Lawvere theories and C-systems[1]

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Abstract

In this paper we consider the class of ℓ-bijective C-systems, i.e., C-systems for which the length function is a bijection. The main result of the paper is a construction of an isomorphism between two categories - the category of ℓ-bijective C-systems and the category of Lawvere theories.

1 Introduction

C-systems in their present form were introduced in [7] by a small modification of the concept of *contextual categories* of John Cartmell [1], [2].

C-systems are among several categorical structures that can be used to define on them systems of operations corresponding to the systems of inference rules of dependent type theories. Another such categorical structure is the structure of a category with families [3]. A C-system together with such system of operations is called a C-system model of the corresponding type theory. In this definition the C-system plays the role analogous to the role that the carrier set plays for a model of a first order theory.

A C-system is a category CC together with a function $\ell : CC \to \mathbb{N}$ called the length function and a number of other structures. An ℓ-bijective C-system is a C-system such that its length function is a bijection.

Lawvere theories were introduced by Bill Lawvere in his thesis under the name algebraic theories [4] as a syntax-free approach to the theory of algebraic theories.

It turns out that ℓ-bijective C-systems are very closely related to Lawvere theories. We construct a functor from the category of Lawvere theories to the category of ℓ-bijective C-systems and a functor in the opposite direction and show in Theorem 6.1 that these functors are mutually inverse isomorphisms of the corresponding categories. We emphasize the unexpected aspect of this result which is that we obtain not simply an equivalence but an actual isomorphism of the categories.

This is essentially the first result on C-systems that includes the description of their homomorphisms.

An algebraic theory, such as the theory of groups, can be presented in the form of a type theory following the idea of Cartmell [2]. Then we find ourselves in a seemingly contradictory situation since C-systems with some operations are *models* of this theory and at the same time, following our bijection, a particular C-system is a form of presentation of the theory itself. We conjecture that given a Lawvere theory the C-system that we construct in this paper is the carrier of the initial model of the type-theoretic inference rules corresponding to this Lawvere theory by Cartmell's construction. Moreover, the C-system structure of the initial C-system model is rich enough to allow on the one hand side for only one possibility for the additional operations and on the other for the reconstruction from it of the theory itself.

Let $\mathbb{F}$ be the category whose set of objects is the set of natural numbers $\mathbb{N}$ and $Mor(m,n) = Fun(stn(m), stn(n))$, where $stn(m) = \{i \in \mathbb{N} \mid i < m\}$ is the standard set with m elements and $Fun(X,Y)$ is the set of functions between two sets.

In Definition 2.1 we define a Lawvere theory, following the definition given by Lawvere himself in [4, Def. on p.62] with $\mathcal{S}_0 = \mathbb{F}$, as a pair $(T, L : \mathbb{F} \to T)$ where T is a category and L a functor that is a bijection on objects, maps 0 to an initial object and standard pushout squares to pushout squares.

Lawvere theories are ubiquitous. It seems intuitively clear that given an object X in a category $\mathcal{C}$ with an initial object $\mathbf{0}$ and coproducts of the form $(Y + X, \iota_1^{Y,X}, \iota_2^{Y,X})$ for all Y we obtain a Lawvere theory in the sense of Definition 2.1 by taking the subcategory of objects of the form $\mathbf{0}, X, X + X, (...(X + X)...) + X$ with the inductively defined functor L. By duality the same construction should work when we have a final object pt and binary products $(Y \times X, \pi_1^{Y,X}, \pi_2^{Y,X})$. This intuitive construction as presented here is incorrect - there is no reason for all the coproducts of X with itself to be different. However, it can be made precise by a composition of two constructions of this paper - Construction 3.5 defines from the data specified above an ℓ-bijective C-system and Construction 5.9 defines from an ℓ-bijective C-system a Lawvere theory $(T, L : \mathbb{F} \to T)$ in the sense of Definition 2.1. One can then prove that if all coproducts the $(...(X + X)...) + X$ are different then the functor $int : T^{op} \to \mathcal{C}$ is an isomorphism to its image.

Since we need to construct an isomorphism of categories we have well defined tasks - to construct the first category, to construct the second category, to construct the functor from the first to the second, to construct the functor from the second to the first and to prove that these two functors are mutually inverse isomorphisms. This precisely described the structure of the paper - it has five sections other than this introduction corresponding to the five tasks specified above.

While writing this paper we had two meta-theories or "foundations of mathematics" in mind. The main meta-theory that the paper can be formalized in is the usual foundation based on the Zermelo-Fraenkel formalism. The secondary meta-theory is UniMath, a univalent foundation based on a small subset of the language of the proof assistant Coq, see [6],[8]. Since the UniMath is constructive we do not use the axiom of excluded middle or the axiom of choice in the paper.

We use the diagrammatic order in writing compositions, i.e., for $f : X \to Y$ and $g : Y \to Z$ we write $f \circ g$ for the composition of f and g.

