Notes on dynamic logic

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The set of names

The set of variables

The set of quantifiers

The set of logical symbols (including equality)

The set of expressions with the subsets of formulas and terms

Immediate state of our language is determined by:

1. A subset in NAMES which consists of names currently in use

2. A map from the set of names in use to the set of KINDS where KINDS is the set whose elements are

"a metafunctions of ... variables", "a metarelation of ... variables", "a metaformula", "a metaexpression",

3. We will refer to the sets of names of a given kind as to the set of metafunctions, set of metaformulas etc.

March 8, 2004 Classically, a language is a subset of the set of all words in a given alphabet. In our case the alphabet is the union of sets of:

names of variables for each n, names of functions of n-variables for each n, names of relations of n-variables standard quantifiers standard logical connectors including equality

The rules which define when a sequence of elementary symbols (elements of the alphabet) is a word (i.e. is "correctly formed") are the usual ones. In addition we have the rules which:

1. establish which words are formulas and which are terms 2. for any formula define the sets of free variables, e-variables and a-variables of this formula. small trouble with the same variable name appearing twice. 3. for a formula without free variables a partial order on the union of the sets of a- and e-variables of this formula.

Our particular language has an additional structure on it. Namely, there are:

1. A map from formulas to n-ary relations where n is the number of free variables of the formula
2. A map from the pairs (formula without free variables, an e-variable of the formula) to n-ary functions where n is the number of a-variables upstream of our e-variable.

These maps should satisfy the following conditions ....

Our language is the smallest (free) language with these additional structure (and the given countable set of variables).

March 11, 2004

Definition 0.1 *[***templates***] A first order template T is the following collection of data:*

1. *a set Var called the set of variables*
2. *for each $n \geq 0$ a set $Pred_n$ called the set of n-ary predicates*
3. *for each $n \geq 0$ a set $Func_n$ called the set of n-ary functions*

Recall that a language is a pair $L = (S, E)$ where S is a set called the alphabet of L and E is a subset in the set of all finite sequences of elements of S. Each first order template defines two languages $LF(T) = (S(T), F(T))$ and $LTm(T) = (S(T), Tm(T))$ which are called the language of formulas and the language of terms generated by T. The set $S(T)$ is the union of the sets Var, $\amalg_{n\geq 0} Pred_n$ and $\amalg_{n\geq 0} Pred_n$ with the finite set of the standard special symbols - quantifiers, connectives and parentheses (see [**?**, p.6]). The sets $F(T)$ and $Tm(T)$ are the sets of (well formed) formulas and terms in the "first order language" defined by T. See [**?**].

For any $F \in F(T)$ define the set $Q(F)$ as the set of all quantifiers occurring in F. Define further the subsets EQ and AQ of e-quantifiers and a-quantifiers in $Q(F)$ inductively as follows:

1. if $F = (G \vee H)$ then $EQ(F) = EQ(G) \amalg EQ(H)$ and $AQ(F) = AQ(G) \amalg AQ(H)$
2. if $F = (G \wedge H)$ then $EQ(F) = EQ(G) \amalg EQ(H)$ and $AQ(F) = AQ(G) \amalg AQ(H)$
3. if $F = \neg G$ then $EQ(F) = AQ(G)$ and $EQ(G) = AQ(F)$
4. if $F = \forall x G$ then this $\forall x$ is added to $EQ(F)$ and $AQ(F)$ remains the same
5. if $F = \exists x G$ then this $\exists x$ is added to $AQ(F)$ and $EQ(F)$ remains the same.

Define further a partial order on the set $Q(F)$ saying that $q1 \prec q2$ if there exists a subformula of F of the form $q1G$ such that $q2$ is contained in G. For a formula F and a quantifier $q \in Q(F)$ define $E(q)$ to be the set of all e-quantifiers q' such that $q' \preceq q$ and $A(q)$ to be the set of all a-quantifiers q' such that $q' \preceq q$. We further denote by $e(q)$ and $a(q)$ the number of elements in $E(q)$ and $A(q)$ respectively.

