Universes in the category of cubical sets

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Introduction

The goal of this note is to give a definition of two universes [1] $p : \tilde{U} \to U$ and $p_F : \tilde{U}_F \to U_F$ in the category of cubical set. This first universe is a universe of cubical sets. The second universe is the universe of "fibrant" cubical sets, and provides a model of type theory with dependent product, sum, identity types and function extensionality. Furthermore, it is fibrant and univalent.

Cubical sets

Definition of the base category

A *de Morgan algebra* is a bounded distributive lattice A, with a top element 1 and a bottom element 0 and with an operation $1-i$ satisfying

$$1-0=1 \quad 1-1=0 \quad 1-(i\vee j)=(1-i)\wedge(1-j) \quad 1-(i\wedge j)=(1-i)\vee(1-j)$$

This notion differs from the one of Boolean algebra by requiring neither $1 = i \vee (1-i)$ nor $0 = i \wedge (1-i)$. A prime example of a de Morgan algebra, which is not a Boolean algebra, is the interval $[0,1]$ with $\max(i,j), \min(i,j)$ operations.

We assume a given (discrete) set of symbols/names/directions, not containing $0,1$. We let $I, J, K, \ldots$ denote finite sets of such symbols. We also assume a function $\mathsf{fresh}(I)$ which selects a name not in I. Let $\mathcal{C}$ be the following precategory. The objects are finite sets of names $I, J, K, \ldots$. A morphism $I \to J$ is a map $I \to \mathsf{dM}(J)$, where $\mathsf{dM}(J)$ is the free de Morgan algebra on J. We think of f as a substitution and may write if the element $f(i)$ in $\mathsf{dM}(J)$. If $f : I \to J$ and $g : J \to K$ we write $fg : I \to K$ the composition of f and g. We write $1_I : I \to I$ the identity map. A *cubical set* is a presheaf on $\mathcal{C}^{opp}$, i.e. a functor $\mathcal{C} \to \mathsf{Set}$.

Another equivalent definition of $\mathcal{C}^{opp}(I,J)$ is the set of monotone maps $2^I \times 2^I \to 2^J \times 2^J$.

We shall need a few notions and properties of the base category that are listed in Appendix I.

A cubical set X is thus given by a family of sets $X(I)$ together with a restriction map

$$X(I) \to X(J)$$

$$u \longmapsto uf$$

such that $u1_I = u$ and $(uf)g = u(fg)$. (We write uf for what is usually written $X(f)(u)$, since we want to think about this operation as a substitution; the elements of $X(I)$ for $I = i_1, \ldots, i_n$ are thought of as elements $u = u(i_1, \ldots, i_n)$ depending on $i_1, \ldots, i_n$ and the restriction uf as a substitution operation.)

If X is a presheaf on the slice category $\mathcal{C}/I$ we let $\Gamma(X)$ be the set $X(I, 1_I)$. If X and Y are two presheaves on $\mathcal{C}/I$ an element of $\Gamma(X \to Y)$ is given by a family of maps $w_{(J,f)} : X(J,f) \to Y(J,f)$ such that $(w_{(J,f)}\ u)g = w_{(K,fg)}\ ug$. We also can consider the presheaf $P(X)$, with $P(X)(J,f) = X(j, J, f\iota_j)$ where $j = \mathsf{fresh}(J)$. We can then define the set $\mathsf{Iso}(X,Y)$ as a subset of

$$\Gamma((X \to Y) \times (Y \to X) \times (X \to P(Y)) \times (Y \to P(X)))$$

The universe of cubical sets

We fix a *set* V of small sets. Each element a in V determines a small set $El(a)$. The pair V, El defines then a (pre)category by taking the set of morphisms between a and b to be $El(a) \to El(b)$. If $\mathcal{D}$ is a (pre)category, we define the *set* of V-valued presheaves on $\mathcal{D}$ to be the set of functors from $\mathcal{D}^{opp}$ to this precategory.

