Notes on type systems

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1 Contextual categories

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Contextual categories. From this point on it will be important for us to distinguish two notions of a category. What is understood by a category by most practicing mathematicians i.e. a category up to an equivalence, will be called, when an explicit distinction is needed, a category of level 1. A category as a combinatorial object i.e. a category up to an isomorphism will be called a set-level category. A set-level category C is a pair of sets $Mor(C)$ and $Ob(C)$ with an additional structure. The structure is given by four maps

$$\partial_0, \partial_1 : Mor(C) \to ob(C)$$

$$Id : ob(C) \to Mor(CC)$$

and the composition map

$$\circ : Mor(CC)_{\partial_1} \times_{\partial_0} Mor(CC) \to Mor(CC)$$

which satisfy the well known conditions.

A contextual category is a set-level category with the following additional structure ([**?**, Def. 1.2, p.47]):

1. a function $lngth : ob(CC) \to \mathbf{N}$ such that there is a unique object pt with $lngth(pt) = 0$ and this object is a final object,
2. a map $ft : Ob(C) \to Ob(C)$ such that $lngth(ft(X)) = max(lngth(X) - 1, 0)$,
3. for each $X \in Ob(C)$ a morphism $p_X : X \to ft(X)$,
4. for each $X \in Ob(C) \backslash \{pt\}$ and each morphism $f : Y \to ft(X)$ a pull-back square of the form

[2009.10.14.eq1]
$$\begin{array}{ccc} f^*X & \xrightarrow{q(f,X)} & X \\ {\scriptstyle p_{f^*X}}\downarrow & & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow{f} & ft(X) \end{array} \qquad (1)$$

such that the squares (1) satisfy the obvious conditions for $f = Id_{ft(X)}$ and $f = f_1 \circ f_2$. (Note that in the squares (1) one has $ft(f^*X) = Y$ since the codomain of the morphism p_{f^*X} is necessarily $ft(f^*X)$).

Note that this structure is given by maps of the form

$$lngth(ob(C)) \to \mathbf{N}$$

$$ft : ob(C) \to ob(C)$$

$$p_{\_} : ob(C) \to Mor(C)$$

and

$$(^*, q) : Mor(C)_{\partial_1} \times_{ft} Ob(C) \to Ob(C) \times Mor(C)$$

This view of the contextual structure will be important below when we consider quotients of contextual categories by equivalence relations.

Contextual subcategories. A subcategory CC' of a contextual category CC is called contextual subcategory if it is closed, in the obvious sense under the operations which define the contextual structure on CC and such that the canonical squares which belong to CC' are pull-back squares in CC'. A contextual subcategory is called non-triviial if it contains at least one other element than pt. A contextual subcategory of a contextual category is itself a contextual category with respect to the induced structure. The following elementary result plays a key role in many constructions of type theory:

Proposition 1.1 *[***2009.10.15.prop1***] Let CC be a contextual category. Then for any family CC_α of contextual subcategories of CC, the intersection $CC' = \cap_\alpha CC_\alpha$ is a contextual sub-category.*

Proof: The only condition to check is that a canonical square which belongs to CC' is a pull-back square in CC'. This follows from the definition of pull-back squares and the fact that fiber products of sets commute with intersections of sets.

Corollary 1.2 *[2009.10.15.cor1] Let CC be a contextual category, C_0 a set of objects of CC and C_1 a set of morphisms of CC. Then there exists the smallest contextual subcategory $[C_1, C_0]$ which contains C_0 and C_1. It is called the contextual subcategory generated by C_0 and C_1.*

Let CC be a contextual category. Let $ob(CC)$ be as usually the set of objects of CC and let $\widetilde{ob}(CC)$ be the set of pairs of the form (X, s) where $X \in ob(CC)$ and $s : ft(X) \to X$ is a section of the canonical morphism $p_X : X \to ft(X)$.

Lemma 1.3 *[2009.10.15.l1] Let CC be a contextual category and CC', CC'' be two contextual subcategories such that $ob(CC') = ob(CC'')$ (as subsets of $ob(CC)$) and $\widetilde{ob}(CC') = \widetilde{ob}(CC'')$ (as subsets of $\widetilde{ob}(CC)$). Then $CC' = CC''$.*

Proof: For any contextual category we have a canonical decomposition $ob(CC) = \amalg_n ob_n(CC)$ such that $ob_0(CC) = \{pt\}$ is the distinguished final object and for $\Gamma \in ob_n$ and $n > 0$, $ft(\Gamma) \in ob_{n-1}$.

Let $f : Y \to X$ be a morphism in CC'. We want to show that it belongs to CC''. Proceed by induction on m where $X \in ob_m$. For $m = 0$ the assertion is obvious. Suppose that $m > 0$. Since CC is a contextual category we have a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{s_f} & (p_X f)^* X & \xrightarrow{q(p_X f, X)} & X \\ \downarrow & & \downarrow p' & & \downarrow p \\ Y & \xrightarrow{=} & Y & \xrightarrow{p_X f} & ft(X) \end{array} \qquad [2009.11.07.oldeq1] \qquad (2)$$

such that $f = q(p_X f, X)\, s_f$. Since the right hand side square is a canonical one, $((p_X f)^*\Gamma', s_f) \in \widetilde{ob}(CC)$ and $ft(X) \in ob_{m-1}(CC)$, the inductive assumption implies that $f \in CC''$.

Remark 1.4 In Lemma 1.3, it is sufficient to assume that $\widetilde{ob}(CC') = \widetilde{ob}(CC'')$. The condition $ob(CC') = ob(CC'')$ is then also satisfied. Indeed, let $X \in ob(CC')$. Then $p_X^* X$ is the product $X \times X$ in CC. Consider the diagonal section $\Delta_X : X \to p_X^* X$ of $p_{p_X^*(X)}$. Since CC' is assumed to be a contextual subcategory we conclude that $\Delta_X \in \widetilde{ob}(CC') = \widetilde{ob}(CC'')$ and therefore $X \in ob(CC'')$. It is however more convenient to think of contextual subcategories in terms of both subsets ob and $\widetilde{ob}$.

Let CC be a contextual category. Let us say that a pair of subsets $C \subset ob(CC)$, $\widetilde{C} \subset \widetilde{ob}(CC)$ is saturated if there exists a contextual subcategory CC' such that $C = ob(CC')$ and $\widetilde{C} = \widetilde{ob}(CC')$. By Lemma 1.3 we have a bijection between contextual subcategories of CC and saturated pairs $(C, \widetilde{C})$.

Let us introduce the following notations. Let $X \in ob(CC)$ and $i \geq 0$. Denote by $p_{X,i}$ the composition of the canonical projections $X \to ft(X) \to \ldots \to ft^i(X)$ such that $p_{X,0} = Id_X$ and $p_{X,1} = p_X$. For $f : Y \to ft^i(X)$ denote by $q(f, X, i) : f^*(X, i) \to X$ the morphism defined inductively by the rule

$$f^*(X, 0) = Y \qquad q(f, X, 0) = f,$$

$$f^*(X, i+1) = q(f, ft(X), i)^*(X) \qquad q(q(f, ft(X), i), X).$$

In other words, $q(f, X, i)$ is the canonical pull-back of the morphism $f : Y \to ft^i(X)$ with respect to the sequence of canonical projections $X \to ft(X) \to \ldots \to ft^i(X)$.

Let $i \geq 1$, $f : Y \to ft^i(X)$ be a morphism and $s : ft(X) \to X$ an element of $\widetilde{ob}(CC)$. Denote by $f^*(s,i)$ the element of $\widetilde{ob}(CC)$ of the form $f^*(ft(X), i-1) \to f^*(X,i)$ which is the pull-back of s with respect to $q(f, ft(X), i-1)$.

Proposition 1.5 *[**2009.10.15.prop2**] A pair $(C, \widetilde{C})$ is saturated if and only if it satisfies the following conditions:*

1. *$pt \in C$,*
2. *if $X \in C$ then $ft(X) \in C$,*
3. *if $(s : ft(X) \to X) \in \widetilde{C}$ then $X \in C$,*
4. *if $(s : ft(X) \to X) \in \widetilde{C}$, $X' \in C$, $i \geq 1$ and $ft^i(X) = ft(X')$ then $q(p_{X'}, ft(X), i-1)^*(s) \in \widetilde{C}$,*
5. *if $(s_1 : ft(X) \to X) \in \widetilde{C}$, $i \geq 1$ and $(s_2 : ft^{i+1}(X) \to ft^i(X)) \in \tilde{C}$ then $q(s_2, ft(X), i-1)^*(s_1) \in \widetilde{C}$,*
6. *if $X \in C$ then the diagonal $s_{id_X} : X \to (p_X)^*(X)$ is in $\widetilde{C}$.*

Conditions (4) and (5) are illustrated by the following diagrams:

$$\begin{array}{ccc}
p^*_{X'}(ft(X), i-1) & \xrightarrow{q(p_{X'}, ft(X), i-1)} & ft(X) \\
\downarrow{\scriptstyle q(p_{X'}, ft(X), i-1)^*(s)} & & \downarrow{\scriptstyle s} \\
p^*_{X'}(X, i) & \xrightarrow{q(p_{X'}, X, i)} & X \\
\downarrow & & \downarrow{\scriptstyle p_X} \\
p^*_{X'}(ft(X), i-1) & \xrightarrow{q(p_{X'}, ft(X), i-1)} & ft(X) \\
\downarrow & & \downarrow \\
\dots & & \dots \\
\downarrow & & \downarrow \\
X' & \xrightarrow{p_{X'}} & ft^i(X)
\end{array}
\qquad
\begin{array}{ccc}
s_2^*(ft(X), i-1) & \xrightarrow{q(s_2, ft(X), i-1)} & ft(X) \\
\downarrow{\scriptstyle q(s_2, ft(X), i-1)^*(s_1)} & & \downarrow{\scriptstyle s_1} \\
s_2^*(X, i) & \xrightarrow{q(s_2, X, i)} & X \\
\downarrow & & \downarrow{\scriptstyle p_X} \\
s_2^*(ft(X), i-1) & \xrightarrow{q(s_2, ft(X), i-1)} & ft(X) \\
\downarrow & & \downarrow \\
\dots & & \dots \\
\downarrow & & \downarrow \\
ft^{i+1}(X) & \xrightarrow{s_2} & ft^i(X)
\end{array}$$

Proof: The "only if" part of the proposition is straightforward. Let us prove that for any $(C, \widetilde{C})$ satisfying the conditions of the proposition there exists a contextual subcategory CC' of CC such that $C = ob(CC')$ and $\widetilde{C} = \widetilde{ob}(CC')$.

For a morphism $f : Y \to X$ let $ft(f) = p_X f : Y \to ft(X)$. Any morphism $f : Y \to X$ in CC has a canonical representation of the form $Y \stackrel{s_f}{\to} X_f \stackrel{q_f}{\to} X$ where $X_f = ft(f)^*(X)$, $q_f = q(ft(f), X)$ and $s_f : Y \to X_f$ is the section of the canonical projection $X_f \to Y$ corresponding to f.

Define a candidate subcategory CC' setting $ob(CC') = C$ and defining the set $Mor(CC')$ of morphisms of CC' inductively by the conditions:

1. $Y \to pt$ is in $Mor(CC')$ if and only if $Y \in C$,
2. $f : Y \to X$ is in $Mor(CC')$ if and only if $X \in ob(C)$, $ft(f) \in Mor(CC')$ and $s_f \in \widetilde{C}$.

(note that the for $(f : Y \to X) \in Mor(CC')$ one has $Y \in C$ since $s_f : Y \to X_f$).

Let us show that if the condition of the proposition are satisfied then $(ob(CC'), Mor(CC'))$ form a contextual subcategory of CC.

The subset $ob(CC')$ contains pt and is closed under ft map by the first two conditions. The following lemma shows that $Mor(CC')$ contains identities and the compositions of canonical projections.

Lemma 1.6 *[**2009.10.16.l1**] Under the assumptions of the proposition, if $X \in C$ and $i \geq 0$ then $p_{X,i} : X \to ft^i(X)$ is in $Mor(CC')$.*

Proof: By definition of contextual categories there exists n such that $ft^n(X) = pt$. Then $p_{X,n} \in Mor(CC')$ by the first constructor of $Mor(CC')$. By induction it remains to show that if $X \in C$ and $p_{X,i} \in Mor(CC')$ then $p_{X,i-1} \in Mor(CC')$. We have $ft(p_{X,i-1}) = p_{X,i}$ and $s_{p_{X,i-1}}$ is the pull-back of the diagonal $ft^{i-1}(X) \to (p_{ft^{i-1}(X)})^*(ft^{i-1}(X))$ with respect to the canonical morphism $X \to ft^{i-1}(X)$. The diagonal is in $\widetilde{C}$ by condition (6) and therefore $s_{p_{X,i-1}}$ is in $\widetilde{C}$ by repeated application of condition (4).

Lemma 1.7 *[**2009.10.16.l3**] Under the assumptions of the proposition, let $X \in C$, $(s : ft(X) \to X) \in \widetilde{C}$, $i \geq 0$, and $(f : Y \to ft^i(X)) \in Mor(CC')$. Then $q(f, ft(X), i-1)^*(s) : ft(f^*(X, i)) \to f^*(X, i)$ is in $Mor(CC')$.*

Proof: Suppose first that $ft^i(X) = pt$. Then $f = p_{Y,n}$ for some n and the statement of the lemma follows from repeated application of condition (4). Suppose that the lemma is proved for all morphisms to objects of length $j-1$ and let the length of $ft^i(X)$ be j. Consider the canonical decomposition $f = q_f s_f$. The morphism q_f is the canonical pull-back of $ft(f)$ and therefore the pull-back of s relative to q_f coincides with its pull-back relative to $ft(f)$ which is $\widetilde{C}$ by the inductive assumption. The pull-back of an element of $\widetilde{C}$ with respect to s_f is in $\widetilde{C}$ by condition (5).

Lemma 1.8 *[**2009.10.16.l4**] Under the assumptions of the proposition, let $g : Z \to Y$ and $f : Y \to X$ be in $Mor(CC')$. then $fg \in Mor(CC')$.*

Proof: If $X = pt$ the the statement is obvious. Assume that it is proved for all f whose codomain is of length $< j$ and let X be of length j. We have $ft(fg) = ft(f)g$ and therefore $ft(fg) \in Mor(CC')$ by the inductive assumption. It remains to show that $s_{fg} \in \widetilde{C}$. We have the following diagram whose squares are canonical pull-back squares

$$\begin{array}{ccccc}
X_{fg} & \longrightarrow & X_f & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_X} \\
Z & \xrightarrow{g} & Y & \xrightarrow{ft(f)} & ft(X)
\end{array}$$

which shows that $s_{fg} = g^*(s_f)$. Therefore, $s_{fg} \in Mor(CC')$ by Lemma 1.7.

Lemma 1.9 *[**2009.10.16.l5**] Under the assumptions of the proposition, let $X \in C$ and let $f : Y \to ft(X)$ be in $Mor(CC')$, then $f^*(X) \in C$ and $q(f, X) \in Mor(CC')$.*

Proof: Consider the diagram

$$
\begin{array}{ccccc}
f^*(X) & \stackrel{q(f,X)}{\longrightarrow} & X & & \\
s_{q(f,X)} \downarrow & & \downarrow s_{Id_X} & & \\
q(f,X)^*(X) & \longrightarrow & p_X^*(X) & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
f^*(X) & \stackrel{q(f,X)}{\longrightarrow} & X & \longrightarrow & ft(X) \\
p_{f^*(X)} \downarrow & & \downarrow p_X & & \\
Y & \stackrel{f}{\longrightarrow} & ft(X) & &
\end{array}
$$

where the squares are canonical. By condition (6) we have $s_{Id} \in \widetilde{C}$. Therefore, by Lemma 1.7, we have $s_{q(f,X)} \in \widetilde{C}$. In particular, $q(f,X)^*(X) \in C$ and therefore $f^*(X) = ft(q(f,X)^*(X)) \in C$. The fact that $q(f,X) \in Mor(CC')$ follows from the fact that $s_{q(f,X)} \in \widetilde{C}$ and $ft(q(f,X)) = f \circ p_{f^*(X)}$ is in $Mor(CC')$ by previous lemmas.

Lemma 1.10 *[***2009.10.16.l6***] Under the assumptions of Lemma 1.9, the square*

$$
\begin{array}{ccc}
f^*(X) & \stackrel{q(f,X)}{\longrightarrow} & X \\
p_{f^*(X)} \downarrow & & \downarrow p_X \\
Y & \stackrel{f}{\longrightarrow} & ft(X)
\end{array}
$$

is a pull-back square in CC'.

Proof: We need to show that for a morphism $g : Z \to f^*(X)$ such that $p_{f^*(X)}g$ and $q(f,X)g$ are in $Mor(CC')$ one has $g \in Mor(CC')$. We have $ft(g) = p_{f^*(X)}g$, therefore by definition of $Mor(CC')$ it remains to check that $s_g \in \widetilde{C}$. The diagram

$$
\begin{array}{ccccc}
(f^*Y)_g & \longrightarrow & f^*Y & \stackrel{q(f,X)}{\longrightarrow} & X \\
\downarrow & & \downarrow & & \downarrow \\
Z & \stackrel{ft(g)}{\longrightarrow} & Y & \stackrel{f}{\longrightarrow} & ft(X)
\end{array}
$$

shows that $s_g = s_{q(f,X)g}$ and therefore $s_g \in Mor(CC')$.

To finish the proof of the proposition it remains to show that $ob(CC') = C$ and $\widetilde{ob}(CC') = \widetilde{C}$. The first assertion is tautological. The second one follows immediately from the fact that for $(s : ft(X) \to X) \in \widetilde{ob}(CC)$ one has $ft(s) = Id_{ft(X)}$ and $s_s = s$.

The sequent axiomatics of contextual categories. Proposition 1.5 suggests that a contextual category CC can be reconstructed from the sets $ob(CC)$ and $\widetilde{ob}(CC)$ together with the structures on these sets which correspond to the conditions of the proposition. Let us show that it is indeed the case. Let $B = ob(CC)$ and $\widetilde{B} = \widetilde{ob}(CC)$. We will further write B_n for $ob(CC)_n$ such that $B_0 = \{pt\}$, $ft : B_n \to B_{n-1}$ for the ft map and $ft^i : B_n \to B_{n-i}$ for the i-th iteration of ft. Let $\partial : \widetilde{B} \to B$ be the map which assigns to a pair $(X, s : ft(X) \to X) \in \widetilde{ob}(CC)$ the object X. It takes values in $B_{\geq 1} = \amalg_{n\geq 1} B_n$. Let $\delta : B_{\geq 1} \to \widetilde{B}$ be the map which sends X to the diagonal section of the projection $p_X^* X \to X$. We will also write $\widetilde{B}_i$ for (X, s) such that $X \in B_i$ with the obvious meaning assigned to $\widetilde{B}_{\geq i}$ etc. Finally, for $X \in B_i$ where $i \geq 1$ we will write $\widetilde{B}(X)$ or $\widetilde{B}_i(X)$ for the set of $r \in \widetilde{B}$ such that $\partial(r) = X$.

