REFEREE REPORT

"LAWVERE THEORIES AND C-SYSTEMS" BY VLADMIR VOEVODSKY

1. Summary and recommendation

The submitted paper is concerned with Lawvere theories and C-systems. Lawvere theories provide a syntax-free alternative to algebraic theories and are one of the fundamental notions in the category-theoretic approach to universal algebra. The notion of a C-system (originally introduced under the name of 'contextual category') provides instead one possible notion of a model for dependent type theory. They have been intensely studied by the author in recent years, with a view towards obtaining a detailed construction of the simplicial model of univalent foundations.

The main result in the paper states that Lawvere theories are equivalent to a natural class of C-systems, namely those whose 'lenght function' is a bijection. More precisely, the author shows that there is an isomorphism between the category of Lawvere theories and the category of C-systems whose length function is bijective.

This is an important and original result. It is imporant because it links precisely two notions that have been studied in separate areas: Lawvere theories have been studied mostly in connection with universal algebra and, more recently, programming language semantics; C-systems, instead, have been studied only in relation to the semantics of dependent type theories. So it is quite interesting to know how C-systems subsume Lawvere theories and it is remarkable that the author succeded in characterizing exactly the C-systems that come from Lawvere theories with a natural condition.

I suspect that the idea that C-systems subsume Lawvere theories may not be completely surprising to experts in the field, since it is known that C-systems have roughly the same expressive power as the so-called 'essentially algebraic theories', which are a generalization of algebraic theories. The latter, in turn, are equivalent to Lawvere theories. But the paper under review has the significant merit of providing a direct comparison between Lawvere theories and C-systems and of including detailed proofs of the results.

However, the presentation of the material in the paper falls very short of the standards of a major mathematical journal. For example, there are very few informal explanations of the ideas involved, a lot of proofs are given by chains of equations without justification, some choices of notation make it very hard to identify the relevant objects, and sometimes definitions and proofs are rather long-winded. These problems are, individually, rather minor, but their combined effect makes the paper very difficult to read.

For these reasons, I recommend that the paper is revised according to the suggestions below and accepted for publication in the Proceedings of the American Mathematical Society only if the revision is carried out satisfactorily.

2. General comments

- Add explanations at the start or end of each of the technical sections to explain what is being done in the section, what are the most delicate aspects, and how what has been done will be used in the reminder of the paper.
- In order to improve notation, I would ask the author to take advantage of the existence of different fonts to help the reader distinguish between different kinds of objects. I include specific suggestion below.
- The paper should include at least one example of the C-system associated to a Lawvere theory (the Lawvere theory corresponding to the algebraic theory of groups, say) and one example of the Lawvere theory associated to a ℓ-bijective C-system (e.g. the bijective C-system generated by the type $\mathbb{N}$ of natural numbers in a Martin-Löf type theory). This could be done in outline, but it would definitely help readers to understand the constructions presented in the paper.

3. Comments on individual sections

Section 1.

- The introduction is too short and does not provide sufficient context and motivation for readers who are not already familiar with the ideas of the paper. It should be explanded significantly (see also general comments above). In particular, add an informal explanation of why there is an isomorphism between ℓ-bijective C-systems and Lawvere theories, which can then be expanded and made more precise in the body of the paper.
- It is not clear how the material plays a role in the simplicial model of univalent foundations. This point should be explained further or omitted.
- Give reference for the 'UniMath language' and the Calculus of Inductive Constructions.
- Give reference for the simplicial model.
- Clarify the sentence that "such a construction itself cannot reply on the univalent foundations...". Other foundational systems are certainly strong enough to formalize (parts of) their metatheory, without violating Gödel's incompleteness by adding suitable assumptions in the metatheory. For example, there has been work in formalizing type theories within Coq. Why is this not the case for univalent foundations?

- Clarify the background setting of the paper. At present, it is written that the paper is developed 'from the perspective of the Zermelo-Frankel formalism', but 'the proof 'doe not use excluded middle or the axiom of choice', which are part of ZFC. In which theory are you working?
- It is claimed that the paper does not use excluded middle (page 1, last line), but maybe some comments in the body of the paper should confirm this point in delicate steps. For example, in construction 4.2 the function ft is defined by cases, and this uses excluded middle at first sight. But I think that excluded middle may be avoided since equality of natural numbers is decidable and there is a unique object X such that $\ell(X) = 0$. This kind of issue should be commented on in the appropriate places.
- The introduction should include an outline of the contents of the paper.

