Universes in the category of cubical sets

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Introduction

The goal of this note is to give a definition of two universes [1] $p : \tilde{U} \to U$ and $p_F : \tilde{U}_F \to U_F$ in the category of cubical set. This first universe is a universe of cubical sets. The second universe is the universe of "fibrant" cubical sets, and provides a model of type theory with dependent product, sum, identity types and function extensionality.

Cubical sets

Definition of the base category

We assume a given (discrete) set of symbols/names/directions, not containing $0, 1$. We let $I, J, K, \dots$ denote finite sets of such symbols. We also assume a function $\mathsf{fresh}(I)$ which selects a name not in I. Let $\mathcal{C}$ be the following precategory. The objects are finite sets of names $I, J, K, \dots$. A morphism $J \to I$ is a monotone map $2^J \to 2^I$. Equivalently, a morphism $J \to I$ can be seen as a map $f : I \to \mathsf{D}(J)$ where $\mathsf{D}(J)$ is the free bounded distributive lattice on J. (This follows from the fact that the free bounded distributive lattice on J is the lattice of monotone maps $2^J \to 2$.)

We write $1_I : I \to I$ the identity map. If $f : J \to I$ and $g : K \to J$ we write $fg : K \to I$ their composition.

A *cubical set* is a presheaf on $\mathcal{C}$, i.e. a functor $\mathcal{C}^{opp} \to \mathsf{Set}$.

We shall need a few notions and properties of the base category that are listed in Appendix I.

A cubical set X is thus given by a family of sets $X(I)$ together with a restriction map

$$X(I) \to X(J)$$

$$u \longmapsto uf$$

such that $u1_I = u$ and $(uf)g = u(fg)$. We write uf for what is usually written $X(f)(u)$.

If X is a presheaf on the slice category $\mathcal{C}/I$ we let $\Gamma(X)$ be the set $X(I, 1_I)$. If X and Y are two presheaves on $\mathcal{C}/I$, we can consider their exponential $X \to Y$ and an element of $\Gamma(X \to Y)$ is given by a family of maps $w_{(J,f)} : X(J, f) \to Y(J, f)$ such that $(w_{(J,f)}\ u)g = w_{(K,fg)}\ ug$ for $g : K \to J$. We also can consider the presheaf of *paths* $P(X)$, with $P(X)(J, f) = X(j, J, f\iota_j)$ where $j = \mathsf{fresh}(J)$. We can then define the set $\mathsf{Iso}(X, Y)$ as a subset of

$$\Gamma((X \to Y) \times (Y \to X) \times (X \to P(Y)) \times (Y \to P(X)))$$

The universe of cubical sets

We fix a *set* V of small sets. Each element a in V determines a small set $El(a)$. The pair V, El defines then a (pre)category by taking the set of morphisms between a and b to be $El(a) \to El(b)$. If $\mathcal{D}$ is a (pre)category, we define the *set* of V-valued presheaves on $\mathcal{D}$ to be the set of functors from $\mathcal{D}^{opp}$ to this precategory.

Given I, we can consider the set of V-valued presheaves on the slice (pre)category $\mathcal{C}/I$. Such a functor A is given by a family of sets $A(J,f)$ in V and restriction maps $El(A(J,f)) \rightarrow El(A(K,fg))$. If A is such a functor, we can define a new functor $P(A)$ by taking $P(A)(I,f)$ to be $A(i,I,f\iota_i)$ where $i = \mathsf{fresh}(I)$. If p is in $P(A)(I,f)$ we can consider $p0 = p(i0)$ and $p1 = p(i1)$ in $A(I,f)$. If a is in $A(I,f)$ we consider the constant path $\overline{a} = a\iota_i$ in $P(A)(I,f)$.

We define $U(I)$ to be the set of all V-valued presheaves on $\mathcal{C}/I$. The restriction map $U(I) \rightarrow U(J)$, $A \longmapsto Af$ is defined by letting Af be the composition of A and the functor $\mathcal{C}/J \rightarrow \mathcal{C}/I$.

If A is an element of the set $U(I)$ we write $\Gamma(A)$ the set of "global element" of A, which is the set $A(I,1_I)$. If $f : J \rightarrow I$, we have $\Gamma(Af) = A(J,f)$ and restriction maps $\Gamma(A) \rightarrow \Gamma(Af)$, $u \longmapsto uf$.

We let $\tilde{U}(I)$ be the set of pairs A, u where A is an element of $U(I)$ and u an element of the set $\Gamma(A)$. We define $(A,u)f = (Af, uf)$. We define the natural transformation $p : \tilde{U} \rightarrow U$ by $p(A,u) = A$.