In addition let us consider the following maps given for all $m \geq n \geq 0$:

1. $T : (B_{n+1})_{ft} \times_{ft^{m+1-n}} (B_{m+1}) \to B_{m+2}$, which sends (Y, X) such that $ft(Y) = ft^{m+1-n}(X)$ to $p_Y^*(X, m+1-n)$,
2. $\widetilde{T} : (B_{n+1})_{ft} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \to \widetilde{B}_{m+2}$, which sends (Y, r) such that $ft(Y) = ft^{m+1-n}\partial(r)$ to $p_Y^*(r, m+1-n)$,
3. $S : (\widetilde{B}_{n+1})_{\partial} \times_{ft^{m+1-n}} (B_{m+2}) \to B_{m+1}$, which sends (s, X) such that $\partial(s) = ft^{m+1-n}(X)$ to $s^*(X, m+1-n)$,
4. $\widetilde{S} : (\widetilde{B}_{n+1})_{\partial} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+2}) \to \widetilde{B}_{m+1}$, which sends (s, r) such that $\partial(s) = ft^{m+1-n}\partial(r)$ to $s^*(r, m+1-n)$.

Note that we have

$$
\begin{aligned}
&ft(T(Y,X)) = \begin{cases} T(Y, ft(X)) & \text{if } l(Y) - l(X) \geq 1 \\ Y & \text{if } l(Y) - l(X) = 0 \end{cases} \\
&\partial(\widetilde{T}(Y,r) = T(Y, \partial(r)) \\
&ft(S(s,X)) = \begin{cases} S(s, ft(X)) & \text{if } l(\partial(s)) - l(X) \geq 1 \\ Y & \text{if } l(\partial(s)) - l(X) = 0 \end{cases} \\
&\partial(\widetilde{S}(s,r)) = S(s, \partial(r))
\end{aligned}
\tag{3}
$$

and

$$[\mathbf{2009.12.27.eq1}]\partial(\delta(X)) = T(X,X) \tag{4}$$

Let us denote by

$$T_j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}} (B_{m+1}) \to B_{m+1+j}$$

$$\widetilde{T}_j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \to \widetilde{B}_{m+1+j}$$

the maps which are defined inductively by

$$T_j(Y,X) = \begin{cases} X & \text{if } j = 0 \\ T(Y,X) & \text{if } j = 1 \\ T_{j-1}(ft(Y), X)) & \text{if } j > 1 \end{cases} \qquad \widetilde{T}_j(Y,r) = \begin{cases} r & \text{if } j = 0 \\ \widetilde{T}(Y,r) & \text{if } j = 1 \\ \widetilde{T}_{j-1}(ft(Y), r)) & \text{if } j > 1 \end{cases} \tag{5}$$

Let $f : Y \to X$ be a morphism such that $Y \in B_n$ and $X \in B_m$. Define a sequence $(s_1(f), \ldots, s_m(f))$ of elements of $\widetilde{B}_{n+1}$ inductively by the rule

$$(s_1(f), \ldots, s_m(f)) = (s_1(ft(f)), \ldots, s_{m-1}(ft(f)), s_f) = (s_{ft^{m-1}(f)}, \ldots, s_{ft(f)}, s_f)$$

where $ft(f) = p_X f$, s_f is defined by the diagram (2) and for $m = 0$ we start with the empty sequence. This construction can be illustrated by the following diagram for $f : Y \to X$ where $X \in B_4$:

$$\begin{array}{ccccccccccc}
Y & \xrightarrow{s_4(f)} & Z_{4,3} & \longrightarrow & Z_{4,2} & \longrightarrow & Z_{4,1} & \longrightarrow & T_n(Y,X) & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & Y & \xrightarrow{s_3(f)} & Z_{3,2} & \longrightarrow & Z_{3,1} & \longrightarrow & T_n(Y,ft(X)) & \longrightarrow & ft(X) \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & Y & \xrightarrow{s_2(f)} & Z_{2,1} & \longrightarrow & T_n(Y,ft^2(X)) & \longrightarrow & ft^2(X) \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & Y & \xrightarrow{s_1(f)} & T_n(Y,ft^3(X)) & \longrightarrow & ft^3(X) \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & Y & \longrightarrow & pt
\end{array} \quad (6)$$

which is completely determined by the condition that the squares are the canonical ones and the composition of morphisms in the i-th arrow from the top is $ft^i(f)$. For the objects Z_i^j we have:

$$\begin{array}{lll}
Z_{4,1} = S(s_1(f), T_n(Y,X)) & Z_{4,2} = S(s_2(f), Z_{4,1}) & Z_{4,3} = S(s_3(f), Z_{4,2}) \\
Z_{3,1} = S(s_1(f), T_n(Y, ft(X))) & Z_{3,2} = S(s_2(f), Z_{3,1}) & \\
Z_{2,1} = S(s_1(f), T_n(Y, ft^2(X))) & &
\end{array} \quad (7)$$

A simple inductive argument similar to the one in the proof of Lemma 1.3 show that if $f, f' : Y \to X$ are two morphisms such that $X \in B_m$ and $s_i(f) = s_i(f')$ for $i = 1, \ldots, m$ then $f = f'$. Therefore, we may consider the set $Mor(CC)$ of morphisms of CC as a subset in $\amalg_{n,m \geq 0} B_n \times B_m \times \widetilde{B}_{n+1}^m$.

Let us show how to describe this subset in terms of the operations introduced above.

Lemma 1.11 *[2009.11.07.l1] An element $(Y, X, s_1, \ldots, s_m)$ of $B_n \times B_m \times \widetilde{B}_{n+1}^m$ corresponds to a morphism if and only if the element $(Y, ft(X), s_1, \ldots, s_{m-1})$ corresponds to a morphism and $\partial(s_m) = Z_{m,m-1}$ where $Z_{m,i}$ is defined inductively by the rule:*

$$Z_{m,0} = T_n(Y, X) \qquad Z_{m,i+1} = S(s_{i+1}(f), Z_{m,i})$$

Proof: Straightforward from the example considered above.

Let us show now how to identify the canonical morphisms $p_{X,i} : X \to ft^i(X)$ and in particular the identity morphisms.

Lemma 1.12 *[2009.11.10.l1] Let $X \in B_m$ and $0 \leq i \leq m$. Let $p_{X,i} : X \to ft^i(X)$ be the canonical morphism. Then one has:*

$$s_j(p_{X,i}) = \widetilde{T}_{m-j}(X, \delta_{ft^{m-j}(X)}) \qquad j = 1, \ldots, m-i$$

Proof: Let us proceed by induction on $m-i$. For $i=m$ the assertion is trivial. Assume the lemma proved for $i+1$. Since $ft(p_{X,i}) = p_{X,i+1}$ we have $s_j(p_{X,i}) = s_j(p_{X,i+1})$ for $j = 1, \ldots, m-i-1$. It remains to show that

$$[\mathbf{2009.11.10.eq1}] s_{m-i}(p_{X,i}) = \widetilde{T}_i(X, \delta_{ft^i(X)}) \tag{8}$$

By definition $s_{m-i}(p_{X,i}) = s_{p_{X,i}}$ and (8) follows from the commutative diagram:

$$\begin{array}{ccccc}
X & \longrightarrow & ft^i(X) & & \\
{\scriptstyle s_p}\big\downarrow & & \big\downarrow{\scriptstyle \delta_{ft^i(X)}} & & \\
p^*_{X,i+1}(ft^i(X)) & \longrightarrow & p^*_{ft^i(X)}(ft^i(X)) & \longrightarrow & ft^i(X) \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle p_{ft^i(X)}} \\
X & \longrightarrow & ft^i(X) & \longrightarrow & ft^{i+1}(X)
\end{array}$$

where $p = p_{X,i}$.

Lemma 1.13 *[**2009.11.10.l2**] Let $(X, s) \in \widetilde{B}_{m+1}$, $Y \in B_n$ and $f : Y \to ft(X)$. Define inductively $(f,i)^*(s) \in \widetilde{B}_{n+m+1-i}$ by the rule*

$$(f,0)^*(s) = \widetilde{T}_n(Y, s)$$

$$(f, i+1)^*(s) = \widetilde{S}(s_{i+1}(f), (f,i)^*(s))$$

Then $f^*(s) = (f,m)^*(s)$.

Proof: It follows from the diagram:

$$\begin{array}{ccccccccccc}
Y & \xrightarrow{s_m(f)} & * & \longrightarrow & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(X) \\
{\scriptstyle f^*(s)}\big\downarrow & & \big\downarrow{\scriptstyle (f,m-1)^*(s)} & & & & \big\downarrow{\scriptstyle (f,1)^*(s)} & & \big\downarrow{\scriptstyle (f,0)^*(s)} & & \big\downarrow{\scriptstyle s} \\
* & \longrightarrow & * & \longrightarrow & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & X \\
\big\downarrow & & \big\downarrow & & & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
Y & \xrightarrow{s_m(f)} & * & \longrightarrow & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(X) \\
 & & \big\downarrow & & & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
 & & Y & \xrightarrow{s_{m-1}(f)} & \ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft^2(X) \\
 & & & & & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
 & & & & & & \ldots & & \ldots & & \ldots \\
 & & & & & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
 & & & & & & Y & \xrightarrow{s_1(f)} & * & \longrightarrow & ft^{m-1}(X) \\
 & & & & & & & & \big\downarrow & & \big\downarrow \\
 & & & & & & & & Y & \longrightarrow & pt
\end{array}$$

Lemma 1.14 *Let $g : Z \to Y$, $f : Y \to X$ and $X \in B_m$. Then $s_i(fg) = g^* s_i(f)$.*

Proof: It follows immediately from the equations $s_{fg} = g^* s_f$ and $ft(fg) = ft(f)g$.

Lemma 1.15 *[2009.11.10.l4] Let $f : Y \to ft(X)$ be a morphism, $Y \in B_n$ and $X \in B_{m+1}$. Define $(f, i)^*(X)$ inductively by the rule:*

$$(f, 0)^*(X) = T_n(Y, X)$$

$$(f, i + 1)^*(X) = S(s_{i+1}(f), (f, i)^*(X))$$

Then $f^*(X) = (f, m)^*(X)$.

Proof: Similar to the proof of Lemma 1.13.

Lemma 1.16 *[2009.11.10.l4] Let $f : Y \to ft(X)$ be a morphism, $Y \in B_n$ and $X \in B_{m+1}$. Then*

$$s_i(q(f, X)) = \begin{cases} \widetilde{T}(f^*X, s_i(f)) & \text{if } i \leq m \\ \widetilde{T}(f^*X, \delta_X) & \text{if } i = m + 1 \end{cases}$$

Proof: We have $s_i(q(f, X)) = s_{ft^{m+1-i}(q(f,X))}$. For $i \leq m$ we have $ft^{m+1-i}(q(f, X)) = ft^{m-i}(f) p_{f^*X}$. Therefore,

$$s_{ft^{m+1-i}(q(f,X))} = s_{ft^{m-i}(f) p_{f^*X}} = p^*_{f^*X} s_{ft^{m-i}(f)} = \widetilde{T}(f^*X, s_i(f))$$

and for $i = m + 1$ we have

$$s_i(q(f, X)) = s_{q(f,X)} = p^*_{f^*X}(\delta_X) = \widetilde{T}(f^*X, \delta_X).$$

The lemmas proved above show that a contextual category can be reconstructed from the pair of sets $B, \widetilde{B}$ connected by the maps ft, ∂, δ, T, $\widetilde{T}$, S and $\widetilde{S}$. While this way of encoding contextual categories may be less convenient than their encoding as a pair of sets Ob and Mor connected by the maps ∂_0, ∂_1, c (composition), id, ft and $qpb : (f, X) \mapsto q(f, X)$, this fact has the following important corollary.

Proposition 1.17 *[2009.11.10.prop1] Let CC, CC' be two contextual categories. Then there is a natural bijection between contextual functors $F : CC \to CC'$ and pairs of maps $F_0 : ob(CC) \to ob(CC')$, $F_1 : \widetilde{ob}(CC) \to \widetilde{ob}(CC')$ which commute in the obvious sense with ft, ∂, δ, T, $\widetilde{T}$, S and $\widetilde{S}$.*

Remark 1.18 Notes on the properties of the maps introduced above:

1. for $Y \in B_{\geq n+2}$, $S(\delta_{ft^{n+1}(Y)}, T(ft^{n+1}(Y), Y)) = Y$.

Remark 1.19 The maps S and T can be defined as $ft\,\partial\,\widetilde{S}\,\delta$ and $ft\,\partial\,\widetilde{T}\,\delta$ respectively.

2 Π-contextual categories

The notion of a Π-contextual category is equivalent to the notion of a contextual category with products of families of types from [?]. We use the name Π-contextual categories to emphasize the fact that we are dealing here with an additional structure on a contextual category rather than with a property of such a category.

Let us recall first the following definition.

Definition 2.1 *[**2009.11.24.def2**] Let $\mathcal{C}$ be a 1-category. Let $g : Z \to Y$, $f : Y \to X$ be a pair of morphisms such that for any $U \to X$ a fiber product $U \times_X Y$ exists. A pair*

$$(w : W \to X, h : W \times_X Y \to Z)$$

such that $g \circ h = pr$ is called a universal pair for (f, g) if for any $U \to X$ the map

$$Hom_X(U, W) \to Hom_Y(U \times_X Y, Z)$$

of the form $u \mapsto h \circ (u \times Id_Y)$ is a bijection.

If a universal pair exists then it is easily seen to be unique up to a canonical isomorphism. We denote such a pair by $(\Pi(g, f), e_{g,f} : \Pi(g, f) \times_X Y \to Z)$. Note that if $f' : Y \to X$ and $pr : Y' \times_X Y \to Y$ is the projection then

$$(\Pi(pr, f), pr' \circ e_{pr,f} : \Pi(g, f) \times_X Y \to Y') = (\underline{Hom}_X(Y, Y'), ev : \underline{Hom}_X(Y, Y') \times_X Y \to Y')$$

so that relative internal Hom-objects are particular cases of universal pairs.

Definition 2.2 *[**2009.11.24.def1**] A Π-contextual category is a contextual category CC together with additional data of the form*

1. *for each $Y \in ob(CC)_{\geq 2}$ an object $\Pi(Y) \in ob(CC)$ such that $ft(\Pi(Y)) = ft^2(Y)$,*
2. *for each $Y \in ob(CC)_{\geq 2}$ a morphism $eval : T(ft(Y), \Pi(Y)) = p^*_{ft(Y)}(\Pi(Y)) \to Y$ over $ft(Y)$,*

such that

(i) for any $f : Z \to ft^2(Y)$ one has $f^*(\Pi(Y)) = \Pi(f^*(Y))$ and $f^*(eval_Y) = eval_{f^*(Y)}$,

(ii) $(\Pi(Y), eval_Y))$ is a universal pair for $(p_Y, p_{ft(Y)})$.

Let us now prove that this definition can be re-written in a less compact but purely equational form. As before let us write B_n for $Ob(CC)_n$, $\widetilde{B}_n$ for $\widetilde{Ob}(CC)_n$ etc.

The contextual category is completely determined by the sets $B_n, \widetilde{B}_{n+1}$, $n \geq 0$ and maps $\partial : \widetilde{B}_{n+1} \to B_{n+1}$, $ft : B_{n+1} \to B_n$, $\delta : B_n \to \widetilde{B}_{n+1}$ and the maps $T_{n+1}, \widetilde{T}_{n+1}, S_{n+1}, \widetilde{S}_{n+1}$ considered above.

Suppose now that we are given a Π-contextual structure on our contextual category. This structure defines maps:

1. $\Pi : B_{n+2} \to B_{n+1}$, $n \geq 0$,

2. $\lambda : \widetilde{B}_{n+2} \to \widetilde{B}_{n+1}$, $n \geq 0$,

3. $ev : (\widetilde{B}_{n+1})_\partial \times_{ft} (B_{n+2})_\Pi \times_\partial (\widetilde{B}_{n+1}) \to \widetilde{B}_{n+1}$, $n \geq 0$

as follows. The map Π is the map from Definition 2.2. Since $(\Pi(Y), eval_Y)$ is a universal pair for $(p_Y, p_{ft(Y)})$ the mapping

$$\phi_Y : \{f \in \widetilde{B}_{n+1} \,|\, \partial(f) = \Pi(Y)\} \to \{s \in \widetilde{B}_{n+2} \,|\, \partial(s) = Y\}$$

given by the formula

$$\phi_Y(f) = eval_Y \circ \widetilde{T}(ft(Y), f)$$

is a bijection. One defines λ_Y as the inverse to this bijection.