Given I, we can consider the set of V-valued functors on the slice (pre)category $\mathcal{C}/I$. Such a functor A is given by a family of sets $A(J, f)$ in V and restriction maps $El(A(J, f)) \to El(A(K, fg))$. If A is such a functor, we can define a new functor $P(A)$ by taking $P(A)(I, f)$ to be $A(i, I, f\iota_i)$ where $i = \mathsf{fresh}(I)$. If p is in $P(A)(I, f)$ we can consider $p0 = p(i0)$ and $p1 = p(i1)$ in $A(I, f)$. If a is in $A(I, f)$ we consider the constant path $\overline{a} = a\iota_i$ in $P(A)(I, f)$.

We define $U(I)$ to be the set of all V-valued presheaves on $\mathcal{C}/I$. The restriction map $U(I) \to U(J),\ A \longmapsto Af$ is defined by letting Af be the composition of A and the functor $\mathcal{C}/J \to \mathcal{C}/I$.

If A is an element of the set $U(I)$ we write $\Gamma(A)$ the set of "global element" of A, which is the set $A(I, 1_I)$. If $f : I \to J$, we have $\Gamma(Af) = A(J, f)$ and restriction maps $\Gamma(A) \to \Gamma(Af),\ u \longmapsto uf$.

We let $\tilde{U}(I)$ be the set of pairs A, u where A is an element of $U(I)$ and u an element of the set $\Gamma(A)$. We define $(A, u)f = (Af, uf)$. We define the natural transformation $p : \tilde{U} \to U$ by $p(A, u) = A$.

Proposition 0.1 *$p : \tilde{U} \to U$ defines a universe [1] in the category of cubical sets.*

Indeed if X is a cubical set and $\sigma : X \to U$ then we can define the cubical set (X, σ) by taking $(X, \sigma)(I)$ to be the set of pairs x, u with x in $X(I)$ and u in $\Gamma(\sigma x)$.

Glueing operation on cubical sets

We assume to have an operation $\mathsf{glue}_I\ A\ \vec{\sigma}$ in $U(I)$, for A in $U(I)$ and a system of maps σ_α in $T_\alpha \to A\alpha$, which satisfies

1. $\mathsf{glue}_I\ A\ \vec{\sigma} = T$ if $\vec{\sigma}$ is $() \mapsto \sigma$ with $\sigma : T \to A$
2. regularity: $\mathsf{glue}_I\ A\ (\vec{\sigma}, \alpha \mapsto id_{A\alpha}) = \mathsf{glue}_I\ A\ \vec{\sigma}$
3. uniformity: $(\mathsf{glue}_I\ A\ \vec{\sigma})f = \mathsf{glue}_J\ Af\ \vec{\sigma}f$ if $f : I \to J$

A system of maps $\vec{\sigma}$ is given by a sieve L in $\mathsf{S}(I)$ (as defined in Appendix I) and a compatible family of maps $\sigma_{(J,f)}$ in $T_{(J,f)} \to Af$ indexed by (J, f) in L. If we have two systems on L and M respectively which coincide on $L \cap M$ then they define a system over the union L, M. For the regularity condition, it is assumed that the system defined by $\alpha \mapsto id_{A\alpha}$ and the system $\vec{\sigma}$ are compatible, that is we have $\sigma_{(J,f)}u = u$ if $f \leqslant \alpha$.

Furthermore any element of $\Gamma(\mathsf{glue}_I\ A\ \vec{\sigma})$ is uniquely determined by a tuple $u, \vec{t}$ with u in $\Gamma(A)$ and t_α in $\Gamma(T_\alpha)$, such that $u\alpha = \sigma_\alpha t_\alpha$. If we write $(u, \vec{t})$ this element, we have $(u, \vec{t})f = (uf, \vec{t}f)$. We also have a map $\delta_I : \mathsf{glue}_I\ A\ \vec{\sigma} \to A$ such that $\delta_I(u, \vec{t}) = u$.

If we don't require the regularity condition, then we can define $\Gamma(\mathsf{glue}_I\ A\ \vec{\sigma})$ to be the set of all such tuple $u, \vec{t}$. (This is actually enough for showing that the universe U_F defined later is univalent; the regularity condition is needed to show that this universe is fibrant.) One possible definition of such a glueing operation satisfying the regularity condition is in Appendix II.