We do not make precise the concept of a *universe U* that we use for some of the statements of the paper. It would be certainly sufficient to assume that U is a Grothendieck universe. However, it seems likely that sets U satisfying much weaker conditions can be used both for the statements and for the proofs of our results.

2 The category of Lawvere theories

In this section start by defining the set $Lw(T)$ of Lawvere theory structures on a category T. Then a Lawvere theory is defined as a pair (T, L) where T is a category and $L \in Lw(T)$. A morphism of Lawvere theories is a functor $T_1 \to T_2$ that is compatible with L_1 and L_2. Restricting ourselves to Lawvere theories (T, L) where T is a category in a given universe U we obtain a category $\mathbf{LW}(U)$ of Lawvere theories in U. At the end of the section we prove a lemma that is used later to simplify the construction of Lawvere theories.

As usual, we let $\mathbb{N}$ denote the set of natural numbers. For $m \in \mathbb{N}$ let $\mathsf{stn}(m) = \{i \in \mathbb{N} \mid 0 \le i < m\}$ be the standard set with m elements.

Let
$$Mor(\mathbb{F}) = \cup_{n,m \in \mathbb{N}} Fun(\mathsf{stn}(n), \mathsf{stn}(m))$$

where $Fun(X, Y)$ is the set of functions from X to Y. Since every function has a well defined domain and codomain we can define a category $\mathbb{F}$ with the set of objects $\mathbb{N}$ and the set of morphisms $Mor(\mathbb{F})$ such that for each n and m the set

$$\mathbb{F}(m, n) := \{f \in Mor(\mathbb{F}) \,|\, dom(f) = m \text{ and } codom(f) = n\}$$

equals to $Fun(\mathsf{stn}(m), \mathsf{stn}(n))$ and composition, when restricted to these subsets, is the composition of functions.

For $m, n \in \mathbb{N}$ let $\iota_1^{m,n} : \mathsf{stn}(m) \to \mathsf{stn}(m+n)$ and $\iota_2^{m,n} : \mathsf{stn}(n) \to \mathsf{stn}(m+n)$ be the injections of the initial segment of length m and the concluding segment of length n.

Definition 2.1 *A Lawvere theory structure on a category T is a functor $L : \mathbb{F} \to T$ such that the following conditions hold:*

1. *L is a bijection on the sets of objects,*

2. *$L(0)$ is an initial object of T,*

3. *for any $m, n \in \mathbb{N}$ the square*

$$\begin{array}{ccc} L(0) & \longrightarrow & L(n) \\ \downarrow & & \downarrow {\scriptstyle L(\iota_2^{m,n})} \\ L(m) & \xrightarrow{L(\iota_1^{m,n})} & L(m+n) \end{array}$$

is a pushout square.

A Lawvere theory is a pair (T, L) where T is a category and L is a Lawvere theory structure on T.

Let us denote the set of Lawvere theory structures on T by $Lw(T)$.

Example 2.2 The identity functor on $\mathbb{F}$ satisfies the condition of Definition 2.1 making $(\mathbb{F}, Id_{\mathbb{F}})$ a Lawvere theory. Many more examples can be constructed using Construction 3.5. In particular, see Example 5.11.

Problem 2.3 *For a universe U, to construct a category $\mathbf{LW}(U)$ of Lawvere theories in U.*

Construction 2.4 Following Lawvere [4, p.61] we define a morphism from a Lawvere theory $\mathbf{T}_1 = (T_1, L_1)$ to a Lawvere theory $\mathbf{T}_2 = (T_2, L_2)$ as a functor $G : T_1 \to T_2$ such that $L_1 \circ G = L_2$. We let $Hom_{\mathbf{LW}}(\mathbf{T}_1, \mathbf{T}_2)$ denote the subset in the set of functors from T_1 to T_2 that are morphisms of the Lawvere theories.

Note that here one uses the equality rather than isomorphism of functors. The composition of morphisms is defined as composition of functors. The identity morphism is the identity functor. The associativity and the left and right unit axioms follow immediately from the corresponding properties of the composition of functors.

We let $Ob(\mathbf{LW}(U))$ denote the set of Lawvere theories in U and $Mor(\mathbf{LW}(U))$ the set

$$Mor(\mathbf{LW}(U)) = \coprod_{\mathbf{T}_1, \mathbf{T}_2 \in Ob(\mathbf{LW}(U))} Hom_{\mathbf{LW}}(\mathbf{T}_1, \mathbf{T}_2)$$

Together with the obvious domain, codomain, identity and composition functions the pair of sets $Ob(\mathbf{LW}(U))$ and $Mor(\mathbf{LW}(U))$ forms a category that we denote $\mathbf{LW}(U)$ and call the category of Lawvere theories in U.