The map ev sends a triple (r, Y, f) such that $\partial(r) = ft(Y)$ and $\partial(f) = \Pi(Y)$ to

$$ev(r, Y, f) = \widetilde{S}(r, eval \circ \widetilde{T}(ft(Y), f))$$

as partially illustrated by the following diagram:

$$\begin{array}{ccccc}
 & & Y & \longleftarrow & S(r,Y) \\
 & & {\scriptstyle p_Y}\downarrow & & \downarrow \\
p^*_{ft(Y)}(\Pi(Y)) & \longrightarrow & ft(Y) & \underset{r}{\longleftarrow} & ft^2(Y) \\
\downarrow & & \downarrow & & \\
\Pi(Y) & \xrightarrow{p_{\Pi(Y)}} & ft^2(Y) & &
\end{array}$$

Lemma 2.3 [**2009.11.30.l1**] *Let $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$ and $f \in \widetilde{B}(\Pi(Y))$. Then one has*

$$g^*(\phi_Y(f), i+2) = \phi_{g^*(Y,i+2)}(g^*(f, i+1))$$

Proof: Let $h_1 = q(g, ft(Y), i+1)$, $h_2 = q(g, ft^(Y), i+2)$. Then one has

$$g^*(\phi_Y(f), i+2) = h_1^*(\phi_Y(f)) = h_1^*(eval_Y \circ \widetilde{T}(ft(Y), f)) = h_1^*(eval_Y) \circ h_1^*(\widetilde{T}(ft(Y), f))$$

$$= eval_{h_1^*(Y)} p^*_{g^*(ft(Y),i+1)}(h_2^*(f)) = \phi_{h_1^*(Y)}(h_2^*(f)) = \phi_{g^*(Y,i+2)}(g^*(f, i+1)).$$

As an immediate corollary of Lemma 2.3 we have:

Lemma 2.4 [**2009.11.30.l2**] *Let $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$ and $r \in \widetilde{B}(Y)$. Then one has*

$$g^*(\lambda(r), i+1) = \lambda(g^*(r, i+2)).$$

Lemma 2.5 [**2009.11.30.l3**] *Let $n \geq i \geq 0$, $Y \in B_{n+2}$, $g : Z \to ft^{i+2}(Y)$, $r \in \widetilde{B}(ft(Y))$ and $f \in \widetilde{B}(\Pi(Y))$. Then one has*

$$g^*(ev(r, Y, f), i+1) = ev(g^*(r, i+2), g^*(Y, i+2), g^*(f, i+1))$$

Proof: Let $h_1 = q(g, ft(Y), i+1)$, $h_2 = q(g, ft^(Y), i+2)$. Then one has:

$$g^*(ev(r, Y, f), i+1) = h_2^*(\widetilde{S}(r, eval \circ \widetilde{T}(ft(Y), f))) = h_2^*(r^*(eval \circ \widetilde{T}(ft(Y), f))) =$$

$$= (h_2^*(r))^* h_1^*(eval \circ \widetilde{T}(ft(Y), f))) = (h_2^*(r))^*(h_1^*(eval) \circ h_1^* p_{ft(Y)}^*(f)) =$$

$$= (g^*(r, i+2))^*(eval \circ p^*_{g^*(ft(Y), i+1)}(h_2^*(f))) = ev(g^*(r, i+2), g^*(Y, i+2), g^*(f, i+1)).$$

Proposition 2.6 *[***2009.11.29.prop1***] Let $CC = (B_n, \widetilde{B}_n, ft, \partial, \delta)$ be a contextual category. Let further $(\Pi, eval)$ be a Π-contextual structure on CC. Then the maps Π, λ, ev defined by this structure satisfy the following conditions:*

1. *for $Y \in B_{n+2}$ one has*
   (a) *$ft\,\Pi(Y) = ft^2(Y)$,*
   (b) *for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}(Y)$, $T(Z, \Pi(Y)) = \Pi(T(Z, Y))$,*
   (c) *for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^{i+1}(Y)$, $S(t, \Pi(Y)) = \Pi(S(t, Y))$,*
2. *for $s \in \widetilde{B}_{n+2}$ one has*
   (a) *$\partial\,\lambda(s) = \Pi\,\partial(s)$,*
   (b) *for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}\,\partial(s)$, $\widetilde{T}(Z, \lambda(s)) = \lambda(\widetilde{T}(Z, s))$,*
   (c) *for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^{i+1}\,\partial(s)$, $\widetilde{S}(t, \lambda(s)) = \lambda(\widetilde{S}(t, s))$,*
3. *for $r \in \widetilde{B}_{n+1}$, $Y \in B_{n+2}$ and $f \in \widetilde{B}_{n+1}$ such that $\partial(r) = ft(Y)$ and $\partial(f) = \Pi(Y)$ one has*
   (a) *$\partial(ev(r, Y, f)) = S(r, Y)$,*
   (b) *for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^{i+1}(Y)$,*
   $$\widetilde{T}(Z, ev(r, Y, f)) = ev(\widetilde{T}(Z, r), T(Z, Y), \widetilde{T}(Z, f)),$$
   (c) *for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^{i+1}(Y)$,*
   $$\widetilde{S}(t, ev(r, Y, f)) = ev(\widetilde{S}(t, r), S(t, Y), \widetilde{S}(t, f)),$$
4. *for $r \in \widetilde{B}_{n+1}$, $s \in \widetilde{B}_{n+2}$ such that $ft(\partial(s)) = \partial(r)$*
   $$ev(r, \partial\,s, \lambda(s)) = \widetilde{S}(r, s)$$
   (β-reduction),
5. *for $Y \in B_{n+2}$, $f \in \widetilde{B}_{n+1}$ such that $\partial(f) = \Pi(Y)$,*
   $$[\mathbf{2009.11.30.oldeq1}]\lambda(ev(\delta_{ft(Y)}, T(ft(Y), Y), \widetilde{T}(ft(Y), f))) = f \qquad (9)$$
   (η-reduction).

Proof: (1a) Follows from Definition 2.2(1). (1b) Follows from Definition 2.2(i) applied to $f = q(p_Z, ft^2(Y), i-1)$. (1c) Follows from Definition 2.2(i) applied to $f = q(t, ft^2(Y), i-1)$.

(2a) Follows from the definition of λ. (2b) Follows from Lemma 2.4 applied to p_Z. (2c) Follows from Lemma 2.4 applied to t.

(3a) Follows from the definition of ev. (3b) Follows from Lemma 2.5 applied to p_Z. (3c) Follows from Lemma 2.5 applied to t.

(4) One has

$$ev(r, \partial\, s, \lambda(s)) = r^*(eval \circ (p^*_{ft(Y)}(\lambda(s)))) = r^*(\phi_Y(s)) = r^*(s) = \widetilde{S}(r, s).$$

(5) Let $T_1 = T(ft(Y), ft(Y))$ and $T_2 = T(ft(Y), Y)$. Then

$$ev(\delta_{ft(Y)}, T(ft(Y), Y), \widetilde{T}(ft(Y), f)) = \delta^*_{ft(Y)}(eval_{T_2} \circ p^*_{T_1}(p^*_{ft(Y)}(f))) =$$

$$= \delta^*_{ft(Y)}(eval_{T_2}) \circ \delta^*_{ft(Y)} p^*_{T_1} p^*_{ft(Y)}(f) = eval_{\delta^*_{ft(Y)}(T_2)} \circ p^*_{ft(Y)}(f) = eval_Y \circ p^*_{ft(Y)}(f) = \phi_Y(f)$$

which implies (9) by definition of λ.

The converse to Proposition 2.6 holds as well. Let $CC = (B_n, \widetilde{B}_n, ft, \partial, \delta)$ be a contextual category and let

1. $\Pi : B_{n+2} \to B_{n+1}$, $n \geq 0$,
2. $\lambda : \widetilde{B}_{n+2} \to \widetilde{B}_{n+1}$, $n \geq 0$,
3. $ev : (\widetilde{B}_{n+1})_\partial \times_{ft} (B_{n+2})_\Pi \times_\partial (\widetilde{B}_{n+1}) \to \widetilde{B}_{n+1}$, $n \geq 0$

be maps satisfying the conclusion of Proposition 2.6. For each $Y \in \widetilde{B}_{n+2}$ define a morphism

$$eval_Y : T(ft(Y), \Pi(Y)) \to Y$$

by the formula

$$eval_Y = q(p_Z, Y) \circ ev(p^*_Z(\delta_{ft(Y)}), T_2(Z, Y), \delta_Z)$$

where $Z = p^*_{ft(Y)}(\Pi(Y))$.

Proposition 2.7 *[**2009.11.30.prop2**] Under the assumption made above the morphisms $eval_Y$ are well defined and $(\Pi, eval)$ is a Π-contextual structure on CC.*

Proof: Let us show that eva_Y is well defined. This requires us to check the following conditions:

1. $ft^2(Y) = ft(\Pi(Y))$, therefore Z is defined,
2. $ft(Z) = ft\partial(\delta_{ft(Y)})$ since $ft(Z) = ft(Y)$, therefore $p^*_Z(\delta_{ft(Y)})$ is defined,
3. $ft^2(Z) = ft^2(Y)$, therefore $T_2(Z, Y)$ is defined,
4. $\partial(p^*_Z(\delta_{ft(Y)}))) = p^*_Z p^*_{ft(Y)}(ft(Y))$, $ft(T_2(Z, Y)) = T_2(Z, ft(Y)) = p^*_Z p^*_{ft(Y)}(ft(Y))$,
5. $\partial(\delta_Z) = p^*_Z(Z) = p^*_Z p^*_{ft(Y)}(\Pi(Y)) = \Pi_{T_2(Z,Y)}$, therefore $ev = ev(p^*_Z(\delta_{ft(Y)}), T_2(Z, Y), \delta_Z)$ is defined,

6.
$$\partial(ev) = (p_Z^*(\delta_{ft(Y)}))^*(T_2(Z,Y)) = (p_Z^*(\delta_{ft(Y)}))^* T(Z, T(ft(Y), Y)) =$$
$$= (p_Z^*(\delta_{ft(Y)}))^*(p_Z)^*((p_{ft(Y)})^*(Y,2),2) = (p_Z^*(\delta_{ft(Y)}))^* q(p_Z, p_Y^*(ft(Y)))^*(p_{ft(Y)})^*(Y,2) =$$
$$= (p_Z^*(\delta_{ft(Y)}))^* q(p_Z, p_Y^*(ft(Y)))^* q(p_{ft(Y)}, ft(Y))^*(Y) =$$
$$= (q(p_{ft(Y)}, ft(Y)) q(p_Z, p_Y^*(ft(Y))) p_Z^*(\delta_{ft(Y)}))^*(Y) = p_Z^*(Y)$$
and $q(p_Z, Y) : p_Z^*(Y) \to Y$. Therefore $eval_Y$ is defined and is a morphism from Z to Y as required by Definition 2.2(2).

We leave the verification of the conditions (i) of (ii) of Definition 2.2 for the later, more mechanized version of this paper.

3 Impredicative Π-universe structures.

Definition 3.1 *[***2009.12.04.def1***] Let $CC = (B, \widetilde{B}, \ldots, \Pi, \ldots)$ be Π-contextual category. An impredicative Π-universe structure on CC is a collection of data of the form*

1. *an object $\widetilde{\Omega} \in B_2$,*
2. *for any $n \geq 0$, $Y \in B_{n+1}$, $g : Y \to ft(\widetilde{\Omega})$ a morphism $\pi_\Omega(g) : ft(Y) \to ft(\widetilde{\Omega})$,*

such that the following conditions hold

(i) for any g as above $\pi_\Omega(g)^*(\widetilde{\Omega}) = \pi(g^*(\widetilde{\Omega}))$,

(ii) for any g as above and $h : Z \to ft(Y)$ one has
$$\pi_\Omega(g) \circ h = \pi_\Omega(g \circ q(h, Y))$$

The sequent presentation of an impredicative Π-structure looks as follows. Given an impredicative Π-universe $(\widetilde{\Omega}, \pi_\Omega)$ denote by Ω the object $ft(\widetilde{\Omega})$. Note that for any $Y \in B_n$ and $Z \in B_1$ the mapping which sends $s \in \widetilde{B}_{n+1}(T_n(Y,Z))$ to $s \circ q(p_{Y,n}, Z)$ defines a bijection $\phi_Y : \widetilde{B}_{n+1}(T_n(Y,Z)) \to Hom_{CC}(Y,Z)$.

For any $n \geq 0$, $Y \in B_{n+1}$, $s \in \widetilde{B}_{n+2}(T_{n+1}(Y,\Omega))$ define $\Pi_\Omega(s) \in \widetilde{B}_{n+1}(T_n(ft(Y),\Omega))$ by the formula
$$\Pi_\Omega(s) = \phi^{-1}_{ft(Y)}(\pi_\Omega(\phi_Y(g))).$$
One verifies immediately that the conditions of Definition 3.1 imply that

1. $S(\Pi_\Omega(s), T_n(ft(Y), \widetilde{\Omega})) = \Pi(S(s, T_{n+1}(Y, \widetilde{\Omega})))$,
2. for $n+1 \geq i \geq 1$, $Z \in B_{n+2-i}$ such that $ft(Z) = ft^i(Y)$, $\widetilde{T}(Z, \Pi_\Omega(s)) = \Pi_\Omega(\widetilde{T}(Z,s))$,
3. for $n+1 \geq i \geq 1$, $t \in \widetilde{B}_{n+1-i}$ such that $\partial(t) = ft^i(Y)$, $\widetilde{S}(t, \Pi_\Omega(s)) = \Pi_\Omega(\widetilde{S}(t,s))$.

Conversely one has:

Proposition 3.2 *[***2009.12.4.prop1***] Let $CC = (B, \widetilde{B}, \ldots, \Pi, \ldots)$ be Π-contextual category. Let $\widetilde{\Omega} \in B_2$, $\Omega = ft(\widetilde{\Omega})$ and*
$$\Pi_\Omega : (B_{n+1})_{T_{n+1}(-,\Omega)} \times_\partial (\widetilde{B}_{n+2}) \to \widetilde{B}_{n+1}$$
be maps satisfying conditions (1),(2),(3) listed above. Then they correspond to a unique impredicative Π-structure on CC.

4 Predicative Π-universe structures.

Definition 4.1 *[***2009.12.1def4***] Let $CC = (B, \widetilde{B}, \dots, \Pi, \dots)$ be Π-contextual category. A predicative Π-universe structure on CC is a collection of data of the form*

1. *an object $\widetilde{\Omega} \in B_2$,*
2. *for any $f : X \to ft(\widetilde{\Omega})$, $g : f^*(\widetilde{\Omega}) \to ft(\Omega)$ a morphism $\Pi_\Omega(f, g) : X \to ft(\widetilde{\Omega})$,*

such that the following conditions hold

(i) for any f, g as above $\Pi_\Omega(f, g)^*(\widetilde{\Omega}) = \Pi(g^*(\widetilde{\Omega}))$,

(ii) for any f, g as above and $h : Z \to X$ one has

$$\Pi_\Omega(f, g) \circ h = \Pi_\Omega(f \circ h, g \circ q(h, f^*(\widetilde{\Omega})))$$

Note that any impredicative universe structure defines a predicative universe structure by the formula $\Pi(f, g) = \Pi(g)$.

The sequent representation of a predicative Π-universe structure looks as follows.

Proposition 4.2 *[***2009.12.4.prop2***] Let $CC = (B, \widetilde{B}, \dots, \Pi, \dots)$ be Π-contextual category. Any predicative Π-universe structure on CC is uniquely determined by a collection of data of the form*

1. *an object $\widetilde{\Omega} \in B_2$ (we will write Ω for $ft(\widetilde{\Omega})$),*
2. *a morphism $\Pi_\Omega : \Pi(T_2(\widetilde{\Omega}, ft(\widetilde{\Omega}))) \to ft(\widetilde{\Omega})$,*

which satisfies the following conditions.

5 Universes with Martin-Loef identity types.

2 Contextual categories defined by universes in 1-categories

Contextual categories $CC(\mathcal{C}, p)$.

Definition 0.1 *[***2009.11.1.def1***] Let $\mathcal{C}$ be a (level 1) category. A universe on $\mathcal{C}$ is a morphism $p : \widetilde{U} \to U$ together with a mapping which assigns to any morphism $f : X \to U$ in $\mathcal{C}$ a pull-back square*

$$\begin{array}{ccc} (X, f) & \xrightarrow{Q(f)} & \widetilde{U} \\ {\scriptstyle p_{(X,f)}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & U \end{array}$$

In what follows we will write $(X, f_1, \dots, f_n)$ for $(\dots((X, f_1), f_2)\dots, f_n)$.

Remark 0.2 A morphism $q : Y \to X$ in $\mathcal{C}$ is called a p-fibration if there a pull-back square of the form

$$\begin{array}{ccc} Y & \longrightarrow & \widetilde{U} \\ {\scriptstyle q}\downarrow & & \downarrow \\ X & \longrightarrow & U \end{array}$$

A morphism q is called p-small if it is a finite composition of p-fibrations.

If $\mathcal{C}$ is a set-level category then we will say that a morphism $f : Y \to X$ is a strict p-fibration if it is of the form $p(f)$.

Let $\mathcal{C}$ be a 1-category, p a pre-universe on $\mathcal{C}$ and pt a final object of $\mathcal{C}$. For such a triple define a contextual category $CC = CC(\mathcal{C}, p)$ as follows. Objects of CC are sequences of the form $(F_1, \dots, F_n)$ where $F_1 \in Hom_{\mathcal{C}}(pt, U)$ and $F_{i+1} \in Hom_{\mathcal{C}}((pt, F_1, \dots, F_i), U)$. Morphisms from $(G_1, \dots, G_n)$ to $(F_1, \dots, F_m)$ are given by

$$Hom_{CC}((G_1, \dots, G_n), (F_1, \dots, F_m)) = Hom_{\mathcal{C}}((pt, G_1, \dots, G_n), (pt, F_1, \dots, F_m))$$

such that the mapping $(F_1, \dots, F_n) \to (pt, F_1, \dots, F_n)$ is a full embedding of the underlying category of CC to $\mathcal{C}$. The image of this embedding consists of objects X for which the canonical morphism $X \to pt$ is a composition of morphisms which are (canonical) pull-backs of p. We will denote this embedding by int.

The final object of CC is the empty sequence (). The map ft sends $(F_1, \dots, F_n)$ to $(F_1, \dots, F_{n-1})$. The canonical morphism $p_{(F_1,\dots,F_n)}$ is the projection

$$p_{((pt,F_1,\dots,F_{n-1}),F_n)} : ((pt, F_1, \dots, F_{n-1}), F_n) \to (pt, F_1, \dots, F_{n-1})$$

For an object $(F_1, \dots, F_{m+1})$ and a morphism $f : (G_1, \dots, G_n) \to (F_1, \dots, F_m)$ the canonical pull-back square is of the form

[2009.10.26.eq3]
$$\begin{array}{ccc} (G_1, \dots, G_n, F_{m+1}f) & \xrightarrow{q(f)} & (F_1, \dots, F_{m+1}) \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_F} \\ (G_1, \dots, G_n) & \xrightarrow{f} & (F_1, \dots, F_m) \end{array} \qquad (10)$$

where $int(p_F) = p((pt, F_1, \dots, F_{n-1}), F_n)$, $int(p_G) = p((pt, G_1, \dots, G_{n-1}), F_{m+1} \circ f)$ and $q(f)$ is the morphism such that $p_F q(f) = f p_G$ and $Q(F_{m+1}) int(q(f)) = Q(F_{m+1} f)$. The unity and composition axioms for the canonical squares follow immediately from the unit and associativity axioms for compositions of morphisms in C.

Let $(\mathcal{C}, p, pt)$ and $(\mathcal{C}', p', pt')$ be two sets of data as above. Let $\Phi : \mathcal{C} \to \mathcal{C}'$ be a functor which takes distinguished squares in $\mathcal{C}$ to pull-back squares in $\mathcal{C}'$, $\phi : \Phi(U) \to U'$, $\widetilde{\phi} : \Phi(\widetilde{U}) \to \widetilde{U}'$ be two morphisms such that

$$\begin{array}{ccc} \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

is a pull-back square and ψ an isomorphism $\psi : pt' \to \Phi(pt)$.

Define a functor $H = H(\Phi, \phi, \widetilde{\phi})$ from $CC(\mathcal{C}, p)$ to $CC(\mathcal{C}', p')$ as follows. We define by induction on n objects $H(F_1, \ldots, F_n) \in CC(\mathcal{C}', p')$ and isomorphisms

$$\psi_{(F_1,\ldots,F_n)} : int'(H(F_1, \ldots, F_n)) \to \Phi(int(F_1, \ldots F_n))$$

where int and int' are the canonical functors $CC(\mathcal{C}, p) \to \mathcal{C}$ and $CC(\mathcal{C}', p') \to \mathcal{C}'$ respectively.