Proposition 0.1 *$p : \tilde{U} \rightarrow U$ defines a universe [1] in the category of cubical sets.*

Indeed if X is a cubical set and $\sigma : X \rightarrow U$ then we can define the cubical set (X,σ) by taking $(X,\sigma)(I)$ to be the set of pairs x, u with x in $X(I)$ and u in $\Gamma(\sigma x)$.

Dependent sets

If X is a cubical set, the *category of element of* X has for object pair (I,ρ) with ρ in $X(I)$ and a morphism between J,ν and I,ρ is a map $f : J \rightarrow I$ such that $\nu = \rho f$. If X is a cubical set, a *dependent set* $X \vdash B$ over X is given by a V-valued presheaf on the category of element of X.

Such a dependent set defines a cubical set $X.B$ by taking $(X.B)(I)$ to be the set of pairs (ρ, v) with ρ in $X(I)$ and v in $B(I,\rho)$.

The universe of fibrant cubical sets

If A is an element of the set $U(I)$ we define the set of composition operations and the set of transport operations for A.

Composition

A *composition operation* for A is given by a family of operations $\mathsf{comp}_f\ u\ \vec{p}$ element in $El(A(J,f))$, u in the set $El(A(J,f))$ and $\vec{p}$ a system for $P(A)$ such that $u\alpha = p_\alpha 0$. We should have

1. $\mathsf{comp}_f\ u\ [() \mapsto p] = p1$
2. *regularity*: $\mathsf{comp}_f\ u\ (\vec{p}, \alpha \mapsto \overline{u\alpha}) = \mathsf{comp}_f\ u\ \vec{p}$
3. *uniformity*: $(\mathsf{comp}_f\ u\ \vec{p})g = \mathsf{comp}_{fg}\ ug\ \vec{p}g$ if $g : K \rightarrow I$.

and a dual family of operations where we swap 0 and 1.

Transport

A *transport operation* for A is given by a family of operations transp^i_f in

$$El(A(J-i, f(i0))) \rightarrow El(A(J-i, f(i1)))$$

for $f : J \rightarrow I$ and i in J. Furthermore we should have

1. *regularity*: $\mathsf{transp}^i_f\ u_0 = u_0$ if we have $f = f(i0)\iota_i$
2. *uniformity*: $(\mathsf{transp}^i_f\ u_0)g = \mathsf{transp}^j_{f(g,i=j)}\ u_0 g$ if $g : J - i \rightarrow K$ and j not in K

and a dual family of operations where we swap 0 and 1.

We let $U_F(I)$ be the set of elements in $U(I)$ together with a composition and transport operation.

If $(\mathsf{comp}_h, \mathsf{transp}^i_h)$ is a composition and transport for A, V-valued presheaf on $\mathcal{C}/I$, then $(\mathsf{comp}^j_{fg}, \mathsf{transp}^j_{fg})$ is a composition and transport for Af, V-valued presheaf on $\mathcal{C}/J$.

We define in this way a restriction map $U_F(I) \to U_F(J)$ and a cubical set U_F.

There is a projection map $U_F \to U$ but U_F is not a subpresehaf of U.

A particular case of regularity can be seen as a kind of α-conversion for transport

$$\mathsf{transp}^i_f u_0 = \mathsf{transp}^j_{f(1_K, i=j)} u_0$$

if $f : K, i \to I$ and j is not in K.

Kan cubical sets

Notice that an element of $U()$ is given by a V-valued presheaf on $\mathcal{C}$ and each transport function

$$\mathsf{transp}^i : A_{I-i} \to A_{I-i}$$

is constant by regularity.

A *Kan cubical set* is a cubical set X together with a composition operation $\mathsf{comp}_I\ u\ \vec{p}$ in $X(I)$ with u in $X(I)$ and $\vec{p}$ a system for X such that

1. $\mathsf{comp}_I\ u\ [() \mapsto p] = p1$
2. *regularity*: $\mathsf{comp}_I\ u\ (\vec{p}, \alpha \mapsto \overline{u\alpha}) = \mathsf{comp}_I\ u\ \vec{p}$
3. *uniformity*: $(\mathsf{comp}_I\ u\ \vec{p})f = \mathsf{comp}^j_J\ uf\ \vec{p}f$

and a dual family of operations where we swap 0 and 1.

Kan fibration

If X is a cubical set, a *Kan fibration* $X \vdash B$ over X is given by a V-valued presheaf on the category of element of X, that is a dependent set over X, which admits composition and transport operations. We should have

1. $\mathsf{comp}_{I,\rho}\ u\ [() \mapsto p] = p1$
2. *regularity*: $\mathsf{comp}_{I,\rho}\ u\ (\vec{p}, \alpha \mapsto \overline{u\alpha}) = \mathsf{comp}_{I,\rho}\ u\ \vec{p}$
3. *uniformity*: $(\mathsf{comp}_{I,\rho}\ u\ \vec{p})f = \mathsf{comp}^j_{J,\rho f}\ uf\ \vec{p}f$ if $g : J \to I$ and j is not in J

for the composition operation, and

1. *regularity*: $\mathsf{transp}^i_{I,\rho}\ u_0 = u_0$ if we have $\rho = \rho(i0)\iota_i$
2. *uniformity*: $(\mathsf{transp}^i_{I,\rho}\ u_0)g = \mathsf{transp}^j_{(J,j),\rho(g,i=j)}\ u_0 g$ if $g : J \to I - i$ and j not in J

for the transport operation, together with the dual family of operations where we swap 0 and 1.