Dependent sets

If X is a cubical set, the *category of element of* X has for object pair (I, ρ) with ρ in $X(I)$ and a morphism between I, ρ and J, ν is a map $f : I \to J$ such that $\nu = \rho f$. If X is a cubical set, a *fibration* $X \vdash B$ over X is given by a V-valued presheaf on the category of element of X.

Such a fibration defines a cubical set $X.B$ by taking $(X.B)(I)$ to be the set of pairs (ρ, v) with ρ in $X(I)$ and v in $B(I, \rho)$.

The universe of fibrant cubical sets

If A is an element of the set $U(I)$ we define the set of composition operations and the set of transport operations for A.

Composition

A *composition operation* for A is given by a family of operations $\mathsf{comp}_f\ u\ \vec{p}$ element in $El(A(I,f))$, u in the set $El(A(I,f))$ and $\vec{p}$ a system for $P(A)$ such that $u\alpha = p_\alpha 0$. We should have

1. $\mathsf{comp}_f\ u\ [() \mapsto p] = p1$
2. *regularity*: $\mathsf{comp}_f\ u\ (\vec{p}, \alpha \mapsto \overline{u\alpha}) = \mathsf{comp}_f\ u\ \vec{p}$
3. *uniformity*: $(\mathsf{comp}_f\ u\ \vec{p})g = \mathsf{comp}^j_{fg}\ ug\ \vec{p}g$ if $g : J \to K$.

Transport

A *transport operation* for A is given by a family of operations transp^i_f in

$$El(A(J-i, f(i0))) \to El(A(J-i, f(i1)))$$

for $f : I \to J$ and i in J. Furthermore we should have

1. *regularity*: $\mathsf{transp}^i_f\ u_0 = u_0$ if we have $f = f(i0)\iota_i$
2. *uniformity*: $(\mathsf{transp}^i_f\ u_0)g = \mathsf{transp}^j_{f(g,i=j)}\ u_0 g$ if $g : J - i \to K$ and j not in K

We let $U_F(I)$ be the set of element in $U(I)$ together with a composition and transport operation.

If $(\mathsf{comp}_h, \mathsf{transp}^i_h)$ is a composition and transport for A, V-valued presheaf on $\mathcal{C}/I$, then $(\mathsf{comp}^j_{fg}, \mathsf{comp}^j_{fg})$ is a composition and transport for Af, V-valued presheaf on $\mathcal{C}/J$.

We define in this way a restriction map $U_F(I) \to U_F(J)$ and a cubical set U_F.

There is a projection map $U_F \to U$ but U_F is not a subpresehaf of U.

A particular case of regularity can be seen as a kind of α-conversion for transport

$$\mathsf{transp}^i_f u_0 = \mathsf{transp}^j_{f(1_K, i=j)} u_0$$

if $f : I \to K, i$ and j is not in K.

Kan cubical sets

Notice that an element of $U()$ is given by a V-valued presheaf on $\mathcal{C}$ and each transport function

$$\mathsf{transp}^i : A_{I-i} \to A_{I-i}$$

is constant by regularity.

A *Kan cubical set* is a cubical set X together with a composition operation $\mathsf{comp}_I\ u\ \vec{p}$ in $X(I)$ with u in $X(I)$ and $\vec{p}$ a system for X such that

1. $\mathsf{comp}_I\ u\ [() \mapsto p] = p1$
2. *regularity*: $\mathsf{comp}_I\ u\ (\vec{p}, \alpha \mapsto \overline{u\alpha}) = \mathsf{comp}_I\ u\ \vec{p}$
3. *uniformity*: $(\mathsf{comp}_I\ u\ \vec{p})f = \mathsf{comp}^j_J\ uf\ \vec{p}f$

Kan fibration

If X is a cubical set, the *category of element of* X has for object pair (I, ρ) with ρ in $A(I)$ and a morphism between I, ρ and J, ν is a map $f : I \to J$ such that $\nu = \rho f$. If X is a cubical set, a *Kan fibration* $X \vdash B$ over X is given by a V-valued presheaf on the category of element of X which admits composition and transport operations. We should have