For $n = 0$ we set $H(pt) = pt$ and $\psi_{()} = \psi$. For $n > 0$ let

$$(F'_1, \ldots, F'_{n-1}) = H(F_1, \ldots, F_{n-1})$$

and let $F_n : int(F_1, \ldots, F_{n-1}) \to U$. Define F'_n as the composition

$$[\textbf{2009.10.26.eq5}] F'_n : int'(F'_1, \ldots, F'_{n-1}) \stackrel{\psi_{(F_1,\ldots,F_{n-1})}}{\longrightarrow} \Phi(int(F_1, \ldots, F_{n-1})) \stackrel{\Phi(F_n)}{\to} \Phi(U) \stackrel{\phi}{\to} U' \quad (11)$$

and let $H(F_1, \ldots, F_n) = (F'_1, \ldots, F'_{n-1}, F'_n)$. Then

$$int'(H(F_1, \ldots, F_n)) = (int'(H(F_1, \ldots, F_n)), F'_n)$$

To define

$$\psi_{(F_1,\ldots,F_n)} : int'(H(F_1, \ldots, F_n)) \to \Phi(int(F_1, \ldots, F_n))$$

observe that by our conditions on $\phi, \widetilde{\phi}$ and Φ the squares of the diagram

$$\begin{array}{ccccc} \Phi(int(F_1, \ldots, F_n)) & \stackrel{\Phi(Q(F_n))}{\longrightarrow} & \Phi(\widetilde{U}) & \longrightarrow & \widetilde{U}' \\ \downarrow & & \downarrow & & \downarrow \\ \Phi(int(F_1, \ldots, F_{n-1})) & \stackrel{\Phi(F_n)}{\longrightarrow} & \Phi(U) & \stackrel{\phi}{\longrightarrow} & U' \end{array}$$

are pull-back. Therefore there is a unique morphism $\psi_{(F_1,\ldots,F_n)}$ such that the diagram

$$[\textbf{2009.10.26.eq2}] \quad \begin{array}{ccccc} int'(H(F_1, \ldots, F_n)) & \stackrel{\psi_{(F_1,\ldots,F_n)}}{\longrightarrow} & \Phi(int(F_1, \ldots, F_n)) & \stackrel{\widetilde{\phi}\Phi(Q(F_n))}{\longrightarrow} & \widetilde{U}' \\ \downarrow & & \downarrow & & \downarrow \\ int'(H(F_1, \ldots, F_{n-1})) & \stackrel{\psi_{(F_1,\ldots,F_{n-1})}}{\longrightarrow} & \Phi(int(F_1, \ldots, F_{n-1})) & \stackrel{\phi\Phi(F_n)}{\longrightarrow} & U' \end{array} \quad (12)$$

commutes and

$$[\textbf{2009.10.26.eq7}] \widetilde{\phi}\Phi(Q(F_n))\psi_{(F_1,\ldots,F_n)} = Q(\phi\Phi(F_n)\psi_{(F_1,\ldots,F_{n-1})}) \quad (13)$$

and this morphism is an isomorphism.

To define H on morphism we use the fact that morphisms $\psi_{(F_1,\ldots,F_n)}$ are isomorphisms and for $f : (F_1, \ldots, F_n) \to (G_1, \ldots, G_m)$ we set

$$[\textbf{2009.10.26.eq6}] H(f) = \psi^{-1}_{(G_1,\ldots,G_m)} \Phi(f) \psi_{(F_1,\ldots,F_n)} \quad (14)$$

The fact that this construction gives a functor i.e. satisfies the unity and composition axioms is straightforward.

It remains to verify that this functor respects the contextual structure. It is clear that it respects the length function and the ft maps. The fact that it takes the canonical projections to canonical projections is equivalent to the commutativity of the left hand side square in (12).

Consider a canonical square of the form (10). Its image is a square of the form

$$
\textbf{[2009.10.26.eq4]} \quad \begin{array}{ccc} (G'_1,\ldots,G'_n,G'_{n+1}) & \xrightarrow{H(q(f))} & (F'_1,\ldots,F'_{m+1}) \\ {\scriptstyle H(p_G)}\downarrow & & \downarrow{\scriptstyle H(p_F)} \\ (G'_1,\ldots,G'_n) & \xrightarrow{H(f)} & (F'_1,\ldots,F'_m) \end{array} \qquad (15)
$$

We already know that the vertical arrows are canonical projections. Therefore, in order to prove that (15) is a canonical square in $CC(\mathcal{C}',p')$ we have to show that $G'_{n+1} = F'_{m+1}int(H(f))$ and

$$
\textbf{[2009.10.26.eq8]} Q(F'_{m+1})int(H(q(f))) = Q(F'_{m+1}int(H(f))) \qquad (16)
$$

By (11) we have

$$
G'_{n+1} = \phi\Phi(F_{m+1}f)\psi_{(G_1,\ldots,G_n)}
$$
$$
F'_{m+1} = \phi\Phi(F_{m+1})\psi_{(F_1,\ldots,F_m)}
$$

and by (14)

$$
int(H(f)) = \psi^{-1}_{(F_1,\ldots,F_m)}\Phi(f)\psi_{(G_1,\ldots,G_n)}
$$
$$
int(H(q(f))) = \psi^{-1}_{(F_1,\ldots,F_{m+1})}\Phi(q(f))\psi_{(G_1,\ldots,G_n,F_{m+1}f)}
$$

Therefore the relation $G'_{n+1} = F'_{m+1}int(H(f))$ follows immediately and the relation (16) follows by application of (13).

Our construction of H shows that if Φ is a full embedding and ϕ and $\widetilde{\phi}$ are isomorphisms then H is an isomorphism of contextual categories. This implies in particular that considered up to a canonical isomorphism $CC(\mathcal{C},p)$ depends only on the equivalence class of the pair $(\mathcal{C},p)$ i.e. that our construction maps level 1 pairs $(\mathcal{C},p)$ to contextual categories which are set level.

Note: for $p:\widetilde{U}\to U$ for which pull-backs exist the notion of a p-small object - an object X such that $X\to pt$ is a composition of morphisms which are pull-backs of p.

Π-universes in lcc categories. Recall that a (level 1) category $\mathcal{C}$ is called a lcc (locally Cartesian closed) category if it has fiber products and all the over-categories $\mathcal{C}/X$ have internal Hom-objects.

Definition 0.3 *[**2009.10.27.def1**] Let $\mathcal{C}$ be an lcc category and let $p_i:\widetilde{U}_i\to U_i$, $i=1,2,3$ be three morphisms in $\mathcal{C}$. A Π-structure on (p_1,p_2,p_3) is a Cartesian square of the form*

$$
\textbf{[Pisq1]} \quad \begin{array}{ccc} \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U_2}) & \xrightarrow{\widetilde{P}} & \widetilde{U}_3 \\ {\scriptstyle p'_2}\downarrow & & \downarrow{\scriptstyle p_3} \\ \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2) & \xrightarrow{P} & U_3 \end{array} \qquad (17)
$$

such that p'_2 is the natural morphism defined by p_2. A Π-structure on $p:\widetilde{U}\to U$ is a Π-structure on (p,p,p).

Remark 0.4 A Π-structure on (p_1,p_2,p_3) corresponds to the rule

$$
\frac{\Gamma, X:U_1, f:X\to U_2 \vdash}{\Gamma, X:U_1, f:X\to U_2\vdash \prod x:X.ev(f,x):U_3}
$$

Let $\mathcal{C}$ be as above, $p : \widetilde{U} \to U$ and let $(\widetilde{P}, P)$ be a Π-structure on (p,p,p). Let us construct a structure of Π-contextual category on $CC = CC(\mathcal{C}, p)$.

We start by recalling some level 1 constructions in $\mathcal{C}$.

Lemma 0.5 *[**2009.11.24.l5**] Consider a pair of pull back squares*

[**2009.11.24.eq3**]
$$\begin{array}{ccccccc} I_2 & \xrightarrow{\widetilde{F}_1} & \widetilde{U}_1 & \qquad & I_3 & \xrightarrow{\widetilde{F}_2} & \widetilde{U}_2 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_1} & & {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ I_1 & \xrightarrow{F_1} & U_1 & & I_2 & \xrightarrow{F_2} & U_2 \end{array} \tag{18}$$

Then there exists a unique morphism $f_{F_1,F_2} : I_1 \to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2)$ such that its composition with the natural morphism to U_1 is F_1 and the composition of its adjoint

$$ev \circ (f_{F_1,F_2} \times_{U_1} \widetilde{U}_1) : I_2 = I_1 \times_{U_1} \widetilde{U}_1 \to U_1 \times U_2$$

with the projection to U_2 is F_2.

Proof: Follows immediately from the definition of internal Hom-objects.

Lemma 0.6 *[**2009.11.24.l3**] In the notation of Lemma 0.5 let*

$$\begin{array}{ccccccc} J_2 & \xrightarrow{\phi_2} & I_2 & \qquad & J_3 & \xrightarrow{\phi_3} & I_3 \\ \downarrow & & \downarrow{\scriptstyle q_1} & & \downarrow & & \downarrow{\scriptstyle q_2} \\ J_1 & \xrightarrow{\phi_1} & I_1 & & J_2 & \xrightarrow{\phi_2} & I_2 \end{array}$$

be two pull-back squares. Then $f_{F_1\phi_1, F_2\phi_2} = f_{F_1,F_2} \circ \phi_1$.

Proof: Straightforward.

Let $p_1 : \widetilde{U}_1 \to U_1$, $p_2 : \widetilde{U}_2 \to U_2$ be a pair of morphisms in an lcc $\mathcal{C}$. Consider a pull-back square of the form

[**2009.11.24.eq4**]
$$\begin{array}{ccc} Fam_2(p_1,p_2) & \longrightarrow & \widetilde{U}_2 \\ {\scriptstyle p_{12}}\downarrow & & \downarrow{\scriptstyle p_2} \\ \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} & \xrightarrow{pr \circ ev} & U_2 \end{array} \tag{19}$$

where

$$ev : \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} \to U_1 \times U_2$$

is the canonical morphism.

Then for any two pull-back squares as in Lemma 0.5, the morphism f_{F_1,F_2} defines factorizations of the pull-back squares (18) of the form

$$\begin{array}{ccccc} I_2 & \xrightarrow{f_{F_1,F_2} \times_{U_1} \widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) \times_{U_1} \widetilde{U_1} & \xrightarrow{pr} & \widetilde{U}_1 \\ {\scriptstyle q_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_1} \\ I_1 & \xrightarrow{f_{F_1,F_2}} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) & \longrightarrow & U_1 \end{array}$$

and

$$\begin{array}{ccccc}
I_3 & \longrightarrow & Fam_2(p_1,p_2) & \longrightarrow & \widetilde{U}_2 \\
{\scriptstyle q_2}\big\downarrow & & \big\downarrow{\scriptstyle p_{12}} & & \big\downarrow{\scriptstyle p_2} \\
I_2 & \xrightarrow{f_{F_1,F_2}\times_{U_1}\widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U_1} & \xrightarrow{pr\circ ev} & U_2
\end{array}$$

respectively and joining the left hand side squares of these diagrams we get a diagram with pull-back squares of the form

$$\begin{array}{ccc}
I_3 & \longrightarrow & Fam_2(p_1,p_2) \\
{\scriptstyle q_2}\big\downarrow & & \big\downarrow{\scriptstyle p_{12}} \\
I_2 & \xrightarrow{f_{F_1,F_2}\times_{U_1}\widetilde{U}_1} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U_1} \\
{\scriptstyle q_1}\big\downarrow & & \big\downarrow{\scriptstyle pr} \\
I_1 & \xrightarrow{f_{F_1,F_2}} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)
\end{array}$$

Let

$$g : \underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times\widetilde{U}_2)\times_{U_1}\widetilde{U_1}\rightarrow Fam_2(p_1,p_2)$$

be the morphism over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U_1}$ whose composition with the projection $Fam_2(p_1,p_2)\rightarrow \widetilde{U}_2$ equals $pr\circ\widetilde{ev}$ where

$$\widetilde{ev} : \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2)\times_{U_1}\widetilde{U_1}\rightarrow U_1\times\widetilde{U}_2$$

is the canonical morphism.

Lemma 0.7 *[***2009.11.24.l2***] The pair*

$$(\underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times\widetilde{U}_2)\rightarrow\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2), g)$$

is universal for (p_{12}, pr)*.*

Proof: For a given $w : Z\rightarrow\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)$, a morphism $Z\rightarrow\underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times\widetilde{U}_2)$ over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)$ is the same as a morphism $Z\times_{U_1}\widetilde{U}_1\rightarrow\widetilde{U}_2$ such that the adjoint of its composition with $p_2 : \widetilde{U}_2\rightarrow U_2$ is w.

A morphism from Z to the universal pair for p_{12} over $\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)$ is a morphism $Z\times_{U_1}\widetilde{U_1}\rightarrow\widetilde{U}_2$ whose composition with p_2 is $(pr\circ ev)\circ(w\times_{U_1} Id_{\widetilde{U_1}})$ which coincides with the condition that the composition of its adjoint with p_2 is w. This can be also seen from the diagram

$$\begin{array}{ccccc}
 & & Fam_2(p_1,p_2) & \longrightarrow & \widetilde{U}_2 \\
 & & {\scriptstyle p_{12}}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\
\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2)\times_{U_1}\widetilde{U}_1 & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2)\times_{U_1}\widetilde{U}_1 & \xrightarrow{pr\circ ev} & U_2 \\
\big\downarrow & & \big\downarrow{\scriptstyle pr} & & \\
\underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2) & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U_1}, U_1\times U_2) & &
\end{array}$$

Lemma 0.8 *[2009.11.24.l4] For two pull back squares as in (18), consider a pull-back square of the form*

$$\begin{array}{ccc} R(F_1,F_2) & \longrightarrow & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2) \\ \downarrow & & \downarrow \\ I_1 & \xrightarrow{f_{F_1,F_2}} & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times U_2) \end{array}$$

and the morphism

$$g_{F_1,F_2} : R(F_1,F_2)\times_{I_1} I_2 \to I_3$$

whose composition with the morphism $I_3\to\widetilde{U}_2$ coincides with the composition

$$R(F_1,F_2)\times_{I_1} I_2 = R(F_1,F_2)\times_{U_1}\widetilde{U}_1 \to \underline{Hom}_{U_1}(\widetilde{U}_1, U_1\times\widetilde{U}_2)\times_{U_1}\widetilde{U}_1 \stackrel{proev}{\to} \widetilde{U}_2$$

Then $(R(F_1,F_2), g_{F_1,F_2})$ is a universal pair for (q_1,q_2).

Proof: It follows from Lemma 0.7 and the fact that in a lcc a pull-back of a universal pair is a universal pair.

Let us now construct a Π-contextual structure on $CC = CC(\mathcal{C}, p)$. Let $n\geq 2$ and $(F_1,\dots,F_n)\in CC$. Denote $(pt, F_1,\dots,F_{n-2})$ by I. Then we have two morphisms $F_{n-1} : I\to U$ and $F_n : (I, F_{n-1})\to U$.

Applying Lemma 0.5 to the corresponding pull-back squares we get a morphism

$$f_{F_{n-1},F_n} : I\to \underline{Hom}_U(\widetilde{U}, U\times U)$$

Set $\Pi(F_1,\dots,F_n) = (I, P\circ f_{F_{n-1},F_n}) = (F_1,\dots,F_{n-2}, P\circ f_{F_{n-1},F_n})$. Since the square (17) is a pull-back square there is a unique morphism $\Pi(F_1,\dots,F_n)\to \underline{Hom}_U(\widetilde{U}, U\times\widetilde{U})$ such that the diagram

$$\begin{array}{ccccc} \Pi(F_1,\dots,F_n) & \longrightarrow & \underline{Hom}_U(\widetilde{U}, U\times\widetilde{U}) & \xrightarrow{\widetilde{P}} & \widetilde{U} \\ \downarrow & & \downarrow & & \downarrow \\ I & \xrightarrow{f_{F_{n-1},F_n}} & \underline{Hom}_U(\widetilde{U}, U\times U) & \xrightarrow{P} & U \end{array}$$

commutes and the composition of the two upper arrows is $Q(f_{F_{n-1},F_n})$. The left hand side square in this diagram is automatically a pull-back square. Applying to this square Lemma 0.8 we obtain a morphism

$$eval_{(F_1,\dots,F_n)} : (I, F_{n-1}, (P\circ f_{F_{n-1},F_n})\circ pr)\to (I, F_{n-1}, F_n)$$

over (I, F_{n-1}) (where $pr : (I, F_{n-1})\to I$ is the projection).

The fact that this construction satisfies the first condition of Definition 2.2 follows from Lemma 0.6. The fact that it satisfies the second condition of this definition follows from Lemma 0.8.

Σ-universes in lcc categories.

Definition 0.9 *[***2009.10.27.def2***] Let $\mathcal{C}$ be an lcc category and $p_i : \widetilde{U}_i \to U_i$, $i = 1, 2, 3$ be three morphisms in $\mathcal{C}$. A Σ-structure on (p_1, p_2, p_3) is a diagram of the form*

$$\begin{array}{ccccc}
\widetilde{U}_2 & \longleftarrow & Fam_{\bullet}(U_1, U_2) & \longrightarrow & \widetilde{U}_3 \\
{\scriptstyle p_2}\downarrow & & \downarrow & & \\
U_2 & \xleftarrow{pr_{U_2} eval} & \widetilde{\widetilde{U}_1} \times_{U_1} \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) & & \downarrow{\scriptstyle p_3} \\
 & & \downarrow{\scriptstyle pr} & & \\
 & & \underline{Hom}_{U_1}(\widetilde{U}_1, U_1 \times U_2) & \xrightarrow{\Sigma} & U_3
\end{array}$$

such that p_2' is the natural morphism defined by p_2, eval is the canonical evaluation morphism and both the square and the vertical rectangle are Cartesian. A Σ-structure on $p : \widetilde{U} \to U$ is a Σ-structure on (p, p, p).