Section 2.

- Say that $\mathbf{N}$ denotes the set of natural numbers (I would rather use $\mathbb{N}$, as usual).
- Use a special font to denote the standard sets with n elements, say $\mathsf{stn}(n)$.
- Since the paper is written in the context of set theory, there is no need to recall (even in outline) the notion of a function. Correspondingly, delete the reference to Bourbaki.
- Use different notation for the category of finite sets and functions, for example **FinSet**. Using the letter F is not sufficiently memorable.
- Define the hom-sets of the category of finite sets more simply, by letting

$$\mathbf{FinSet}[m, n] = \mathbf{Set}[\mathsf{stn}(n), \mathsf{stn}(m)]$$

  where **Set** is the usual category of sets and functions (which should be introduced before).
- Use a special font and simpler notation for the canonical functions currently denoted $ii_1^{m,n}$ and $ii_2^{m,n}$, for example $\mathsf{i}_1^{m,n}$ and $\mathsf{i}_2^{m,n}$, or $\iota_1^{m,n}$ and $\iota_2^{m,n}$. The second option would match the use of π_1 and π_2 for projections.
- Use a special font to denote categories, for example $\mathcal{T}$ for the undelying category of a Lawvere theory and $\mathcal{C}$ for the underlying category of a C-system.
- Definition 2.1: I think this is a non-standard, but essentially equivalent, definition of a Lawvere theory. Usually, Lawvere theories are defined as categories with finite products having the natural numbers as objects. Explain the relationship.
- The diagram in item 3 of Definition 2.1 (and others in the following) seems to be displayed rather oddly, with the top arrow being quite far away from its domain and codomain. Use a different way of typesetting diagrams.
- Page 3, Line 5: replace 'isomorphism' with 'natural isomorphism'.

- Page 3, Line 10 and onward: this is very long-winded and unnecessary. Either you choose to define the basic data of a category to be a pair of sets (objects and morphisms) related by functions (source and target), or to be a set (objects) and a family of sets indexed by pairs of objects (morphisms with specified domain and codomain). There is no need to invoke coercions; the abuse of notation is common in standard mathematical practice.
- Sentence before Lemma 2.4: give a precise forward reference for where Lemma 2.4 is used.
- Lemma 2.4: typo in the conclusion ('The' instead of 'Then').
- Page 4, Proof of Lemma 2.4: give number to first display, for reference later in the proof.
- Page 4, Proof of Lemma 2.4: say 'duality' instead of 'inversion of the direction of arrows'.
- Page 4, Proof of Lemma 2.4 (and elsewhere in the paper): replace 'push-out' with 'pushout' (and similarly, replace 'pull-back' with 'pullback').

Section 3.

- Page 4, Line 7: as already mentioned above, use a different font for categories (for example, $\mathcal{C}$) and special fonts for the operators that are part of the data of a C-system, for example

$$\mathsf{Cs} = (\mathsf{pt}, \mathsf{p}, \mathsf{q}, \mathsf{ft}, \mathsf{s}, \ell)$$

  I would leave the length function as the last component of the tuple, since it is the only one that does not refer to the categorical structure of $\mathcal{C}$, but rather links it with an external object (the set of natural numbers). Note that, for clarity, I would use ℓ rather than l to denote the length function.
- Page 4: since the definition of Lawvere theory is given, it would be good to recall, at least in outline, the definition of a C-system. For example, the domains and codomains of the maps given by the operators listed above could be recalled. For example, say that we have pullback squares

$$\begin{tikzcd} f^*(X) \arrow[r, "\mathsf{q}_{f,X}"] \arrow[d, "\mathsf{p}_{f^*(X)}"'] & X \arrow[d, "\mathsf{p}_X"] \\ Y \arrow[r, "f"'] & \mathsf{ft}(X) \end{tikzcd}$$

- Use special font to denote the category of C-systems, say **CSys**. The category of ℓ-bijective C-systems could be denoted $\mathbf{CSys}_{\mathbb{N}}$ or similar. In Construction 3.3, mention that, in particular, the homomorphisms of C-systems preserve the length function.
- After Theorem 3.4: address my comment above about giving examples.

Section 4.