Whenever $X \vdash B$ we can define the associated total space $X.B$ by taking $(X.B)(I)$ to be the set of pairs ρ, u with ρ in $A(I)$ and u in $B(I, \rho)$ and $(\rho, u)f = \rho f, uf$. We have a projection $p : X.B \to X$ defined by $p(\rho, u) = \rho$.

Definition of $\tilde{U}_F \to U_F$

By change of base of $p : \tilde{U} \to U$ along the projection $U_F \to U$ we get a map $\tilde{U}_F \to U_F$. Concretely, an element of $\tilde{U}_F(I)$ is given by an element A in $U(I)$, a composition and a transport operation on A, and an element of $\Gamma(A)$. This corresponds to a dependent set $U_F \vdash B$.

We define $\mathsf{comp}_{I,A}\ u\ \vec{u}$ to be $\mathsf{comp}_{1_I}\ u\ \vec{u}$ and $\mathsf{transp}^i_{I,A}$ to be $\mathsf{transp}^i_{1_I}$.

Proposition 0.2 *With these operations of composition and transport, $U_F \vdash B$ is a Kan fibration.*

Glueing operation on cubical sets

We now define an operation $\mathsf{glue}_I\ A\ \vec{\sigma}$ in $U(I)$, for A in $U(I)$ and a system of maps σ_α in $T_\alpha \to A\alpha$, which satisfies

1. $\mathsf{glue}_I\ A\ \vec{\sigma} = T$ if $\vec{\sigma}$ is $() \mapsto \sigma$ with $\sigma : T \to A$
2. $\mathsf{glue}_I\ A\ \vec{\sigma} = A$ if $\vec{\sigma}$ is empty
3. uniformity: $(\mathsf{glue}_I\ A\ \vec{\sigma})f = \mathsf{glue}_J\ Af\ \vec{\sigma}f$ if $f : I \to J$

A system of maps $\vec{\sigma}$ is given by a sieve L in $\mathsf{S}(I)$ (as defined in Appendix I; in particular L is determined by a finite discrete set of faces on I) and a compatible family of maps $\sigma_{(J,f)}$ in $T_{(J,f)} \to Af$ indexed by (J, f) in L.

We assume that V, El is rich enough so that we can define the following operation. Given a system of element t_α in V and u in V we can form the element $\mathsf{glue}(u, \vec{t})$ with

1. $\mathsf{glue}(u, \vec{t}) = u$ if L is empty
2. $\mathsf{glue}(u, \vec{t}) = t$ if L is the total sieve
3. $\mathsf{glue}(u, \vec{t})$ is the tuple $(u, \vec{t})$ otherwise

We define then $(\mathsf{glue}_I\ A\ \vec{\sigma})_{(J,f)}$ to be the set of elements $\mathsf{glue}(u, \vec{t})$ with u in Af and $\vec{t}$ a Lf-system compatible with u.

Any isomorphism in $\mathsf{Iso}(A, T)$ defines in particular a map $T \to A$. So we can define a glueing operation $\mathsf{glue}_I\ A\ \vec{\sigma}$ where now $\vec{\sigma}$ is a system of isomorphisms σ_α in $\mathsf{Iso}(A\alpha, T_\alpha)$.

Given A and B in $U_F(I)$ we let $\mathsf{Id}(I, A, B)$ be the set of elements E in $U_F(I, i)$ satisfying $E(i0) = A$ and $E(i1) = B$, where $i = \mathsf{fresh}(I)$. We also have defined already the set $\mathsf{Iso}(I, A, B)$. This defines two dependent sets $U_F \times U_F \vdash \mathsf{Id}$ and $U_F \times U_F \vdash \mathsf{Iso}$ over $U_F \times U_F$ that is two presheaves on the category of elements of $U_F \times U_F$.

Theorem 0.3 *We have a natural transformation* $\mathsf{Iso}(I, A, B) \to \mathsf{Id}(I, A, B)$.