A Σ-structure on (p_1, p_2, p_3) corresponds to the rule

$$\frac{\Gamma, X : U_1, f : X \to U_2 \vdash}{\Gamma, X : U_1, f : X \to U_2 \vdash \sum x : X.ev(f, x) : U_3}$$

Definition 0.10 *[***2009.11.2.def1***] Let $\mathcal{C}$ be an lcc category and $p : \widetilde{U} \to U$ be a morphism. A Prop-structure on p is a collection of data of the following form:*

1. *two pull-back squares*

$$\begin{array}{ccc}
P & \longrightarrow & \widetilde{U} \\
\downarrow & & \downarrow{\scriptstyle p} \\
pt & \longrightarrow & U
\end{array}
\qquad\qquad
\begin{array}{ccc}
\widetilde{P} & \longrightarrow & \widetilde{U} \\
{\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\
P & \longrightarrow & U
\end{array}$$

2. *a Π-structure on (p, p_0, p_0).*

A *Prop*-structure on p corresponds to the rules:

$$\frac{}{x : P \vdash} \qquad \frac{}{x : P, y : x \vdash} \qquad \frac{\Gamma, f : X \to P \vdash}{\Gamma, f : X \to P \vdash \prod x : X.ev(f, x) : P}$$

3 Universes in the category of simplicial sets

1 Well-ordered morphisms of simplicial sets

Let X, Y be simplicilal sets. A well-ordered morphism $p : Y \to X$ is a pair which consists of a morphism $Y \to X$ (also denoted by p) and of a function which assigns to each $n \geq 0$ and each $\sigma \in X_n$ a well-ordering on $p^{-1}(\sigma) \subset Y_n$.

Note that there is a unique well-ordering on any isomorphism but, for example, the morphism $pt \amalg pt \to pt$ has uncountably many well-orderings since $pt_n = pt$ for all n and we require no compatibility conditions for well orderings of the fibers over different simplexes of the target.

If $p : Y \to X$, $p' : Y' \to X$ are two well-ordered morphisms then we define a standard isomorphism from Y to Y' over X as an isomorphism over X such that for each $n \geq 0$ and each $\sigma \in X_n$ the bijection $p^{-1}(\sigma) \to (p')^{-1}(\sigma)$ is order-preserving. Since there is at most one order-preserving bijection between two well-ordered sets, there is at most one standard isomorphism between two well-ordered simplicial sets over X.

Let $WOM(X, < \alpha)$ be the set of standard isomorphism classes of well-ordered simplicial sets $p : Y \to X$ over X such that for each $n \geq 0$ and each $\sigma \in X_n$ the fiber $p^{-1}(\sigma)$ has cardinality $< \alpha$. For any $f : X' \to X$ the pull-back $p' : Y' = X' \times_X Y \to X'$ of a well-ordered morphism has a natural well-ordering which makes $WOM(X, < \alpha)$ into a functor from $\Delta^{op}Sets$ to $Sets$.

Consider $WOM(\Delta^n, < \alpha)$. These sets depend on Δ^n functorially and therefore define a simplicial set $WOM(< \alpha)$. Let $\widetilde{WOM}(\Delta^n, < \alpha)$ be the set of pairs $p : Y \to \Delta^n$, $s \in Y_n$ where $p \in WOM(\Delta^n, < \alpha)$ and $s \in p^{-1}(\sigma_n)$ where σ_n is the non-degenerate n-simplex of Δ^n. These sets also depend on Δ^n functorially and define a simplicial set $\widetilde{WOM}(< \alpha)$.

Since $p^{-1}(\sigma)$ carries a well-ordering the natural projection $\widetilde{WOM}(< \alpha) \to WOM(< \alpha)$ carries a natural well-ordering.

Proposition 1.1 *[***2009.12.10.pr1***] The morphism $\widetilde{WOM}(< \alpha) \to WOM(< \alpha)$ is a universal well-ordered morphism with fibers of cardinality $< \alpha$. In particular, $WOM(< \alpha)$ represents the functor $WOM(-, < \alpha)$.*

Proof: Straightforward.

Note that $WOM(< \alpha)$ is obviously a contractible Kan simplicial set for any $\alpha > 0$.

Let us consider now the sub-object $WOF(< \alpha)$ of $WOM(< \alpha)$ which classifies well-ordered Kan fibrations whose fibers have cardinality $< \alpha$ and let $\widetilde{WOF}(< \alpha) \to WOF(< \alpha)$ be the corresponding universal fibration.

The idea of the proof of the following result and in general the idea to use minimal fibrations is due to A. Bousfield and reached me through Peter May and Rick Jardine.

Proposition 1.2 *[***2009.12.8.prop1***] Let α be an infinite cardinal. Then the simplicial set $WOF(< \alpha)$ is Kan.*

Proof: One can easily see that it is sufficient to show that for any horn inclusion $\Lambda^n_k \to \Delta^n$ and any Kan fibration $p : B \to \Lambda^n_k$ there exists a pull-back square of the form

$$\textbf{[2009.12.8.eq1]} \quad \begin{array}{ccc} B & \longrightarrow & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \Lambda^n_k & \longrightarrow & \Delta^n_k \end{array} \qquad (20)$$

where q is a Kan fibration whose fibers have cardinality $< \alpha$. By Quillen's Lemma ([**?**]) there is a factorization of p of the form $B \stackrel{p'}{\to} B' \stackrel{p''}{\to} \Lambda^n_k$ where p' is a trivial fibration and p'' is a minimal fibration. Since trivial fibrations are surjective, both p' and p'' have fibers of cardinality $< \alpha$. By

[?, Cor. 11.7, p.45] the fibration p'' is isomorphic to a fibration $F\times\Lambda^n_k\to\Lambda^n_k$ where F is a Kan simplicial set. Together with Lemma 1.4 it shows that there is a diagram of the form

$$\begin{array}{ccc} B & \longrightarrow & C \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle q'} \\ F\times\Lambda^n_k & \longrightarrow & F\times\Delta^n \\ \downarrow & & \downarrow \\ \Lambda^n_k & \longrightarrow & \Delta^n \end{array}$$

with pull-back squares such that q' is a trivial fibration with fibers of cardinality $<\alpha$. The external square of this diagram has the required form (20).

Lemma 1.3 *[**2009.12.11.l1**] Let α be an infinite cardinal. Let $p: Y\to X$ be a map of simplicial sets such that for each $n\ge 0$, $x\in X_n$ one has $|p^{-1}(x)\cap Y_n^{nd}|<\alpha$ where Y_n^{nd} is the subset of non-degenate simplexes in Y_n. Then for each $n\ge 0$, $\sigma\in X_n$ one has $|p^{-1}(x)|<\alpha$.*

Proof: Since for any surjection s the map $s^*: X_m\to X_n$ is an inclusion and there are only finitely many surjections of the form $[n]\to[m]$ (where $[n]=\{0,\dots,n\}$) there exists only finitely many pair-wise distinct pairs (x_i,s_i) where $x_1,\dots,x_d\in X_{m_i}$ and $s:[n]\to[m_i]$ is a surjection, such that $s_i^*(x_i)=x$.

Consider the map

$$[\mathbf{2009.12.11.eq1}]\ \amalg_i s_i^*:\amalg_i(p^{-1}(x_i)\cap Y_{m_i}^{nd})\to p^{-1}(x) \tag{21}$$

If $y\in p^{-1}(x)$ then there exists $s:[n]\to[m]$ and $y'\in Y_m^{nd}$ such that $s^*(y')=y$. Then $s^*p(y')=p\,s^*(y')=x$ and therefore $s=s_i$ for $i=1,\dots,d$. We conclude that the map (21) is surjective and therefore $|p^{-1}(x)|<\alpha$.

Lemma 1.4 *[**2009.12.8.l4**] Let $\alpha>\aleph_0$ be an cardinal. Let $j:A\to X$ be a cofibration (monomorphism) and $p:B\to A$ be a trivial Kan fibration with fibers of cardinality $<\alpha$. Then there exists a pull-back square of the form*

$$[\mathbf{2009.12.8.eq2}]\quad \begin{array}{ccc} B & \longrightarrow & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{j} & X \end{array} \tag{22}$$

such that q is a trivial Kan fibration with fibers of cardinality $<\alpha$.

Proof: Define inductively squares

$$[\mathbf{2009.12.8.eq3}]\quad \begin{array}{ccc} B & \longrightarrow & B_i \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_i} \\ A & \xrightarrow{j} & X \end{array} \tag{23}$$

setting $p_0=p$ and defining B_{i+1} by the push-out square of the form

$$[\mathbf{2009.12.11.eq2}]\quad \begin{array}{ccc} \amalg_n\amalg_{Q_{n,i}}\partial\Delta^n & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ \amalg_n\amalg_{Q_{n,i}}\Delta^n & \longrightarrow & B_{i+1} \end{array} \tag{24}$$

where $Q_{n,i}$ is the set of commutative squares of the form

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & X \end{array}$$

such that its base simplex i.e. the simplex corresponding to the map $\Delta^n \to X$ does not belong to A.

Since for such a map f one has $f^{-1}(A) \subset \partial\Delta^n$ the squares (23) are pull-back squares. Define Y as $colim\, B_i$. Then one verifies easily that (22) is a pull-back square and q is a fibration. Let us show that the fibares of q have cardinality $< \alpha$. Since $\alpha > \aleph_0$ it is sufficient to show that, assuming that the fibers of $B_i \to X$ are of cardinality $< \alpha$, the fibers of B_{i+1} are. The squares (24) show that for each n and $x \in X_n$ the fiber $p_{i+1}^{-1}(x) \cap (B_{i+1})_n^{nd}$ is of the form $(p_i^{-1}(x) \cap (B_i)_n^{nd}) \amalg Q(n,i;x)$ where $Q(n,i;x)$ is the subset in $Q(n,i)$ which consists of squares whose base simplex $\Delta^n \to X$ is x. It remains to observe that the number of such squares is $< \alpha^{n+1}$ and to apply Lemma 1.3.

The category $\Delta^{op}Sets$ is a topos and in particular an lcc. The relative internal Hom-objects in $\Delta^{op}Sets$ can be explicitly described as follows.

Lemma 1.5 *[2009.12.8.l5] Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be morphisms of simplicial sets. Consider the simplicial set $H(p_1, p_2)$ whose set of n-simplexes is the set of pairs of the form $(f, \widetilde{f})$ where $f : \Delta^n \to B$ and $\widetilde{f} : f^*(p_1) \to p_2$ is a morphism over B.*

Let $H(p_1, p_2) \to B$ be the morphism $ev : (f, \widetilde{f}) \mapsto f$ and let $H(p_1, p_2) \times_B E_1 \to E_2$ be the morphism which sends $((f, \widetilde{f}), \sigma)$ to $\widetilde{f}(\sigma)$. Then $(H(p_1, p_2), ev)$ is an internal Hom-object from E_1 to E_2 over B.

Lemma 1.6 *[2009.12.8.l6] Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be Kan fibrations. Then $H(p_1, p_2) \to B$ is a Kan fibration.*

Proof: It follows immediately from definitions and the fact that for a fibration $p_1 : E_1 \to B$ and an anodyne morphism $A \to X$ over B, the morphism $A \times_B E_1 \to X \times_B E_1$ is anodyne.

Lemma 1.7 *[2009.12.9.l1] Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be Kan fibrations and $f : E_1 \to E_2$ a morphism over B which is a weak equivalence. Then for any $g : B' \to B$ the pull-back $f' : B' \times_B E_1 \to B' \times_B E_2$ is a weak equivalence.*

Proof: Using the factorization of f into a trivial cofibration and a trivial fibration and the fact that the pull-back of a trivial fibration is a trivial fibration we may assume that f is a trivial cofibration. A trivial cofibration between two fibrant objects (in the category over B) is a homotopy equivalence and the pull-back of a homotopy equivalence is a homotopy equivalence.

Lemma 1.8 *[2009.12.9.l3] Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ be Kan fibrations and $f : E_1 \to E_2$ a morphism over B. Suppose that for any $n \geq 0$ and any simplex $\sigma : \Delta^n \to B$ the pull-back $f_\sigma : \Delta^n \times_B E_1 \to \Delta^n \times_B E_2$ is a weak equivalence. Then f is a weak equivalence.*

Proof: Replacing p_1, p_2 by minimal fibrations we may assume that p_1, p_2 are minimal. Then our condition implies that f_σ is an isomorphism for each σ and therefore is an isomorphism globally.

Let p_1, p_2 be Kan fibrations as above. Consider the internal Hom-object $H(p_1, p_2)$. A morphism $f : A \to H(p_1, p_2)$ defines a morphism $pr(f) : A \to B$ and a morphism $fib(f) : A \times_B E_1 \to A \times_B E_2$. Let $Eq(p_1, p_2)_n$ be the subset of simplexes $\sigma : \Delta^n \to H(p_1, p_2)$ such that $fib(\sigma)$ is a weak equivalence. Lemma 1.7 implies that these subsets form a simplicial subset in $H(p_1, p_2)$ which we denote by $Eq(p_1, p_2)$ or $Eq_B(p_1, p_2)$.

Lemma 1.9 *[***2009.12.9.l2***] Let p_1, p_2 be Kan fibrations as above and $f : A \to H(p_1, p_2)$ a morphism. The $fib(f)$ is a weak equivalence if and only if $Im(f) \subset Eq(p_1, p_2)$.*

Proof: Straightforward using Lemmas 1.7 and 1.8.

Lemma 1.10 *[***2009.12.9.l4***] Let p_1, p_2 be Kan fibrations as above, $f : E_1 \to E_2$ a morphism over B and $b \in B$. Assume that B is connected and that $p_1^{-1}(b) \to p_2^{-1}(b)$ is a weak equivalence. Then f is a weak equivalence.*

Proof: In view of Lemma 1.8 we may assume that $B = \Delta^n$. Since the pull-back of a weak equivalence along a fibration is a weak equivalence and $b : \Delta^0 \to \Delta^n$ is a weak equivalence, we conclude that $p_1^{-1}(b) \to E_1$ and $p_2^{-1}(b) \to E_2$ are weak equivalences. Therefore, if $f_b : p_1^{-1}(b) \to p_2^{-1}(b)$ is a weak equivalence then so is f.

Lemma 1.11 *[***2009.12.9.l5***] Let p_1, p_2 be Kan fibrations as above. Then $Eq(p_1, p_2)$ is a union of connected components of $H(p_1, p_2)$ i.e. if (A, a) is a connected pointed simplicial set and $f : A \to H(p_1, p_2)$ a morphism such that $f(a) \in Eq(p_1, p_2)$ then $Im(f) \subset Eq(p_1, p_2)$.*

Proof: Follows immediately from Lemma 1.10.

Let $p : E \to B$ be a fibration. Let $p_1 : E \times B \to B \times B$ and $p_2 : B \times E \to B \times B$ be the obvious projections. Consider the space $H(p_1, p_2)$ over $B \times B$. The natural isomorphism $p_1^{-1}(\Delta(B)) = p_2^{-1}(\Delta(B))$ where Δ is the diagonal defines a morphism $B \to H(p_1, p_2)$ over $B \times B$ which, by Lemma 1.9, takes values in $Eq(p_1, p_2)$. Let us denote this morphism by $mm_p : B \to Eq(p_1, p_2)$.

Definition 1.12 *[***2009.12.9.def1***] A Kan fibration $p : E \to B$ is called univalent if the morphism $mm_p : B \to Eq(p_1, p_2)$ defined above is a weak equivalence.*

Theorem 1.13 *[***2009.12.9.th1***] The Kan fibration*

$$p_{fib} : \widetilde{WOF}(< \alpha) \to WOF(\alpha)$$

is univalent.

Proof: Let $E = \widetilde{WOF}(< \alpha)$ and $B = WOF(< \alpha)$. Let $P_1 : E \times B \to B \times B$, $P_2 : B \times E \to B \times B$ be the projections. Proposition 1.1 implies easily that the space $H(P_1, P_2)$ represents the functor which sends X into the set of (standard isomorphism classes of) triples of the form $p_1 : Y_1 \to X$,

$p_2 : Y_2 \to X$, $f : Y_1 \to Y_2$ where p_1, p_2 are well ordered Kan fibrations with fibers of cardinality $< \alpha$ and f is a morphism over X. The subspace $Eq(P_1, P_2)$ classifies triples such that f is a weak equivalence.

Consider now the morphism $r : B \to Eq(P_1, P_2) \to B \times B \stackrel{pr_2}{\to} B$. To prove the theorem it is sufficient to show that the composition $Eq(P_1, P_2) \to B \to Eq(P_1, P_2)$ is homotopic to the identity. This composition represents the functor morphism which sends (p_1, p_2, f) to (p_2, p_2, id).

Applying Lemma 1.14 to the universal equivalence of fibrations over $Eq(P_1, P_2)$ and using the axiom of choice we construct the required homotopy.

Lemma 1.14 *[**2009.12.11.l3**] Let $p_1 : Y_1 \to X$, $p_2 : Y_2 \to X$ be two Kan fibrations and $f : Y_1 \to Y_2$ be a morphism over X which is a weak equivalence. Then there exists a fibration $q : Z \to X \times \Delta^1$ and a morphism $F : Z \to Y_2 \times \Delta^1$ over X such that the fiber of F over $X \times \{0\}$ is isomorphic to f and the fiber over $X \times \{1\}$ is isomorphic to Id_{Y_2}.*

In addition if $\alpha > \aleph_0$ is a cardinal and the fibers of p_1 and p_2 have cardinality $< \alpha$ then we can choose q such that its fibers have cardinality $< \alpha$.

Proof: Let $Y_1 \stackrel{p_1'}{\to} Y_1' \stackrel{p_1''}{\to} X$, $Y_1 \stackrel{p_1'}{\to} Y_1' \stackrel{p_1''}{\to} X$ be factorizations of p_1 and p_2 such that p_i' is a trivial fibration and p_i'' a minimal fibration which exist by [**?**]. If s_1 is a section of p_1' (which exist since all simplicial sets are cofibrant) then $p_2' f\, s$ is a weak equivalence between two minimal fibrations over X and therefore an isomorphism. Let us denote it by $f' : Y_1' \to Y_2'$.

Applying Lemma 1.4 to the trivial fibration $p_1' \amalg p_2' : Y_1 \amalg Y_2 \to Y_1' \amalg Y_2'$ and monomorphism $j = (i_0\, f' \amalg i_1) : Y_1' \amalg Y_2' \to Y_2' \times \Delta^1$ we obtain a pull-back square of the form

$$\begin{array}{ccc} Y_1 \amalg Y_2 & \stackrel{k}{\longrightarrow} & Z \\ {\scriptstyle p_1' \amalg p_2'}\downarrow & & \downarrow{\scriptstyle q} \\ Y_1' \amalg Y_2' & \stackrel{j}{\longrightarrow} & Y_2' \times \Delta^1 \end{array}$$

Consider now the square

$$\begin{array}{ccc} Y_1 \amalg Y_2 & \stackrel{i_0\, f \amalg i_1}{\longrightarrow} & Y_2 \times \Delta^1 \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle p_2' \times Id} \\ Z & \underset{q}{\longrightarrow} & Y_2' \times \Delta^1 \end{array}$$

Since k is a cofibration (monomorphism) and $p_2' \times Id$ is a trivial fibration, there is a morphism $F : Z \to Y_2 \times \Delta^1$ which splits this square into two commutative triangles. One verifies easily that the pair (Z, F) satisfies the conditions of the lemma.

