Comments on the work by Thierry Coquand, Simone Huber, Anders Mortberg and Cyril Cohen on cubical sets.

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Comments on uu.pdf version from May 14, 2015:

Last paragraph on p.1. It would be good to have it as a lemma. Mathematically speaking the statement is as follows.

First some generalities. For any category $\mathcal{C}$ the category of presheaves $PS = PreShv(\mathcal{C})$ can be given the structure of a cartesian closed category. For a definition of the structure of a cartesian closed category see [**?**, p.96]. As a part of the cartesian closed structure one has to specify for all $X, Y \in PS$ an object $X \times Y$ and an object $\underline{Hom}(X, Y)$. We will sometimes denote the latter object by $(X \to Y)$.

In the usual definition of this structure on the presheaf categories $X \times Y$ is defined as the presheaf that, on objects, takes $I \in \mathcal{C}$ to $X(I) \times Y(I)$[3].

The object $\underline{Hom}(X, Y)$ can be defined in various ways (such definitions will produce isomorphic but not equal presheaves). One definition is as follows.

Definition 0.1 *[***2015.05.15.def1***] Define an element of $\underline{Hom}(X, Y)(I)$ as a collection of data of the form:*

1. *for all $J \in \mathcal{C}$, $f : J \to I$, $u \in X(J)$, an element $\phi(J, f, u) \in Y(J)$,*
2. *for all $K, J \in \mathcal{C}$, $g : K \to J$, $f : J \to I$, $u \in X(J)$, equality*

$$\phi(K, g \circ f, X(g)(u)) = Y(g)(\phi(J, f, u))$$

The the action of $\underline{Hom}(X, Y)$ on morphisms $h : I' \to I$ is given by

$$\underline{Hom}(X, Y)(h)(\phi)(J, f : J \to I', u) = \phi(J, f \circ h, u)$$

One can prove that this data defines a presheaf. One can then construct natural transformations $\underline{Hom}(X, Y) \to \underline{Hom}(X', Y)$ for $f : X' \to X$, to show that they satisfy the identity and composition axioms, to construct maps

$$ev^W_{X,Y} : Hom_{PS}(W, \underline{Hom}(X, Y)) \to Hom_{PS}(W \times X, Y)$$

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[3]Later when one considers V-values presheaves this will require a choice of the product function $V \to V \to V$ on V and similarly for other constructions discussed below.

to show that these maps are bijections and that they are natural in X. Having done all that one would show that so defined the presheaves $\underline{Hom}(X, Y)$ form a part of a cartesian closed structure on PS as defined in [**?**].

This is actually not very hard, the verifications involved are often very simple and writing it up in UniMath should be a good exercise. It does not involve any use of squash types or quotients and therefore does not require any resizing rules. It will, most likely, require function extensionality to make sure that the h-levels of various "forall" types are what they should be. It does not require any other use of the univalence axiom.

Many of the proofs can probably be clarified by defining $\underline{Hom}(X, Y)(I)$ not directly as was done above but by the formula

$$[\mathbf{2015.05.15.eq1}]\underline{Hom}'(X, Y)(I) := Hom_{PS}(y(I) \times X, Y) \tag{1}$$

where $y(I)$ is the presheaf represented by I.

Now back to the last paragraph in uu.pdf p.1. To me, to make sense of it, I had to solve the following problem:

Problem 0.2 *[**2015.05.15.l1**] Let $\mathcal{C}$ be a category, $I \in \mathcal{C}$ and let X, Y be two presheaves on $\mathcal{C}/I$. Let $\Phi(X, Y)$ be the set of collections of data of the form*

1. *for all $J \in \mathcal{C}$, $f : J \to I$ a map $w_{(J,f)} : X(J, f) \to Y(J, f)$,*
2. *for all $K \in \mathcal{C}$, $g : K \to J$ an equality*

   $$Y(g)(w_{(J,f)}(u)) = w_{(K,f\circ g)}(X(g)(u))$$

   or, in the notations of uu.pdf

   $$(w_{(J,f)}u)g = w_{(K,f\circ g)}ug$$

to construct a bijection

$$\rho : \Phi(X, Y) \to \underline{Hom}(X, Y)(I, 1_I)$$

Construction 0.3 *[**2015.05.15.constr1**] This is easy to do using the fact that $(I, 1_I)$ is a final object of $\mathcal{C}/I$ - it is actually easier to do it formally then to write it in English.*

*For me it was also easier to do it using (**??**) rather than Definition **??** since then from the fact that $(I, 1_I)$ is the final object we have a bijection from $\underline{Hom}'(X, Y)(I, 1_I)$ to $Hom_{PreShv(\mathcal{C}/I)}(X, Y)$ and the later set is equal to (probably definitionally in Coq) the set $\Phi(X, Y)$.*