1. $\mathsf{comp}_{I,\rho}\ u\ [() \mapsto p] = p1$
2. *regularity*: $\mathsf{comp}_{I,\rho}\ u\ (\vec{p}, \alpha \mapsto \overline{u\alpha}) = \mathsf{comp}_{I,\rho}\ u\ \vec{p}$
3. *uniformity*: $(\mathsf{comp}_{I,\rho}\ u\ \vec{p})f = \mathsf{comp}^j_{J,\rho f}\ uf\ \vec{p}f$ if $g : I \to J$ and j is not in J

for the composition operation, and

1. *regularity*: $\mathsf{transp}^i_{I,\rho}\ u_0 = u_0$ if we have $\rho = \rho(i0)\iota_i$
2. *uniformity*: $(\mathsf{transp}^i_{I,\rho}\ u_0)g = \mathsf{transp}^j_{(J,j),\rho(g,i=j)}\ u_0 g$ if $g : I - i \to J$ and j not in J

for the transport operation.

Whenever $X \vdash B$ we can define the associated total space $X.B$ by taking $(X.B)(I)$ to be the set of pairs ρ, u with ρ in $A(I)$ and u in $B(I, \rho)$ and $(\rho, u)f = \rho f, uf$. We have a projection $p : X.B \to X$ defined by $p(\rho, u) = \rho$.

Definition of $\tilde{U}_F \to U_F$

By change of base of $p : \tilde{U} \to U$ along the projection $U_F \to U$ we get a map $\tilde{U}_F \to U_F$. Concretely, an element of $\tilde{U}_F(I)$ is given by an element A in $U(I)$, a composition and a transport operation on A, and an element of $\Gamma(A)$.

We define $\mathsf{comp}_{I,A}\ u\ \vec{u}$ to be $\mathsf{comp}_{1_I}\ u\ \vec{u}$ and $\mathsf{transp}^i_{I,A}$ to be $\mathsf{transp}^i_{1_I}$.

Proposition 0.2 *With these operations of composition and transport, $U \vdash B$ is a Kan fibration.*

Given A and B in $U_F(I)$ we can define $\mathsf{Id}(I, A, B)$ to be the set of elements E in $U_F(I, i)$ satisfying $E(i0) = A$ and $E(i1) = B$, where $i = \mathsf{fresh}(I)$. We also have defined already the set $\mathsf{Iso}(I, A, B)$. This corresponds to two fibrations $U_F \times U_F \vdash \mathsf{Id}$ and $U_F \times U_F \vdash \mathsf{Iso}$.

Theorem 0.3 *We have a natural transformation* $\mathsf{Iso}(I, A, B) \to \mathsf{Id}(I, A, B)$.

Theorem 0.4 *U_F has a structure of a Kan cubical set.*

All operations are defined in term of the glueing operation on cubical sets.

Appendix I: Properties of the base precategory

We say that a map $f : I \to J$ is *strict* if if is neither 0 nor 1 for all i in I. One key remark is the following.

Lemma 0.5 *If $f : I \to J$ is strict and ψ in $\mathsf{dM}(I)$ such that $\psi f = b$ (where b is 0 or 1) then already $\psi = b$.*

(This does not hold if we work with Boolean algebra instead of de Morgan algebra. For instance the map $(i = j) : \{i, j\} \to \{j\}$ is strict and $(i \wedge (1 - j))(i = j) = 0$ in a Boolean algebra, but $i \wedge (1 - j)$ is neither 0 nor 1.)

A face map $\alpha : I \to I_\alpha$ is a map such that $i\alpha$ is either $0, 1$ or i for all i in I. We write I_α the subset of element i such that $i\alpha = i$, and $dom(\alpha) = I - I_\alpha$ is the *domain* of α. If $\iota_\alpha : I_\alpha \to I$ is the inclusion, we have $\iota_\alpha \alpha = 1$ and hence any face map α is *epi*. If $f : I \to J$ we write $f \leqslant \alpha$ to mean that there exists a map f' (uniquely determined) such that $f = \alpha f'$. This means that $if = i\alpha$ for all i in the domain of α. This defines a poset structure on the set of face maps $\alpha : I \to I_\alpha$ and this poset is a partial meet-semilattice: if α and β are compatible then they have a meet $\gamma = \alpha \wedge \beta$ with $I_\gamma = I_\alpha \cap I_\beta$.