Let $p' : E \to B$, $p : \widetilde{U} \to U$ be two Kan fibrations. For a simplicial set X denote by $HInd(p', p)(X)$ the set of pairs $(\widetilde{f}, f)$ where $\widetilde{f} : E \times X \to \widetilde{U}$, $f : B \times X \to \widetilde{U}$ are morphisms such that the square

[2009.12.23.eq1]
$$\begin{array}{ccc} E \times X & \stackrel{\widetilde{f}}{\longrightarrow} & \widetilde{U} \\ {\scriptstyle p \times Id_X}\downarrow & & \downarrow{\scriptstyle p} \\ B \times X & \stackrel{f}{\longrightarrow} & U \end{array} \qquad (25)$$

is a homotopy pull-back square i.e. such that $p \circ \widetilde{f} = f \circ (p' \times Id_X)$ and the obvious morphism $E \times X \to (B \times X) \times_U \widetilde{U}$ is a weak equivalence. Since p' and p are fibrations, the composition of a homotopy pull-back square of the form (25) with a pull-back square

$$\begin{array}{ccc} E \times X' & \longrightarrow & E \times X \\ \downarrow & & \downarrow \\ B \times X' & \longrightarrow & B \times X \end{array}$$

defined by any morphism $f : X' \to X$, is a homotopy pull-back square. Therefore $HInd(p', p)(-)$ is a contravariant functor on $\Delta^{op}Sets$ and Lemma 1.8 implies easily that it is represented by the simplicial set $HInd(p', p)$ whose set of n-simplexes is $HInd(p', p)(\Delta^n)$.

Proposition 1.15 *[***2009.12.23.prop1***] A Kan fibration $p : \widetilde{U} \to U$ such that U is a Kan simplicial set is univalent if and only if for any Kan fibration $p' : E \to B$, $(HInd(p', p) \neq \emptyset) \Rightarrow (HInd(p', p)$ is contractible).*

Proof: Let $p' : E \to B$ be a Kan fibration such that $HInd(p', p) \neq \emptyset$ i.e. such that there exists a pull-back square of the form

$$\begin{array}{ccc} E & \xrightarrow[\widetilde{f}]{} & \widetilde{U} \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \xrightarrow[f]{} & U \end{array}$$

Let X be a simplicial set. Then a morphism $X \to HInd(p', p)$ is given by a pair of a morphism $f_X : B \times X \to U$ and a weak equivalence $E \times X \to (B \times X)_{f_X} \times_p \widetilde{U}$ over $B \times X$. The morphism $E \times X \to B \times X$ is canonically isomorphic to the projection $(B \times X)_{f \circ pr_B} \times_p \widetilde{U} \to B \times X$. Therefore, morphisms $X \to HInd(p', p)$ correspond to morphisms $B \times X \to Eq(p \times Id_U, Id_U \times p)$ whose composition with $Eq(p \times Id_U, Id_U \times p) \stackrel{p_{Eq}}{\to} U \times U \stackrel{pr_2}{\to} U$ equals $f \circ pr_B : B \times X \to B \stackrel{f}{\to} U$.

Since U is assumed to be a Kan simplicial set the morphism $pr_2 \circ p_{Eq}$ is a Kan fibration. If p is univalent it is a trivial Kan fibration and from the previous description of $HInr(p', p)$ we conclude that for any cofibration $X \subset Y$ and a morphism $F = (\widetilde{f}, f) : X \to HInt(p', p)$ there exists an extension of F to Y i.e. that $HInt(p', p) \to pt$ is a trivial Kan fibration.

To prove the other implication consider the case when $B = pt$. Then our considerations show that $HInd(p', p)$ is isomorphic to the fiber of $pr_2 \circ p_{Eq}$ over $f(pt) \in U_0$. Since any Kan fibrations with contractible fibers is a trivial Kan fibration we conclude that the required implication holds.

2 Well-ordered simplicial sets

We consider a triple (ST, ST', M) where ST, ST' are ZF-like set-theories and M is a model of ST and ST'. These data defines "the set of all ST sets" as an ST'-set. Similarly, these data provides an unambiguous definition for objects such as "the set of isomorphism classes of simplicial sets" etc.

Our first step is to choose a convenient set-level model of the 1-category of simplicial sets.

Definition 2.1 *[***2009.12.8.def1***] A well-ordered simplicial set is a simplicial set $(X_n)_{n \geq 0}$ together with well orderings $\prec$ on each of X_n.*

Note that the well orderings on X_n are note assumed to be compatible with the boundary or degeneracy maps. By a morphism between two well-ordered simplicial sets we will mean a morphism between the corresponding simplicial sets without any regard for orderings. A morphism which preserves well-orderings on each of X_n will be called a standard morphism.

The standard facts about well-ordered sets imply that there is at most one standard isomorphism between any two well-ordered simplicial sets. Therefore, we may consider a set level model C of $\Delta^{op}Sets$ where $ob(C)$ is the set of standard isomorphism classes of well-ordered simplicial sets and $Mor(X,Y)$ is the set of all morphisms from X to Y. The uniqueness of standard isomorphisms implies that the composition of morphisms is well defined.

For well-ordered simplicial sets X, Y denote by $X \times Y$ the well-ordered simplicial set whose terms $X_n \times Y_n$ are well-ordered with respect to the lexicographical ordering such that the projection to X (but not to Y) is a standard morphism.

For $f : X' \to X$ and $p : Y \to X$ define the standard pull-back square

$$\begin{array}{ccc} f^*(Y,p) & \xrightarrow{q(f,Y,p)} & Y \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f'} & X \end{array}$$

setting $f^*(Y,p)$ to be the subset in $X' \times Y$ defined by the usual equations with the induced well-ordering.

One verifies easily the following results.

Lemma 2.2 *[***2009.12.8.l1***] For any p the morphism p_f is standard and $p : Y \to X$ is standard if and only if $Id_X^*(p) = p$.*

Lemma 2.3 *[***2009.12.8.l2***] For any $p : Y \to X$ and $g : X'' \to X'$, $f : X' \to X$ one has $(fg)^*(p) = g^*f^*(p)$, $q(fg,Y,p) = q(g,f^*(p),p_f)q(f,Y,p)$ and $p_{fg} = (p_f)_g$.*

Note that $q(f,Y,p)$ need not be standard even if both p and f are standard (consider e.g. the case when $X = pt$).

In what follows we choose a well-ordering on the sets Δ_i^n and consider the standard simplexes as objects of C with respect to this ordering.

4 Type systems

1 Systems of expressions

Free systems of expressions. Let M be a set and let $T(M)$ be the set of finite rooted trees whose vertices (including the root) are labeled by elements of M and such that for any vertex the set of edges leaving this vertex is ordered. Note that such ordered trees have no symmetries. We will use the following notations. For $T \in T(M)$ let $Vrtx(T)$ be the set of vertices of T and for $v \in Vrtx(T)$ let $lbl(v) = lbl(v)_T \in M$ be the label on v. We will sometimes write $v \in T$ instead of $v \in Vrtx(T)$. For $v \in Vrtx(T)$ let $[v] = [v]_T \in T(M)$ be the subtree in T which consists of

v and all the vertices under v. Let $val(v)$ be the valency of v i.e. the number of edges leaving v and $ch_1(v), \ldots, ch_{val(v)}(v) \in Vrtx(T)$ be the "children" of v i.e. the end points of these edges. Let further $br_i(v) = [ch_i(v)]$ be the branches of $[v]$. We write $v \leq w$ (resp. $v < w$) if $v \in [w]$ (resp. $v \in [w] - w$). We say that two vertices v and w are independent if $v \notin [w]$ and $w \notin [v]$.

For three sets A, B and $Cont$ let

$$AllExp(A, B; Con) = T(A \amalg B \amalg (Con \times (\amalg_{n \geq 0} B^n)))$$

Elements of $AllExp(A, B; Con)$ are called expressions over the alphabet Con (or with a set of constructors Con), free variables from A and bound variables from B.

An expression is called unambiguous if it satisfies the following conditions:

1. if $lbl(v) \in A \amalg B$ then $val(v) = 0$,
2. (a) if $v < v'$, $lbl(v) = (c; x_1, \ldots, x_n)$ and $lbl(v') = (c'; x'_1, \ldots, x'_{n'})$ then $\{x_1, \ldots, x_n\} \cap \{x'_1, \ldots, x'_{n'}\} = \emptyset$,
   (b) if $lbl(v) = (c; x_1, \ldots, x_n)$ then $x_i \neq x_j$ for $i \neq j$,
3. if $lbl(v) = (c; x_1, \ldots, x_n)$ and $lbl(v') \in \{x_1, \ldots, x_n\}$ then $v' \in [v]$.

The first conditions says that a vertex labeled by a variable is a leaf. The second one is equivalent to saying that if the same variable is bound at two different vertices v, v' then these vertices are independent i.e. $[v] \cap [v'] = \emptyset$ and that a vertex can not bind the same variable twice. The third one says that all the leaves labeled by a bound variable lie under the vertex where it is boud. We let $UAExp(A, B; Con)$ denote the subset of unambiguous expressions in $AllExp(A, B; Con)$. Note that for for any $T \in UAExp(A, B; Con)$ and $v \in Vrtx(T)$ there is a subset $Ext(v) \subset B$ such that

$$[v] \in UAExp(A \amalg Ext(v), B \backslash Ext(v); Con)$$

Any triple of maps $f_{Con} : A \to A'$, $f_B : B \to B'$, $f_{Con} : Con \to Con'$ define a map

$$f_* = (f_A, f_B, f_{Con})_* : AllExp(A, B; Con) \to AllExp(A', B'; Con')$$

which changes labels in the obvious way. If f_B is injective then f_* maps unambiguous expressions to unambiguous ones.

An element T of $UAExp(A, B; Con)$ is said to be strictly unambiguous if for any $v \neq v'$ in $Vrtx(T)$ such that $lbl(v) = (c; x_1, \ldots, x_n)$ and $lbl(v') = (c'; x'_1, \ldots, x'_{n'})$ one has $\{x_1, , \ldots, x_n\} \cap \{x'_1, \ldots, x'_{n'}\} = \emptyset$ i.e. if the names of all bound variables are different. We let $SUAExp(A, B; Con)$ denote the subset of strictly unambiguous expressions in $UAExp(A, B; Con)$.

An element T of $UAExp(A, B; Con)$ is said to be α-equivalent to an element T' of $UAExp(A, B'; Con)$ if there is a set B'' , an element $T'' \in UAExp(A, B''; Con)$ and two maps $f : B'' \to B$, $f' : B'' \to B'$ such that $T = (Id, f, Id)_*(T'')$ and $T' = (Id, f', Id)_*(T'')$. The following lemma is straightforward:

Lemma 1.1 *[**2009.09.08.l1**] For any two sets A and Con one has:*

1. *α-equivalence is an equivalence relation,*
2. *for any set B and any element $T \in UAExp(A, B; Con)$ there exists an element $T' \in UAExp(A, \mathbf{N}; Con)$ such that $T \stackrel{\alpha}{\sim} T'$ and T' is strictly unambiguous,*

3. *fwo strictly unambiguous elements $T, T' \in UAExp(A, B; Con)$ are α-equivalent if and only if there exists a permutation $f : B \to B$ such that $(Id, f, Id)_*(T) = T'$.*

We let $Exp_\alpha(A; Con)$ denote the set of α-equivalence classes in $\amalg_B UAExp(A, B; Con)$. In view of Lemma 1.1 this set is well defined and can be also defined as the set of equivalence classes in $SUAExp(A, \mathbf{N}; Con)$ modulo the equivalence relation generated by the permutations on $\mathbf{N}$.

Note that for two α-equivalent expressions T_1, T_2 and a vertex $v \in V(T_1) = V(T_2)$ the expressions $[v]_{T_1}$ and $[v]_{T_2}$ need not be α-equivalent since some of the variables which are bound in T_1 may be free in $[v]$.

The maps $(f_A, f_B, f_{Con})_*$ respect α-equivalence. Therefore for any $f_A : A \to A'$ and $f_{Con} : Con \to Con'$ there is a well defined map

$$(f_A, f_{Con})_* : Exp_\alpha(A; Con) \to Exp(A'; Con')$$

which make $Exp_\alpha(-; -)$ into a covariant functors from pairs of sets to sets. In addition there is a well defined notion of substitution on $Exp_\alpha(-; Con)$ which may be considered as a collection of maps of the form:

$$Exp_\alpha(A; Con) \times (\prod_{a \in A} Exp_\alpha(X_a; Con)) \to Exp_\alpha(\amalg_{a \in A} X_a; Con)$$

given for all pairs $(A; \{X_a\}_{a \in A})$ where A is a set and $\{X_a\}_{a \in A}$ a family of sets parametrized by A. Alternatively, the substitution structure can be seen as a collection of maps

$$Exp_\alpha(Exp_\alpha(A; Con); Con) \to Exp_\alpha(A; Con)$$

given for all A and Con. These maps make the functor $Exp_\alpha(-; Con)$ into a monad (triple) on the category of sets which functorially depends on the set Con.

Example 1.2 [**lambda**] The mapping which sends a set X to the set of α-equivalence classes of terms of the untyped λ-calculus with free variables from X is a sub-triple of $Exp_\alpha(-; Con)$ where $Con = \{\lambda, ev\}$. Elements T of $UAExp(X, \mathbf{N}; \{\lambda, ev\})$ which belong to this sub-triple are characterized by the following "local" conditions:

1. for each $v \in T$, $lbl(v) \in X \amalg \mathbf{N} \amalg \{ev\} \amalg \{\lambda\} \times \mathbf{N}$
2. if $lbl(v) \in \{\lambda\} \times \mathbf{N}$ then $val(v) = 1$
3. if $lbl(v) = ev$ then $val(v) = 2$.

Example 1.3 [**propositional**]The mapping which sends a set X to the set of terms of the propositional calculus with free variables from X is a sub-triple of $Exp_\alpha(-; C_0)$ where $C_0 = \{\vee, \wedge, \urcorner, \Rightarrow\}$. Elements T of $UAExp(X, \mathbf{N}; C_0)$ which belong to this sub-triple are characterized by the following "local" conditions:

1. for all $v \in T$, $lbl(v) \in X \amalg C_0$
2. if $lbl(v) \in \{\vee, \wedge, \Rightarrow\}$ then $val(v) = 2$
3. if $lbl(v) = \urcorner$ then $val(v) = 1$.

Example 1.4 [**multisorted**] Consider first order logic with several sorts $GS = \{S_1, \ldots, S_n\}$. Let GP be the set of generating predicates and GF the set of generating functions. Let $C_1 = C_0 \amalg \{\forall, \exists\}$ and $C_2 = C_1 \amalg GP \amalg GF \amalg GS$. We can identify the α-equivalence classes of formulas of the first order language defined by GS and GF with free variables from a set X with a subset in $Exp_\alpha(X, \mathbf{N}; C_2)$. Vertices which are labeled by $(\forall; x)$ and $(\exists; x)$ have valency two. For such a vertex v, the first branch of $[v]$ is one vertex labeled by an element of GS giving the sort over which the quantification occurs and the second branch is the expression which is quantified. Now however, these subsets do not form a sub-triple of Exp_α since not all substitutions are allowed. By allowing all substitutions irrespectively of the sort we get (for each X) a subset in $Exp_\alpha(X; C_2)$ whose elements will be called pseudo-formulas.

The following operations on expressions are well defined up to the α-equivalence:

1. If $T_1, \ldots, T_m \in Exp_\alpha(A; Con)$, $a_1, \ldots, a_n$ are pair-wise different elements of A and $M \in Con$ we will write $(M, a_1, \ldots, a_n)(T_1, \ldots, T_m)$ for the expression whose root v is labeled by $(M, a_1, \ldots, a_n)$, $val(v) = n$ and $br_i(v) = T_i$.
2. For $T_1, T_2 \in Exp_\alpha(A; Con)$ and $v \in T_1$ we let $T_1(T_2/[v])$ be the expression obtained by replacing $[v]$ in T_1 with T_2' where T_2' is obtained from T_2 by the change of bound variables such that the bound variables of T_2' do not conflict with the variables of T_1.
3. For $T, R_1, \ldots, R_n \in Exp_\alpha(A; Con)$ and $y_1, \ldots, y_n \in A$ we let $T(R_1/y_1, \ldots, R_n/y_n)$ denote the expression obtained by changing R_i's by α-equivalent R_i' such that $bnd(R_i') \cap bnd(R_j)' = \emptyset$ for $i \neq j$, changing T to an α-equivalent T' such that $bnd(T') \cap (var(R_1') \cup \ldots \cup var(R_n')) = \emptyset$ and then replacing all the leaves of T' marked by y_i by R_i'.

In all the examples considered above, these operations correspond to the usual operations on formulas. The first operation can be used to directly associate expressions in our sense with the formulas. For example, the expression associated with the formula $\forall x : S.P(x, y)$ in a multi-sorted predicate calculus is $(\forall, x)(S, P(x, y))$ where as was mentioned above we use the same notation for an element of $A \amalg B \amalg (Con \times (\amalg_{n \geq 0} B^n))$ and the one vertex tree with the corresponding label.

Note: about representing elements of $AllExp(A, B; Con)$ by linear sequences of elements of $A \amalg B \amalg$??.

Reduction structures. Another component of the structure present in systems of expressions used in formal systems is the reduction relation. It is very important for our approach to type systems that the reduction relation is defined on all pseudo-formulas and is compatible with the substitution structure even when not all psedu-formulas are well formed formulas. In what follows we will consider, instead of a particular syntactic system, a pair $(S, \triangleright)$ where S is a continuous triple on the category of sets and $\triangleright$ is a reduction structure on S i.e. a collection of relations $\triangleright_X$ on $S(X)$ given for all finite sets X satisfying the following two conditions:

1. if $E \in S(\{x_1, \ldots, x_n\})$, $f_1, \ldots, f_n, f_i' \in S(\{y_1, \ldots, y_m\})$ and $f_i \triangleright_{\{y_1, \ldots, y_m\}} f_i'$ then
$$E(f_1/x_1, \ldots, f_i/x_i, \ldots f_n/x_n) \triangleright_{\{x_1, \ldots, x_n\}} E(f_1/x_1, \ldots, f_i'/x_i, \ldots f_n/x_n),$$
2. if $E, E' \in S(\{x_1, \ldots, x_n\})$, $f_1, \ldots, f_n \in S(\{y_1, \ldots, y_m\})$ and $E \triangleright_{\{x_1, \ldots, x_n\}} E'$ then
$$E(f_1/x_1, \ldots, f_n/x_n) \triangleright_{\{x_1, \ldots, x_n\}} E'(f_1/x_1, \ldots, f_n/x_n).$$

The following two results are obvious but important.

