Univalent models of type systems and resizing axioms

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1 Introduction

We consider the simplest type system where we can formulate the main homotopy notions and some of the resizing axioms. For that we need the standard dependent products, dependent sums, Martin-Lof identity types and at least two universes $U0$ and $U1$ embedded into each other. We will build such a type system in several steps. The first type system we consider has only $U0$, $U1$ and dependent products and does not have any reduction rules (other than α-conversion i.e. the renaming of bound variables). It will be denoted TS_0.

2 Type system TS_0 and its models

1 Contexts and judgements of TS_0

Below are the rules for the formation of contexts and judgements in TS_0. We use bold font to denote names of variables and the usual font to denote names of expressions. As usual Γ, Δ etc. denote an arbitrary context or a segment of a context and it is understood that any expression has as free variables only the variables which have been declared and assigned types strictly prior to the use of this expression. We use commas to separate variable declarations in a context and dots in the expressions for dependent sums and products. The notation $\mathbf{x} \notin \Gamma$ means that the variable name $\mathbf{x}$ has not been declared in Γ.

Empty context, diagonals

$$\frac{}{\vdash} \qquad \frac{\Gamma, \mathbf{x} : T, \Gamma' \vdash}{\Gamma, \mathbf{x} : T, \Gamma' \vdash \mathbf{x} : T}$$

Universe structure

$$\frac{\Gamma \vdash}{\Gamma, \mathbf{x} : U0 \vdash} \qquad \frac{\Gamma \vdash T : U0}{\Gamma, \mathbf{x} : T \vdash} \qquad (\mathbf{x} \notin \Gamma)$$

$$\frac{\Gamma \vdash}{\Gamma, \mathbf{x} : U1 \vdash} \qquad \frac{\Gamma \vdash T : U1}{\Gamma, \mathbf{x} : T \vdash} \qquad (\mathbf{x} \notin \Gamma)$$

$$\frac{\Gamma \vdash T : U0}{\Gamma \vdash T : U1} \qquad \frac{\Gamma \vdash}{\Gamma \vdash U0 : U1}$$

Dependent Products

$$\frac{\Gamma, \mathbf{x} : R, \mathbf{y} : S \vdash}{\Gamma, \mathbf{z} : \prod \mathbf{x} : R. S \vdash} \qquad (\mathbf{z} \notin \Gamma)$$

$$\frac{\Gamma, \mathbf{x} : R \vdash s : S}{\Gamma \vdash \lambda \mathbf{x} : R.s : \prod \mathbf{x} : R.S} \qquad \frac{\Gamma \vdash f : \prod \mathbf{x} : R.S \quad \Gamma \vdash r : R}{\Gamma \vdash f\ r : S[\mathbf{x}/r]}$$

Dependent products and universes

$$\frac{\Gamma \vdash R : U0 \quad \Gamma, \mathbf{x} : R \vdash S : U0}{\Gamma \vdash \prod \mathbf{x} : R. S : U0}$$

$$\frac{\Gamma \vdash R : U1 \quad \Gamma, \mathbf{x} : R \vdash S : U1}{\Gamma \vdash \prod \mathbf{x} : R. S : U1}$$

One could re-write our specification given so far using three universes $U0$, $U1$, $U2$ and only using judgements but not "free" contexts. Then the system described so far will look very similar to the PTS of Barndregt (see [1][p. 214]) with the "signature"

$$\mathcal{S} = (U0, U1, U2)$$

$$\mathcal{A} = (U0 : U1, U1 : U2)$$

$$\mathcal{R} = (U0 : U0, U1 : U1, U2 : U2)$$

and without β-reduction. However, we also have inclusions $U0 \subset U1 \subset U2$ which Barendregt apparently does not consider. We prefer the specification given above since it emphasizes the fact that the third universe $U2$ can not be a part of any constructions.

2 Proof that TS_0 is a type system

References

[1] H. P. Barendregt. Lambda calculi with types. In *Handbook of logic in computer science, Vol. 2*, volume 2 of *Handb. Log. Comput. Sci.*, pages 117–309. Oxford Univ. Press, New York, 1992.