- Problem 4.1: I do not think that the notation LC is very good to denote a functor, since it may be interpreted as denoting a functor L being applied to an object C. Maybe simply use $F : \mathbf{Law}(\mathcal{T}) \to \mathbf{CSys}(\mathcal{T}^{\mathrm{op}})$ and G for its inverse (defined in Section 5).
- Construction 4.2: address my earlier comment on excluded middle.
- Construction 4.2: you also need to show that $\mathsf{ft}(f^*(X)) = Y$, which is not discussed yet. Note that this is necessary to make the diagram at the top of page 7 be of the form in (2), as claimed.
- Page 6, Line -7: I do not understand what is Y doing here, since we are considering the identity map on X. Please fix or clarify.
- Page 6, Line -5: you need to consider also $f : X \to Y$.
- Page 6, Line -5: when you say that 'we have to verify...', you are presupposing that the two arrows have the same domain and codomain, but this seems to follow from the next sentence. Please rephrase.
- Page 7, 2nd display: give some justification for these equations.
- Page 7, Line 2: Please check numbering of lemmas in the reference (the ArXiv version has it as Proposition 2.3).
- Page 7, Proof of Lemma 4.3: too many linebreaks.
- Page 7, Line, -6: Missing 'The' at the start of the sentence. Here it would be appropriate to draw some diagrams, so as to help the reader with explicit information about domains and codomains of the morphisms.
- Page 7, last display and Page 8, first display: the non-trivial equations should be justified. It may also be helpful for the reader to have the equations displayed in the format:

$$\begin{aligned} A &= B && \text{(by (3.1))} \\ &= C && \text{(by (3.6))} \\ &\vdots \end{aligned}$$

Section 5.

- Problem 5.1: See my comment on Problem 4.1. After the statement, say that the problem will be solved by Construction 5.9.
- Page 8, display (6): say that this is a pullback (this fact is used implicitly later).
- Problem 5.2: are these arrows making $\ell^{-1}(n)$ into the product of n copies of $\ell^{-1}(1)$ in $\mathcal{T}$? I feel that similar explanations, about the universal properties of various objects, may be missing in the paper. It would be helpful to add them, where appropriate.

- Page 9, Problem 5.4: before the statement, say a few words about the fact that we are about to define the action of L on arrows.
- Page 9, Line -5: fix the grammar in this sentence.
- Page 9, Last line: Say instead 'The lemma is now proved' (also later, in proof of Lemma 5.7).
- Page 10, Proof of Lemma 5.8: say which one between Lemma 5.6 and Lemma 5.7 proves (i) and which one proves (ii). If necessary, change the order of the items in Lemma 5.8 to match the order of the auxiliary lemmas, so that part (i) follows from Lemma 5.6 and part (ii) follows from Lemma 5.7.
- Page 10, Line -3: Add 'We now verify the conditions for a Lawvere theory in Definition 2.1'. Then use the numbering of that definition rather than saying 'the first condition', etc...
- Page 11, Lines 1 and 2: the expression 'the canonical square of the C-system...' is not clear; the notion of a canonical square has not been defined. Give a diagram to help the reader.
- Page 11, Lemma 5.10 and Lemma 5.11: notation for homomorphisms of C-systems is not the same (H in one, G in the other).
- Page 11, Line -4: missing 'the'.

Section 6.

- Page 12: Section title should be 'The isomorphism theorem'.
- Page 12, Line -11: Surely $L'(f) = L(f)$ should be discussed after you verify that the two functors are inverse to each other on objects (which is currently mentioned after).
- Page 13, Line 1: should be 'this equality'.
- Page 13, Line 6: there is an extra bracket in what should be $\pi^n_{f(0)}$.
- Page 13, 2nd display: extra brackets in what should be $\pi^n_{f(0)}$.
- Page 13, 3rd display: justify last equation.
- Page 13, Line -1: replace 'a' with 'an'.
- Page 13, Line -1: say 'a ... C-structure $\mathsf{cs} = (\mathsf{pt}, \mathsf{ft}, \mathsf{p}, \mathsf{q}, \mathsf{s}, \ell)$, one has ...'.
- Page 14, Lines 3 and 4: you write that the ft is determined by ℓ. Maybe this should be pointed out earlier, in a remark, to which you can then refer here.
- Page 14, Line 7: typo in what should be $\ell^{-1}(\ell(X))$.
- Page 14, Line 13: I do not understand what you mean by 'Applying this equation to (18) for q...'. Clarify.
- Page 14, Last line: Is the reference to Carnegie Mellon University correct?

References.

- [4]: I do not understand what the ‘and it’ refers to. Has the paper been published 3 times?
- [6]: fix number.