Proposition 1.5 *[***2009.10.17.prop1***] Let S be a continuous triple on $Sets$ and $\triangleright_\alpha$ be a family of reduction structures on S. Then the intersection $\cap_\alpha \triangleright_\alpha : X \mapsto \cap_\alpha \triangleright_{\alpha,X}$ is a reduction structure on S.*

Corollary 1.6 *[***2009.10.17.cor1***] For any family (X_α, pre_α) of pairs of the form (X, pre) where X is a set and pre is a relation on $S(X)$ (i.e. a subset of $S(X) \times S(X)$) there exists the smallest reduction structure $\triangleright = \triangleright(X_\alpha, pre_\alpha)$ on S such that for each α and each $(f, g) \in pre_\alpha$ one has $f \triangleright g$.*

2 Type systems

Contextual categories defined by a triple. Let S be a continuous triple on $Sets$. Let $S - cor$ be the full subcategory of the Kleisli category of S whose objects are finite sets. Recall, that the set of morphisms from X to Y in $S - cor$ is the set of maps from X to $S(Y)$ i.e. $S(Y)^X$ (in other words, $S - cor$ is the category of free, finitely generated S-algebras). We will construct two contextual categories $C(S)$ and $CC(S)$which are based on $(S - cor)^{op}$.

Examples:

1. If $S = Id$ i.e. $S(X) = X$ the $S - cor = FSets$ is the category of finite sets. It is easy to see that the category of finite sets is the free category with finite coproducts generated by one object. Therefore, $(FSets)^{op}$ can be thought of the free category with finite products generated by one object.

2. Let S be given by $S(X) = X \amalg A$ where A is a set. This corresponds to the system of expressions where all expressions are either variables or constants and the set of constants is A. The category $(S - cor)^{op}$ can be though of as the free category with finite products generated by an object U and the set A of morphisms $pt \to U$.

The categories of sets, finite sets or even the category of finite linearly ordered sets and their isomorphisms are all level 1 categories and so is the category $S - cor$. We can get a set-level model $C(S)$ for $(S - cor)^{op}$ by setting $ob(C(S)) = \mathbf{N}$ and $Hom_{C(S)}(n, m) = S(\{1, \ldots, n\})^m$.

The category $C(S)$ has a contextual structure which is defined as follows. The final object is 0. The map ft is given by

$$ft(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{if } n > 0 \end{cases}$$

The canonical projection $n \to n - 1$ is given by the sequence $(1, \ldots, n - 1)$. For $f = (f_1, \ldots, f_m) : n \to m$ the canonical square build on f and the canonical projection $m + 1 \to m$ is of the form

$$\begin{array}{ccc} n+1 & \xrightarrow{(f_1,\ldots,f_m,n+1)} & m+1 \\ \downarrow & & \downarrow \\ n & \xrightarrow{(f_1,\ldots,f_m)} & m \end{array}$$

Any morphism of triples $S \to S'$ defines a contextual functor $C(S) \to C(S')$. Non-trivial contextual subcategories of $C(S)$ are in one-to-one correspondence with continuous sub-triples of S.

Note: add notes that a continuous sub-triple of S is exactly the same as a subcategory in $S-cor$ which contains all (isomorphism classes of) objects. Intersection of two sub-triples is a sub-triple which allows us to speak of sub-triples (systems of expressions etc.) generated by a set of expressions. For the construction of type systems the category $S-cor$ is replaced by the contextual category $CC(S,X)$.

Note: that continuous triples on $Sets$ are the same as category structures on $\mathbf{N}$ which extend the a category structure of finite sets and where the addition remains to be coproduct.

Let now $CC(S)$ be the set-level category whose set of objects is $ob(CC(S)) = \amalg_{n\geq 0} ob_n$ where

$$ob_n = S(\emptyset) \times \ldots \times S(\{1,\ldots,n-1\})$$

and the set of morphisms is

$$mor(CC(S)) = \coprod_{n,m\geq 0} ob_n \times ob_m \times S(\{1,\ldots,n\})^m$$

with the obvious domain and codomain maps. The composition of morphisms is defined in the same way as in $C(S)$ such that the mapping $ob(CC(S)) \to \mathbf{N}$ which sends all elements of ob_n to n, is a functor. The associativity of compositions follows immediately from the associativity of compositions in $S-cor$.

Note that if $S(\emptyset) = \emptyset$ then $CC(S) = \emptyset$ and otherwise the functor $CC(S) \to (S-cor)^{op}$ is an equivalence, so that in the second case $C(S)$ and $CC(S)$ are indistinguishable as level 1 categories. However, as set level categories they are quite different.

The category $CC(S)$ is given a contextual structure as follows. The final object is the only element of ob_0, the map ft is defined by the rule

$$ft(T_1,\ldots,T_n) = (T_1,\ldots,T_{n-1}).$$

The canonical pull-back square defined by an object $(T_1,\ldots,T_{m+1})$ and a morphism $(f_1,\ldots,f_m) \in S(\{1,\ldots,n\})^m$ from $(R_1,\ldots,R_n)$ to $(T_1,\ldots,T_m)$ is of the form

[**2009.11.05.oldeq1**]
$$\begin{array}{ccc} (R_1,\ldots,R_n,T_{m+1}(f_1/1,\ldots,f_m/m)) & \xrightarrow{(f_1,\ldots,f_m,n+1)} & (T_1,\ldots,T_{m+1}) \\ \downarrow & & \downarrow \\ (R_1,\ldots,R_n) & \xrightarrow{(f_1,\ldots,f_m)} & (T_1,\ldots,T_m) \end{array} \qquad (26)$$

Proposition 2.1 *[***2009.10.01.prop2***] With the structure defined above $CC(S)$ is a contextual category.*

Proof: Straightforward.

Note that the natural projection $CC(S) \to C(S)$ is a contextual functor. It's contextual sections are in one-to-one correspondence with $S(\emptyset)$ such that $U \in S(\emptyset)$ corresponds to the section which takes the object n of $C(S)$ to the object $(U,\ldots,U)$ of $CC(S)$.

Any morphism of triples $S \to S'$ defines a contextual functor $CC(S) \to CC(S')$. Contextual subcategories of $CC(S)$, which are discussed in more detail below, provide an important class of type systems over S.

There is another construction of a category from a continuous triple S which takes as an additional parameter a set Var which is called the set of variables. Let $F_n(Var)$ be the set of sequences of length n of pair-wise distinct elements of Var. Define the category $CC(S, Var)$ as follows. The set of objects of $CC(S, Var)$ is

$$ob(CC(S, Var)) = \amalg_{n\geq 0} \amalg_{(x_1,\ldots,x_n)\in F_n(Var)} S(\emptyset) \times \ldots \times S(\{x_1, \ldots, x_{n-1}\})$$

For notational compatibility with the traditional type theory we will write the elements of $ob(CC(S, X))$ as sequences of the form $x_1 : E_1, \ldots, x_n : E_n$. The set of morphisms is given by

$$Hom_{CC(S,,Var)}((x_1 : E_1, \ldots, x_n : E_n), (y_1 : T_1, \ldots, y_m : T_m)) = S(\{x_1, \ldots, x_n\})^m$$

The composition is defined in such a way that the projection

$$(x_1 : E_1, \ldots, x_n : E_n) \mapsto (E_1, E_2(1/x_1), \ldots, E_n(1/x_1, \ldots, n-1/x_{n-1}))$$

is a functor from $CC(S, X)$ to $CC(S)$. This functor is clearly an equivalence. There is an obvious final object and $frth$ map on $CC(S, X)$. There is however a real problem in making it into a contextual category which is due to the following. Consider an object $(y_1 : T_1, \ldots, y_{m+1} : T_{m+1})$ and a morphism $(f_1, \ldots, f_m) : (x_1 : R_1, \ldots, x_n : R_n) \to (y_1 : T_1, \ldots, y_m : T_m)$. In order for the functor to $CC(S)$ to be a contextual functor the canonical square build on this pair should have the form

$$\begin{array}{ccc}
(x_1 : R_1, \ldots, x_n : R_n, x_{n+1} : T_{m+1}(f_1/1, \ldots, f_m/m)) & \xrightarrow{(f_1,\ldots,f_m,n+1)} & (y_1 : T_1, \ldots, y_{m+1} : T_{m+1}) \\
\downarrow & & \downarrow \\
(x_1 : R_1, \ldots, x_n : R_n) & \xrightarrow{(f_1,\ldots,f_m)} & (y_1 : T_1, \ldots, y_m : T_m)
\end{array}$$

where x_{n+1} is an element of X which is distinct from each of the elements $x_1, \ldots, x_n$. Moreover, we should choose x_{n+1} in such a way the the resulting construction satisfies the contextual category axioms for $(f_1, \ldots, f_m) = Id$ and for the compositions $(g_1, \ldots, g_n) \circ (f_1, \ldots, f_m)$. One can easily see that no such choice is possible for a finite set X. At the moment it is not clear to me whether or not such it is possible for an infinite X.

Contextual subcategories of $CC(S)$. Let TS be a contextual subcategory of $CC(S)$. By Lemma 1.3, TS is determined by the subsets $ob(TS)$ and $\widetilde{ob}(TS)$ in $ob(CC(S))$ and $\widetilde{ob}(CC(S))$. By definition we have

$$ob(CC(S)) = \coprod_{n\geq 0} S(\emptyset) \times S(\{1\}) \times \ldots \times S(\{1, \ldots, n-1\}) \times S(\{1, \ldots, n\})$$

An element of $\widetilde{ob}(CC(S))$ is given by a pair (Γ, s) where $\Gamma \in ob(CC(S))$ is an object and $s : ft(\Gamma) \to \Gamma$ is a section of the canonical morphism $p_\Gamma : \Gamma \to ft(\Gamma)$. It follows immediately from the definition of $CC(S)$ that for $\Gamma = (E_1, \ldots, E_{n+1})$, a morphism $(f_1, \ldots, f_{n+1}) \in S(\{1, \ldots, n\})^{n+1}$ from $ft(\Gamma)$ to Γ is a section of p_Γ if an only if $f_i = i$ for $i = 1, \ldots, n$. Therefore, any such section is determined by its last component f_{n+1} and mapping $((E_1, \ldots, E_{n+1}), (f_1, \ldots, f_{n+1}))$ to $(E_1, \ldots, E_n, E_{n+1}, f_{n+1})$ we get a bijection

$$[\textbf{2009.10.15.eq2}]\widetilde{ob}(CC(S)) \cong \coprod_{n\geq 0} S(\emptyset) \times S(\{1\}) \times \ldots \times S(\{1, \ldots, n-1\}) \times S(\{1, \ldots, n\})^2 \quad (27)$$

Let $Seq_0(TS) = ob(TS)$ and let $Seq_1(TS)$ be the image of $\widetilde{ob}(TS)$ under the bijection (27). Traditionally, one writes $E_1, E_2, \ldots, E_n \vdash_{TS}$ if $(E_1, \ldots, E_n)$ is in $Seq_0(TS)$ and $E_1, E_2, \ldots, E_n \vdash_{TS} t : T$ if $(E_1, \ldots, E_n, T, t)$ is in $Seq_1(TS)$. When no confusion is possible we will write $\vdash$ instead of $\vdash_{TS}$.

The following result is an immediate corollary of Proposition 1.5.

Proposition 2.2 *[**2009.10.16.prop3**] Let S be a continuous triple on Sets. A pair of subsets Seq_0, Seq_1 of the form*

$$\textbf{[2009.10.01.eq3]} \quad \begin{array}{c} Seq_0(TS) \subset \coprod_{n\ge 0} S(\emptyset) \times S(\{1\}) \times \ldots \times S(\{1,\ldots,n-1\}) \times S(\{1,\ldots,n\}) \\ Seq_1(TS) \subset \coprod_{n\ge 0} S(\emptyset) \times S(\{1\}) \times \ldots \times S(\{1,\ldots,n-1\}) \times S(\{1,\ldots,n\})^2 \end{array} \tag{28}$$

defines a contextual subcategory of $CC(S)$ if and only if the following conditions hold:

1. *$\vdash$,*
2. *if $E_1, \ldots, E_n \vdash$ then $E_1, \ldots, E_{n-1} \vdash$,*
3. *if $E_1, \ldots, E_n \vdash t : T$ then $E_1, \ldots, E_n, T \vdash$,*
4. *if $E_1, \ldots, E_n \vdash t : T$, $i = 1, \ldots, n$ and $E_1, \ldots, E_i, E'_{i+1} \vdash$ then $E_1, \ldots, E_i, E'_{i+1}, E_{i+1}, \ldots, E_n(i+2/i+1, \ldots, n/n-1) \vdash t(i+2/i+1, \ldots, n+1/n) : T(i+2/i+1, \ldots, n+1/n)$,*
5. *if $E_1, \ldots, E_n \vdash t : T$, $i = 1, \ldots, n$ and $E_1, \ldots, E_i \vdash r : E_{i+1}$ then $E_1, \ldots, E_i, E_{i+2}(r/i+1), \ldots, E_n(r/i+1, i+1/i+2, \ldots, n-2/n-1) \vdash t(r/i+1, i+1/i+2, \ldots, n-2/n-1) : T(r/i+1, i+1/i+2, \ldots, n-2/n-1)$,*
6. *$E_1, \ldots, E_n \vdash$ then $E_1, \ldots, E_n \vdash n : E_n$.*

Note that conditions (4) and (5) together with condition (6) and condition (3) imply the following

4a if $E_1, \ldots, E_n \vdash$, $i = 1, \ldots, n$ and $E_1, \ldots, E_i, E'_{i+1} \vdash$ then $E_1, \ldots, E_i, E'_{i+1}, E_{i+1}, \ldots, E_n(i+2/i+1, \ldots, n/n-1) \vdash$,

5a if $E_1, \ldots, E_n \vdash$, $i = 1, \ldots, n$ and $E_1, \ldots, E_i \vdash r : E_{i+1}$ then $E_1, \ldots, E_i, E_{i+2}(r/i+1), \ldots, E_n(r/i+1, i+1/i+2, \ldots, n-2/n-1) \vdash$.

Lemma 2.3 *[**2009.11.05.l1**] Let S be as above and let $Seq_0 = ob(TS)$, $Seq_1 \cong \widetilde{ob}(TS)$ be subsets corresponding to a contextual subcategory TS. Let further $(E_1, \ldots, E_n), (T_1, \ldots, T_m) \in ob(TS)$ and $(f_1, \ldots, f_m) \in S(\{1, \ldots, n\})^m$. Then*

$$(f_1, \ldots, f_m) \in Hom_{TS}((E_1, \ldots, E_n), (T_1, \ldots, T_m))$$

if and only if $(f_1, \ldots, f_{m-1}) \in Hom_{TS}((E_1, \ldots, E_n), (T_1, \ldots, T_{m-1}))$ and

$$(E_1, \ldots, E_n, T_m(f_1/1, \ldots, f_{m-1}/m-1), f_m) \in Seq_1$$

Proof: Straightforward using the fact that the canonical pull-back squares in $CC(S)$ are given by (26).

Note: describe all type systems over the triple $X \mapsto X_+$. (Note that there are no non-trivial type systems over the identity triple since $Id(\emptyset) = \emptyset$.

Remark 2.4 One can also define type systems as sub-quotients rather than just subobjects of $C_1(S, X)$. This requires one to consider "type equality sequents" and "term equality sequents" in addition to the type sequents and term sequents which we consider. Some sort of syntactic equality is necessary in order to define well behaved type systems. The question is whether or not one needs to consider context dependent syntactic equalities which make it necessary to pass to equivalence classes on the level of contextual categories. The example of most of the type systems which are practically used suggests that it is sufficient to consider context-independent syntactic equalities (i.e. equivalence classes of abstract expressions) and to use equality types for the context dependent equalities. For the approach with context dependent syntactic equalities see [**?**].

Type systems over $(S, \triangleright)$**.** Let S be a continuous triple on $Sets$ and $\triangleright$ a relation on S (see above). One defines type systems over $(S, \triangleright)$ as follows.

Definition 2.5 *[***2009.10.01.def2***] Let S be a continuous triple on the category of sets. A type system over $(S, \triangleright)$ is a contextual subcategory TS of $CC(S)$ such that*

1. *if $(E_1, \ldots, E_n) \in ob_n(TS)$, $i = 1, \ldots, n$ and $E'_i \in S(\{1, \ldots, i-1\})$ is such that $E_i \triangleright E'_i$ then $(E_1, \ldots, E'_i, \ldots, E_n) \in ob_n(TS)$,*
2. *if $(f_1, \ldots, f_m) \in Hom_{TS}((E_1, \ldots, E_n), (T_1, \ldots, T_m))$, $i = 1, \ldots, m$ and $f'_i \in S(\{1, \ldots, n\})$ is such that $f_i \triangleright f'_i$ then $(f_1, \ldots, f'_i, \ldots, f_m) \in Hom_{TS}((E_1, \ldots, E_n), (T_1, \ldots, T_m))$,*
3. *if $(E_1, \ldots, E_n), (T_1, \ldots, T_m) \in ob(TS)$, $(f_1, \ldots, f_m) \in S(\{1, \ldots, n\})^m$, $i = 1, \ldots, n$ and $E'_i \in S(\{1, \ldots, i-1\})$ is such that $E_i \triangleright E'_i$ then*
$$(f_1, \ldots, f_m) \in Hom_{TS}((E_1, \ldots, E_n), (T_1, \ldots, T_m))$$
if and only if $(f_1, \ldots, f_m) \in Hom_{TS}((E_1, \ldots, E'_i, \ldots, E_n), (T_1, \ldots, T_m))$,
4. *if $(E_1, \ldots, E_n), (T_1, \ldots, T_m) \in ob(TS)$, $(f_1, \ldots, f_m) \in S(\{1, \ldots, n\})^m$, $i = 1, \ldots, m$ and $T'_i \in S(\{1, \ldots, i-1\})$ is such that $T_i \triangleright T'_i$ then*
$$(f_1, \ldots, f_m) \in Hom_{TS}((E_1, \ldots, E_n), (T_1, \ldots, T_m))$$
if and only if $(f_1, \ldots, f_m) \in Hom_{TS}((E_1, \ldots, E_n), (T_1, \ldots, T'_i, \ldots, T_m))$,

We will write $TS(S, \triangleright)$ for the set of type systems over $(S, \triangleright)$. Type systems over $(S, =)$ are just the contextual subcategories of $CC(S)$.