Corollary 0.6 *If $fg \leqslant \alpha$ and g is strict then $f \leqslant \alpha$.*

Proof. For any i in the domain of α we have $i\alpha = ifg$ and so $i\alpha = if$ since $i\alpha = 0$ or 1 and by Lemma 0.5. □

Any map $f : I \to J$ can be written uniquely as the composition $f = \alpha h$ of a face map $\alpha : I \to I_\alpha$ and a map $h : I_\alpha \to J$ which is strict.

If i not in I we write $\iota_i : I \to I, i$ the inclusion. If i is in I we write $(i0) : I \to I - i$ and $(i1) : I \to I - i$ the two face operation for i.

Lemma 0.7 *If we have $\alpha f = \beta g$ with $f : I_\alpha \to J$ and $g : I_\beta \to J$ then α and β are compatible. If γ is the meet of α and β, then there exists a unique $h : I_\gamma \to J$ such that $\alpha f = \gamma h = \beta g$. If we write $\alpha\alpha_1 = \gamma = \beta\beta_1$ then $\alpha_1 f = h = \beta_1 g$.*

Systems

We define $\mathsf{S}(I)$ to be the set of sieves L over I such that f is in L whenever fg is in L and g is strict. Such a sieve L is completely characterised by its subset of face maps $\alpha : I \to I_\alpha$. This defines a cubical set S.

If A is a presheaf on the slice category $\mathcal{C}^{opp}/I$ and L in $\mathsf{S}(I)$ a *L-system for A* is given by a family $a_{(J,f)}$ in Af for (J, f) in L such that $a_{(J,f)}g = a_{(K,fg)}$ for all $g : J \to K$.

If $f : I \to J$ and we have a L-system $\vec{a}$ we define a Lf system $\vec{b} = \vec{a}f$ by taking $b_{(K,g)} = a_{(K,fg)}$.

A L-system for A is completely determined by the family $[\alpha \mapsto a_\alpha]$ for α face in L. If L is the union of M and N, and we have a M-system $\vec{a}$ and a N-system $\vec{b}$ that coincide on $M \cap N$ then they define a system $\vec{a}, \vec{b}$ on the union M, N.

Appendix II: Definition of the glueing operation

An *I-element* in V is a tuple (u_α) indexed by the face operations of I. An *I-set* is a set all element of which are I-elements. If X is an I-set, and u is an element of X we can consider the element u_α for each face operation α on I.

We refine the definition of $U(I)$ by imposing $A(J, f)$ to be a J-set *and* the restriction operation $A(J, f) \to A(J\beta, f\beta),\ u \longmapsto u\beta$ to satisfy $u\beta_\gamma = u_{\beta\gamma}$ for each face operation γ on $J\beta$.

With this refinement, we can define, given u in $\Gamma(A)$ and t_α in $\Gamma(T_\alpha)$ such that $\sigma_\alpha t_\alpha = u\alpha$, a tuple $(u, \vec{t})$ by $(u, \vec{t})_\beta = t_\beta$ if $\beta \leqslant L$ and $(u, \vec{t})_\beta = u_\beta$ otherwise. Notice that we have $(u, \vec{t}) = u$ if all σ_α are identity maps. We then define $\Gamma(\mathsf{glue}_I\ A\ \vec{\sigma})$ to be the I-set of elements $(u, \vec{t})$.

If v is in $\Gamma(\mathsf{glue}_I\ A\ \vec{\sigma})$ we define $(\delta_I v)_\beta = \sigma_\beta v_\beta$ if $\beta \leqslant L$ and $(\delta_I v)_\beta = v_\beta$ otherwise.

References

[1] V. Voevodsky. Notes on type systems. github.com/vladimirias/old_notes_on_type_systems, 2009-2012.