This uses the glueing operation defined above. We "lift" this glueing operation on cubical sets to an operation on Kan cubical sets. We define an operation $\mathsf{glue}_I\ A\ \vec{\sigma}$ in $U_F(I)$, for A in $U_F(I)$ and a system of maps σ_α in $\mathsf{Iso}(A\alpha, T_\alpha)$ which satisfies

1. $\mathsf{glue}_I\ A\ \vec{\sigma} = T$ if $\vec{\sigma}$ is $() \mapsto \sigma$ with σ in $\mathsf{Iso}(A, T)$
2. $\mathsf{glue}_I\ A\ \vec{\sigma} = A$ if $\vec{\sigma}$ is empty
3. uniformity: $(\mathsf{glue}_I\ A\ \vec{\sigma})f = \mathsf{glue}_J\ Af\ \vec{\sigma}f$ if $f : I \to J$

This is the first main point to be formalized.

Once this operation is defined, for any isomorphism σ in $\mathsf{Iso}(A, T)$ we can consider

$$\mathsf{glue}_{I,i}\ A\iota_i\ [(i1) \mapsto \sigma]$$

which is an element of $\mathsf{Id}(I, A, T)$.

Appendix I: Properties of the base precategory

A map $f : J \to I$ can be seen as a map $I \to \mathsf{D}(J)$. We think of f as a substitution and can thus consider the element $f(i)$ in $\mathsf{D}(J)$ for i in I and the element ψf in $\mathsf{D}(J)$ if ψ is an element in $\mathsf{D}(I)$. We say that a map $f : J \to I$ is *strict* if $f(i)$ is neither 0 nor 1 for all i in I.

Lemma 0.4 *If $f : J \to I$ is strict and ψ in $\mathsf{D}(I)$ such that $\psi f = b$ (where b is 0 or 1) then already $\psi = b$.*

A face map $\alpha : I\alpha \to I$ is a map such that $\alpha(i)$ is either $0, 1$ or i for all i in I. We write $I\alpha$ the subset of element i such that $\alpha(i) = i$, and $dom(\alpha) = I - I_\alpha$ is the *domain* of α. If $\iota_\alpha : I\alpha \to I$ is the inclusion, we have $\iota_\alpha \alpha = 1$ and hence any face map α is *mono*. If $f : J \to I$ we write $f \leqslant \alpha$ to mean that there exists a map f' (uniquely determined) such that $f = \alpha f'$. This means that $f(i) = \alpha(i)$ for all i in the domain of α. This defines a poset structure on the set of face maps $\alpha : I\alpha \to I$ and this poset is a partial meet-semilattice: if α and β are compatible then they have a meet $\gamma = \alpha \wedge \beta$ with $I\gamma = I\alpha \cap I\beta$.

Corollary 0.5 *If $fg \leqslant \alpha$ and g is strict then $f \leqslant \alpha$.*

Proof. For any i in the domain of α we have $\alpha(i) = g(f(i))$ and so $\alpha(i) = f(i)$ since $\alpha(i) = 0$ or 1 and by Lemma 0.4. □

Any map $f : J \to I$ can be written uniquely as the composition $f = \alpha h$ of a face map $\alpha : I\alpha \to I$ and a map $h : J \to I\alpha$ which is strict.

If i not in I we write $\iota_i : I, i \to I$ the projection $2^{I,i} \to 2^I$. If i is in I we write $(i0) : I - i \to I$ and $(i1) : I - i \to I$ the two face operation for i.

Lemma 0.6 *If we have $\alpha f = \beta g$ with $f : J \to I\alpha$ and $g : J \to I\beta$ then α and β are compatible. If γ is the meet of α and β, then there exists a unique $h : J \to I\gamma$ such that $\alpha f = \gamma h = \beta g$. If we write $\alpha\alpha_1 = \gamma = \beta\beta_1$ then $\alpha_1 f = h = \beta_1 g$.*

Systems

We define $\mathsf{S}(I)$ to be the set of sieves L over I such that f is in L whenever fg is in L and g is strict. Such a sieve L is completely characterised by its subset of face maps $\alpha : I\alpha \to I$, and we require this subset to be decidable. This defines a cubical set S.

If A is a presheaf on the slice category $\mathcal{C}/I$ and L in $\mathsf{S}(I)$ a *L-system for A* is given by a family $a_{(J,f)}$ in Af for (J, f) in L such that $a_{(J,f)}g = a_{(K,fg)}$ for all $g : J \to K$.

If $f : I \to J$ and we have a L-system $\vec{a}$ we define a Lf system $\vec{b} = \vec{a}f$ by taking $b_{(K,g)} = a_{(K,fg)}$.

A L-system for A is completely determined by the family $[\alpha \mapsto a_\alpha]$ for α face in L. If L is the union of M and N, and we have a M-system $\vec{a}$ and a N-system $\vec{b}$ that coincide on $M \cap N$ then they define a system $\vec{a}, \vec{b}$ on the union M, N.

References

[1] V. Voevodsky. Notes on type systems. github.com/vladimirias/old_notes_on_type_systems, 2009-2012.