Define relations $\triangleright_{TS}$ on $ob(TS)$ and on the sets $Hom_{TS}(-, -)$

Proposition 2.6 *[***2009.10.18.prop1***] Let S be a continuous triple on Sets and $\triangleright$ a reduction structure on S. A pair of subsets Seq_0, Seq_1 of the form (28) defines a type system over $(S, \triangleright)$ if and only of it satisfies the conditions of Proposition 2.2 together with the following additional conditions:*

1. *if* $(E_1,\ldots,E_n)\in Seq_0$*,* $i=1,\ldots,n$ *and* $E_i \triangleright E_i'$ *then* $(E_1,\ldots,E_i',\ldots,E_n)\in Seq_0$*,*

2. *if* $(E_1,\ldots,E_{n+1},t)\in Seq_1$ *and* $t\triangleright t'$ *then* $(E_1,\ldots,E_{n+1},t')\in Seq_1$*,*

3. *if* $(E_1,\ldots,E_{n+1},t)\in Seq_1$*,* $i=1,\ldots,n+1$ *and* $E_i\triangleright E_i'$ *then* $(E_1,\ldots,E_i',\ldots,E_{n+1},t)\in Seq_1$*,*

4. *if* $(E_1,\ldots,E_{n+1})\in Seq_0$*,* $i=1,\ldots,n+1$*,* $E_i\triangleright E_i'$ *and* $(E_1,\ldots,E_i',\ldots,E_{n+1},t)\in Seq_1$ *then* $(E_1,\ldots,E_i,\ldots,E_{n+1},t)\in Seq_1$*.*

Proof: Since $Seq_0 = ob(TS)$ the first condition of the proposition is equivalent to the first condition of Definition 2.5.

Suppose that Seq_0 and Seq_1 correspond to a contextual subcategory TS satisfying the conditions of Definition 2.5. A sequence $(E_1,\ldots,E_{n+1},t)$ is in Seq_1 if and only if $(1,\ldots,n,t)$ is a morphism from $(E_1,\ldots,E_n)$ to $(E_1,\ldots,E_{n+1})$ in TS. If $(E_1,\ldots,E_{n+1},t)\in Seq_1$ and $t\triangleright t'$ then $(E_1,\ldots,E_{n+1},t')\in Seq_1$ by condition (2) of Definition 2.5. If $(E_1,\ldots,E_{n+1},t)\in Seq_1$, $i=1,\ldots,n$ and $E_i\triangleright E_i'$ then by $(E_1,\ldots,E_i',\ldots,E_{n+1},t)\in Seq_1$ conditions (3) and (4) of Definition 2.5. If $(E_1,\ldots,E_{n+1})\in Seq_0$, $i=1,\ldots,n+1$, $E_i\triangleright E_i'$ and $(E_1,\ldots,E_i',\ldots,E_{n+1},t)\in Seq_1$ then $(E_1,\ldots,E_i,\ldots,E_{n+1},t)\in Seq_1$ again by conditions (3), (4) of Definition 2.5.

Suppose now that the conditions (1)-(4) of the proposition are satisfied and Seq_0, Seq_1 correspond to a contextual subcategory TS. Let us show that TS satisfies conditions (2)-(4) of Definition 2.5. Let $(f_1,\ldots,f_m)\in Hom_{TS}((E_1,\ldots,E_n),(T_1,\ldots,T_m))$, $i=1,\ldots,m$ and $f_i'\in S(\{1,\ldots,n\})$ is such that $f_i\triangleright f_i'$. Then $(f_1,\ldots,f_i',\ldots,f_m)\in Hom_{TS}((E_1,\ldots,E_n),(T_1,\ldots,T_m))$ by Lemma 2.3, the definition of the reduction structure on S and conditions (2), (3) of our proposition.

Let $(E_1,\ldots,E_n),(T_1,\ldots,T_m)\in ob(TS)$, $(f_1,\ldots,f_m)\in S(\{1,\ldots,n\})^m$, $i=1,\ldots,n$ and $E_i'\in S(\{1,\ldots,i-1\})$ is such that $E_i\triangleright E_i'$. Then

$$(f_1,\ldots,f_m)\in Hom_{TS}((E_1,\ldots,E_n),(T_1,\ldots,T_m))$$

if and only if $(f_1,\ldots,f_m)\in Hom_{TS}((E_1,\ldots,E_i',\ldots,E_n),(T_1,\ldots,T_m))$ by obvious induction, Lemma 2.3 and conditions (3), (4) of the proposition.

Similarly, if $(E_1,\ldots,E_n),(T_1,\ldots,T_m)\in ob(TS)$, $(f_1,\ldots,f_m)\in S(\{1,\ldots,n\})^m$, $i=1,\ldots,m$ and $T_i'\in S(\{1,\ldots,i-1\})$ is such that $T_i\triangleright T_i'$ then

$$(f_1,\ldots,f_m)\in Hom_{TS}((E_1,\ldots,E_n),(T_1,\ldots,T_m))$$

if and only if $(f_1,\ldots,f_m)\in Hom_{TS}((E_1,\ldots,E_n),(T_1,\ldots,T_i',\ldots,T_m))$, by obvious induction, Lemma 2.3, the definition of reduction structure and conditions (3), (4) of the proposition.

Definition 2.7 *[***2009.11.4.def1***] Let* S*,* $\triangleright$ *and* TS *be as above. Let further* $(\mathcal{C},p)$ *be a category with a pre-universe structure. A closed model of* TS *with values in* $(\mathcal{C},p)$ *is a contextual functor*

$$M: TS\to CC(\mathcal{C},p)$$

which is compatible with $\triangleright$ *i.e. such that the following conditions hold:*

1. *if* $(E_1,\ldots,E_n)\in ob(TS)$*,* $i=1,\ldots,n$ *and* $E_i'\in S(\{x_1,\ldots,x_{i-1}\})$ *is such that* $E_i\triangleright E_i'$ *then* $M(E_1,\ldots,E_n)=M(E_1,\ldots,E_i',\ldots,E_n)$*,*

2. *if* $(f_1, \ldots, f_m) \in Hom_{TS}((E_1, \ldots, E_n), (T_1, \ldots, T_m))$, $i = 1, \ldots, m$ *and* $f'_i \in S(\{1, \ldots, n\})$ *is such that* $f_i \triangleright f'_i$ *then*

   $$M((f_1, \ldots, f_m); (E_1, \ldots, E_n); (T_1, \ldots, T_m)) = M((f_1, \ldots, f'_i, \ldots, f_m); (E_1, \ldots, E_n); (T_1, \ldots, T_m)$$

3. *if* $(E_1, \ldots, E_n), (T_1, \ldots, T_m) \in ob(TS)$, $(f_1, \ldots, f_m) \in S(\{1, \ldots, n\})^m$, $i = 1, \ldots, n$ *and* $E'_i \in S(\{1, \ldots, i-1\})$ *is such that* $E_i \triangleright E'_i$ *then*

   $$M((f_1, \ldots, f_m); (E_1, \ldots, E_n); (T_1, \ldots, T_m)) = M((f_1, \ldots, f_m); (E_1, \ldots, E'_i, \ldots, E_n); (T_1, \ldots, T_m))$$

4. *if* $(E_1, \ldots, E_n), (T_1, \ldots, T_m) \in ob(TS)$, $(f_1, \ldots, f_m) \in S(\{1, \ldots, n\})^m$, $i = 1, \ldots, m$ *and* $T'_i \in S(\{1, \ldots, i-1\})$ *is such that* $T_i \triangleright T'_i$ *then*

   $$M((f_1, \ldots, f_m); (E_1, \ldots, E_n); (T_1, \ldots, T_m)) = M((f_1, \ldots, f_m); (E_1, \ldots, E_n); (T_1, \ldots, T'_i, \ldots, T_m))$$

... are called the subset of type sequents and the subset of term sequents of a type system. By Lemma 1.3 they uniquely determine the type system.

Elements of Seq_0 are called contexts and elements of Seq_1 are called judgements. Proposition 2.2 shows that for any type system TS and any $(E_1, \ldots, E_n, t, T)$ in $Seq_1(TS)$ the sequence $(E_1, \ldots, E_n)$ is in Seq_0 i.e. the first part of a judgement should be a valid context.

One also often uses the notation $E_1, E_2, \ldots, E_n \vdash T : Type$ which is equivalent to $E_1, E_2, \ldots, E_n, T \vdash$. The meaning assigned to these subsets is as follows:

1. $E_1, E_2, \ldots, E_n \vdash$ means that E_1 is a well formed closed type expression and for $i > 1$, $E_i(1, \ldots, i-1)$ is a well formed type expression in the context where variables $1, \ldots, i-1$ have types $E_1, \ldots, E_{i-1}$ respectively,
2. $E_1, E_2, \ldots, E_n \vdash t : T$ means that $E_1, E_2, \ldots, E_n, T \vdash$ and in the context where variables $1, \ldots, n$ are of the types $E_1, \ldots E_n$ respectively, $t(1, \ldots, n)$ is a well formed term expression of type $T(1, \ldots, n)$.

3 Rules and parsing schemes in contextual categories

Rules. Let $CC = (B, \widetilde{B}, ...)$ be a contextual category. We will write Seq for $B \amalg \widetilde{B}$.

Definition 3.1 *[***2009.11.11.def1***] A rule of valency* n *on* CC *is a subset* $R \subset Seq^n \times Seq$. *A rule is called functional if it defines a partial function* $f_R : U_R \to Seq$.

Definition 3.2 *[***2009.11.11.def2***] A subset* $Sq \subset Seq$ *is said to satisfy a rule* R *if for any* $(S_1, \ldots, S_n; S') \in R$ *such that* $S_1, \ldots, S_n \in Sq$ *one has* $S' \in Sq$.

Let $\mathcal{R}$ be a set of rules on CC. We say that $Sq \subset Seq(CC)$ satisfies $\mathcal{R}$ if it satisfies all the rules from $\mathcal{R}$. The intersection of all the subsets in Seq which satisfy $\mathcal{R}$ is the smallest subset which satisfies $\mathcal{R}$ and we denote it by $Seq(\mathcal{R})$. We will also write $Seq_0(\mathcal{R})$ and $Seq_1(\mathcal{R})$ for the intersections of $Seq(\mathcal{R})$ with B and $\widetilde{B}$ respectively.

Let us consider the condition on $\mathcal{R}$ which are required for the pair of subsets $(Seq_0(\mathcal{R}), Seq_1(\mathcal{R}))$ to be saturated i.e. to correspond to a contextual subcategory.

A contextual subcategory CC' of CC satisfies $\mathcal{R}$ if $Seq(CC')$ satisfies $\mathcal{R}$ as a subset of $Seq(CC)$.

Lemma 3.3 *[2009.11.11.l1] Let CC be a contextual category and $\mathcal{R}$ be a set of rules on CC. Then there exists the smallest contextual subcategory $CC(\mathcal{R})$ which satisfies $\mathcal{R}$.*

Proof: The whole CC clearly satisfy all possible rules. Consider the intersection of all contextual subcategories which satisfy $\mathcal{R}$. It is a contextual subcategory by Proposition 1.1. Since the intersection of any family of subcategories satisfying a rule R obviously satisfies this rule as well, we conclude that this subcategory satisfies $\mathcal{R}$. Since it is clearly the smallest possible subcategory satisfying $\mathcal{R}$ this proves the lemma.

The contextual subcategory $CC(\mathcal{R})$ is called the subcategory generated by the set of rules $\mathcal{R}$.

Parsing schemes.

Definition 3.4 *[2009.11.11.def2] Let $CC = (B, \widetilde{B}, \ldots)$ be a contextual category. A parsing scheme ζ on CC is a triple $n \geq 0$, $U_\zeta \subset Seq$ and $\zeta : U_\zeta \to Seq^n$.*

Definition 3.5 *[2009.12.1.def1] Let $R \subset Seq^n \times Seq$ be a rule of valency n on CC and $\zeta = (n, V_\zeta, \zeta)$ be a parsing scheme. Then ζ is said to be a strict parsing scheme for R if one has:*

1. *for any $S \in V_\zeta$, $(\zeta(S), S) \in R$,*
2. *for any $(S_1, \ldots, S_n, S') \in R$, $S' \in V_\zeta$ and $\zeta(S') = (S_1, \ldots, S_n)$.*

If only the first of the two conditions is satisfied then ζ is called a weak parsing scheme for R.

Note that if a strict parsing scheme for R exists then R is necessarily functional. Note also that all the structures such as ft, δ, T, S, Π etc. which we considered above are examples of functional rules on CC.

Definition 3.6 *[2009.12.01.def2]*

1. *A Π-shaped parsing scheme on CC is a parsing scheme ζ such that $V_\zeta \subset \amalg_{n\geq 0} B_{n+1}$ and for $Z \in V_\zeta \cap B_{n+1}$ one has*
   (a) *$\zeta(Z) \in B_{n+2}$,*
   (b) *$ft^2(\zeta(Z)) = ft(Z)$,*
   (c) *for any $n+1 \geq i \geq 1$, $A \in B_{n+2-i}$ such that $ft(A) = ft^i(Z)$ one has $T(A, Z) \in V_\zeta$ and $\zeta(T(A, Z)) = T(A, \zeta(Z))$,*
   (d) *for any $n+1 \geq i \geq 1$, $a \in \widetilde{B}_{n+1-i}$ such that $\partial(a) = ft^i(Z)$ one has $S(a, Z) \in V_\zeta$ and $\zeta(S(a, Z)) = S(a, \zeta(Z))$.*
2. *A (Π, λ)-shaped pair of parsing schemes on CC is a Π-shaped parsing scheme ζ together with a parsing scheme ξ such that $V_\xi \subset \amalg_{n\geq 0} \widetilde{B}_{n+1}$ and for $f \in V_\xi \cap \widetilde{B}_{n+1}$ one has:*
   (a) *$\xi(f) \in \widetilde{B}_{n+2}$,*
   (b) *$\partial(f) \in V_\zeta$,*

(c) $\partial(\xi(f)) = \zeta(\partial(f))$,

(d) for any $n+1 \geq i \geq 1$, $A \in B_{n+2-i}$ such that $ft(Y) = ft^i(\partial(f))$ one has $\widetilde{T}(A,f) \in V_\xi$ and $\xi(\widetilde{T}(A,f)) = \widetilde{T}(A,\xi(f))$,

(e) for any $n+1 \geq i \geq 1$, $a \in \widetilde{B}_{n+1-i}$ such that $\partial(a) = ft^i(\partial(f))$ one has $\widetilde{S}(a,f) \in V_\xi$ and $\xi(\widetilde{S}(a,f)) = \widetilde{S}(a,\xi(f))$.

3. A (Π, λ, ev)-shaped triple of parsing schemes is a (Π, λ)-pair (ζ, ξ) together with a parsing scheme ϵ such that $V_\epsilon \subset \amalg_{n\geq 0}\widetilde{B}_{n+1}$, $V_\epsilon \cap V_\xi = \emptyset$ and for $s \in V_\epsilon \cap \widetilde{B}_{n+1}$ one has:

(a) $\epsilon(s) = (f, Y, r) \in \widetilde{B}_{n+1} \times B_{n+2} \times \widetilde{B}_{n+1}$, $\partial(f) = ft(Y)$, $\partial(r) \in V_\zeta$, $\zeta(\partial(r)) = Y$,

(b) $\partial(s) = S(r, Y)$,

(c) for any $n+1 \geq i \geq 1$, $A \in B_{n+2-i}$ such that $ft(Y) = ft^i(\partial(s))$ one has $\widetilde{T}(A,s) \in V_\epsilon$ and $\epsilon(\widetilde{T}(A,s)) = \widetilde{T}(A,\epsilon(s))$,

(d) for any $n+1 \geq i \geq 1$, $a \in \widetilde{B}_{n+1-i}$ such that $\partial(a) = ft^i(\partial(s))$ one has $\widetilde{S}(a,s) \in V_\epsilon$ and $\epsilon(\widetilde{S}(a,s)) = \widetilde{S}(a,\epsilon(s))$.

Definition 3.7 *[**2009.12.1.def3**] Let $CC = (B, \widetilde{B}, \ldots)$ be a contextual category and $i \geq 1$. A weakly adapted (δ, i)-shaped parsing scheme on CC is a parsing scheme (ϱ, i) such that $V_{(\varrho,i)} \subset \amalg_{n\geq i}\widetilde{B}_{n+1}$ and for $r \in V_{(\varrho,i)} \cap \widetilde{B}_{n+1}$ one has*

1. $(\varrho, i)(r) = ft(\partial(r))$,

2. $r = \delta((\varrho, i)(r), i)$ *where for* $Y \in B_n$, $\delta(Y, i) = \widetilde{T}_{n-i}(Y, \delta(ft^{n-i}(Y)))$.

Let us define two more classes of parsing schemes.

Definition 3.8 *[**2009.12.2.def1**] Let $CC = (B, \widetilde{B}, \ldots)$ be a contextual category, $\widetilde{\Omega} \in B_2$ and $\Omega = ft(\widetilde{\Omega})$. Then*

1. A weakly adapted Ω-shaped parsing scheme is a parsing scheme ω such that $V_\omega \subset \amalg_{n\geq 0}B_{n+1}$ and for $X \in V_\omega \cap B_{n+1}$ one has

(a) $\omega(X) = ft(X)$,

(b) $X = T_n(ft(X), \Omega)$.

2. A weakly adapted $\widetilde{\Omega}$-shaped parsing scheme on CC is a parsing scheme $\widetilde{\omega}$ such that $V_{\widetilde{\omega}} \subset \amalg_{n\geq 0}B_{n+1}$ and for $X \in V_{\widetilde{\omega}} \cap B_{n+1}$ one has

(a) $\widetilde{\omega}(X) \in \widetilde{B}_{n+1}$,

(b) $\partial(\widetilde{\omega}(X)) = T_n(ft(X), \Omega)$,

(c) $X = S(\widetilde{\omega}(X), T_n(ft(X), \widetilde{\Omega}))$.

3. A very weakly adapted $(\widetilde{\Omega}, \Pi_\Omega)$-shaped pair of parsing schemes is a weakly adapted $\widetilde{\Omega}$-shaped parsing scheme $\widetilde{\omega}$ and a parsing scheme κ such that $V_\kappa \subset \amalg_{n\geq 0}\widetilde{B}_{n+1}$ and for $p \in V_\kappa \cap \widetilde{B}_{n+1}$ one has

(a) $\kappa(p) = (r, s) \in \widetilde{B}_{n+1} \times \widetilde{B}_{n+2}$, $\partial(s) = T_{n+1}(ft(\partial(s)), \Omega)$, $ft(\partial(s)) \in V_{\widetilde{\omega}} \cap \widetilde{B}_{n+1}$ and $r = \omega(ft(\partial(s)))$, *(b)* $ft(\partial(r)) = ft(\partial(p))$,

(c) $\partial(p) = T_n(ft(\partial(p)), \Omega))$.

Lemma 3.9 *[**2009.11.11.l3**] Let CC and $\mathcal{R}$ be as above. Let CC' be a subcategory satisfying $\mathcal{R}$. Suppose that there exists a parsing scheme on $Seq_0(CC') \amalg Seq_1(CC')$ relative to $\mathcal{R}$. Then CC' is generated by $\mathcal{R}$.*

Proof: Using induction on the length of the parsing tree from $S \in Seq_0(CC') \amalg Seq_1(CC')$ one can easily see that any such S belongs to the subcategory generated by $\mathcal{R}$.

Let $\mathcal{R}$ be a collection of rules and let $Seq(\mathcal{R})$ be the smallest subset in Seq which satisfies $\mathcal{R}$.