Definition 0.2 *[**fdependence**] Let T be a first order template and let $F_0(T)$ be the set of formulas without free variables defined by T. A functional dependence structure on T is a map ϕ which assigns to a pair (F, q) where F is a formula and q is an e-quantifier in F, a set $\phi(F, q)$ of families of the form $(f_{q'})_{q' \in E(q)}$ where $f_{q'}$ is an element of $Func_{a(q')}(T)$. This map should satisfy the following conditions:*

1. *??take equivalent to the same*
2. *??$f_{q'}$ do not appear in F*

Definition 0.3 *[**rdependence**] Let T be a first order template. A relational dependence structure on T is a map ρ which assigns to any formula F in $F(T)$ an element of $Pred_n$ where n is the number of free variables in F. This map should satsify the following conditions:*

1. *...*

March 13, 2004 We define a system of reasoning as a first order template together with a relational dependance structure and a functional dependence structure.

We want to encode (represent) mathematics as a text in the associated formula language (i.e. as a set of formulas). Given any text satisfying certain compatibility properties we want to be able to navigate it according to certain rules. These rules may be approximately as follows:

1. There is a catalog of all predicates used in the text. These predicates are intuitively understood as types of objects (i.e. there may be a predicate called "isagroup" corresponding to groups).
2. For each predicate there is a catalog of all functions which have the validity of the predicate as a prerequisite - they are intuitively understood as the constructions which can be performed on objects of a given type. E.g. there may be a function "underlying-set-of-a-group" or a function "the-automorphism-group-of-a-group".
3. Similarly for each predicate there is a catalog of functions which produce objects of the corresponding type i.e. which have the predicate in post-requisites. E.g. there may be "the-automorphism-group-of-a-set" or "the-mean-value-of-a-random-variable".

In fact we should think about encoding for an individual mathematical paper (article).

March 16,2004 Let T be a first order template. Let further $F_n(T)$ be the set of formulas in T with n free variables and let $\pi_n : Pred_n \to F_n(T)$, $n \geq 0$ be a family of maps. For such a structure let us try to define a three-valued interpretation of (T, π) as follows.

Let us recall first the usual notion of an interpretation of a first order template T in a topos with a distinguished element M. Such an interpretation is given by "primary" maps:

$$[\mathbf{f1}]Func_n(T) \to Hom(M^n, M) \tag{1}$$

$$[\mathbf{f2}]Pred_n(T) \to Hom(M^n, \Omega) \tag{2}$$

where Ω is the subobject classifier. Any interpretation defines "secondary" maps

1. $Tm_n(T) \to Hom(M^n, M)$
2. $F_n(T) \to Hom(M^n, \Omega)$

which extend (**??**) and (**??**). More precisely one should talk about the maps $Tm_A(T) \to Hom(M^A, M)$ for all subsets $A \subset Var$ and similarly for $Pred$. A closed formula defines a map $M^0 = pt \to \Omega$ and one says that it is true (with respect to the given interpretation) if this map is "true" and false if this map is "false". Classically, the topos is the topos of sets where Ω iis the two point set $\{0, 1\}$ in which 0 is "false" and 1 is "true".

We now return to our set-up where we have the prerequisite maps $\pi_n : Pred_n \to F_n$. We want to see first how we can extend π_n to a map $F_n \to F_n$. We have:

1. $\pi(\neg P) = \pi(P)$
2. $\pi(P \vee Q) = P \vee Q \vee (\pi(P) \wedge \pi(Q))$
3. $\pi(P \wedge Q) = \neg P \vee \neg Q \vee (\pi(P) \wedge \pi(Q))$
4. $\pi(\forall_x P) = \forall_x \pi(P)$
5. $\pi(\exists_x P) = \forall_x \pi(P)$

March 26,2004 Have to talk about the "foundations of math". The following approach seems inevitable. We need the category of sets as soon as we want to be able to talk about models of the "elementary" languages which we work with. Since there does not seem to be a universally accepted choice for such a category we do not want to fix one.

Given a definition we want to be able to assign to it a (first order ?) theory t (with prerequisites). The class of objects specified by the definition is then the class of models of this theory in our category of sets Xth. The models in Xth_0 can be represented (internalized) giving rise to an object T in Xth whose elements are models of t in Xth_0.

A construction from t_1 to t_2 (e.g. the underlying multiplicative monoid of a ring) internalizes into a morphism $T_1 \to T_2$.