (B x -> Type))(q:forall x:A, forall y:B x, forall z: D x y, A):= c: forall b:B a, forall f : (forall d: D a b, IC A (q a b d) B D q), IC A a B D q .

and introduced by the inference rule

$$\frac{\Gamma \vdash a : A \quad \Gamma, x : A, y : B, z : D \vdash q : A}{\Gamma, w : [IC](A, a, x.B, x.y.D, x.y.z.q) \rhd}$$

for the type and

$$\frac{\Gamma \vdash a : A \quad \Gamma, x : A, y : B, z : D \vdash q : A \quad \Gamma \vdash b : B[a/x] \quad \Gamma, z : D[a/x, b/y] \vdash f : ICqa}{\Gamma \vdash [c](A, a, x.B, x.y.D, x.y.z.q, b, z.f) : [IC](A, a, x.B, x.y.D, x.y.z.q)}$$

for the constructor. The inference rule for the eliminator can be more or less copied from the type of IC_{rect} printed by Coq. Similarly one can write down the computation rule (definitional equality associated with the eliminator) from the general such rules used in Coq.

The type IC is fibrant if all the arguments are fibrant i.e. if one has

$$\Gamma \vdash A \quad Fib \quad \Gamma, x : A \vdash B \quad Fib \quad \Gamma, x : A, b : B \vdash D \quad Fib$$

There are obvious versions of IC on the level of the universes $\mathcal{U}$ and $\mathcal{U}_F$.

In particular, in this system the datatypes such as the types of natural numbers or binary trees can be expressed through the W-types obtaining types which have both the computation rules obtained in the strictly positive formalism as well as recursive η-rules which allow to prove definitional equality by induction.

It will be very interesting to extend to this system the formalism of more general inductive definitions (such as inductive-inductive and inductive-recursive).

Multiple universes

The system of multiple universes connected by explicit inclusions as in the specification of TS0 is easy to add to HTS. There will be two universes for each universe - one for all types and one for fibrant types.

Univalence axiom

Univalence axiom can be formulated in the usual way. It should use path equalities and $\mathcal{U}_F$.

Elementary properties of the system

Lemma 2.1 *[**strars**] The inference rules which are marked with (*) follow from the other inference rules.*

Proof: Straightforward by transforming definitional equality into exact equality, using J and transforming the exact equality back to the definitional one using the "reflexion rule".

Lemma 2.2 *[**uip**] In any type system with exact equality $[Id], [refl], [J]$ satisfying inference rules given above one has:*

$$\frac{\Gamma \vdash e_1 : [Id](o_1, o_2) \quad \Gamma \vdash e_2 : [Id](o_1, o_2)}{\Gamma \vdash e_1 \stackrel{d}{=} e_2 : [Id](o_1, o_2)}$$

Proof: It is sufficient to apply J with P defined by

$$\Gamma, x_1 : X, x_2 : X, e : [Id](x_1, x_2), \phi : [Id](e, [refl](x_1)) \rhd$$

which is derivable since

$$\Gamma, x_1 : X, x_2 : X, e : [Id](x_1, x_2) \vdash [refl](x_1) : [Id](x_1, x_2)$$

is derivable using the "reflection rule".

Lemma 2.3 *[**funext**] In any type system with the dependent products and exact equality satisfying the inference rules as above the functional extensionality holds for the exact equality.*

Proof: Straightforward from the rule ensuring that definitional equality propagates through λ.