Note that a morphism from $\mathbf{T}_1$ to $\mathbf{T}_2$ in this category is not a morphism of Lawvere theories but an iterated pair $((\mathbf{T}_1, \mathbf{T}_2), G)$ where G is a morphism of Lawvere theories. However, there is an obvious bijection

$$Mor_{\mathbf{LW}(U)}(\mathbf{T}_1, \mathbf{T}_2) \to Hom_{\mathbf{LW}}(\mathbf{T}_1, \mathbf{T}_2)$$

and by the common abuse of notation every time we have an expression which denotes an element of one of these sets in a position where an element of the other is expected it is assumed to be replaced by its image under this bijection.

The following lemma will be used below in Construction 5.9.

Lemma 2.5 *Let T be a category and $L : \mathbb{F} \to T$ a functor such that the following conditions hold:*

1. *$L(0)$ is an initial object of T,*

2. *for any $m \in \mathbb{N}$ the square*

$$\begin{array}{ccc} L(0) & \longrightarrow & L(1) \\ \downarrow & & \downarrow {\scriptstyle L(\iota_2^{m,1})} \\ L(m) & \xrightarrow{L(\iota_1^{m,1})} & L(m+1) \end{array} \qquad (1)$$

   is a pushout square.

Then for any $m, n \in \mathbb{N}$ the square

$$\begin{array}{ccc} L(0) & \longrightarrow & L(n) \\ \downarrow & & \downarrow {\scriptstyle L(\iota_2^{m,n})} \\ L(m) & \xrightarrow{L(\iota_1^{m,n})} & L(m+n) \end{array}$$

is a pushout square.

Proof: Let $m, n \in \mathbb{N}$. Consider first the diagram

$$\begin{array}{ccccc} L(0) & \longrightarrow & L(n) & \xrightarrow{L(\iota_2^{m,n})} & L(m+n) \\ \downarrow & & \downarrow {\scriptstyle L(\iota_1^{n,1})} & & \downarrow {\scriptstyle L(\iota_1^{m+n,1})} \\ L(1) & \xrightarrow{L(\iota_2^{n,1})} & L(n+1) & \xrightarrow{L(\iota_2^{m,n+1})} & L(m+n+1) \end{array}$$

The first square is the reflection relative to the diagonal of a square of the form (1) and therefore it is a pushout square.

We have $\iota_2^{n,1} \circ \iota_2^{m,n+1} = \iota_2^{m+n,1}$. Therefore the large square is the reflection relative to the diagonal of a square of the form (1) and therefore it is a pushout square.

The right hand side square is commutative.

By the general properties of pushout squares which are obtained from the similar properties of the pullback squares by duality we conclude that the right hand side square is a pushout square.

To prove the lemma proceed now by induction on $n \in \mathbb{N}$.

For $n = 0$ the horizontal arrows are isomorphisms and since the square commutes it is a pushout square.

For $n = 1$ it is a square of the form (1).

For the successor consider the diagram

$$
\begin{array}{ccccc}
L(0) & \longrightarrow & L(n) & \xrightarrow{\;L(\iota_1^{n,1})\;} & L(n+1) \\
\downarrow & & \downarrow{\scriptstyle L(\iota_2^{m,n})} & & \downarrow{\scriptstyle L(\iota_2^{m,n+1})} \\
L(m) & \xrightarrow{\;L(\iota_1^{m,n})\;} & L(m+n) & \xrightarrow{\;L(\iota_1^{m+n,1})\;} & L(m+n+1)
\end{array}
$$

The first square is a pushout by the inductive assumption. The second square is pushout by the first part of the proof. Therefore the ambient square is pushout. Since $\iota_1^{m,n} \circ \iota_1^{m+n,1} = \iota_1^{m,n+1}$ this completes the proof of the lemma.

3 The category of ℓ-bijective C-systems

In this section we first recall some facts about C-systems and give references to the papers where they are described in full detail. Then we define the concept of an ℓ-bijective C-system and provide a construction of ℓ-bijective C-systems from categories equipped with a simple additional data. Later this construction, composed with the construction of a Lawvere theory from an ℓ-bijective C-system, is used to make precise an important intuitive approach to Lawvere theories. At the end of the section we remind some facts about homomorphisms of C-systems, construct the category $\mathbf{CSys}(U)$ of C-systems in a universe U and define its full subcategory $\mathbf{CSys}_{\mathbb{N}}(U)$ of ℓ-bijective C-systems $\mathbf{CSys}(U)$.

For the definition of a C-system see [1], [2] (where they are called contextual categories) as well as [7]. A C-system structure on a category CC is a six-tuple $cs = (\ell, \mathsf{pt}, \mathsf{ft}, \mathsf{p}, \mathsf{q}, \mathsf{s})$ where ℓ, ft and p are functions, pt an element of $Ob(CC)$, q a partial function on pairs and s a partial function. To be a C-system structure these objects must satisfy the conditions of [7, Definitions 2.1 and 2.3]. One of the main structures of a C-system is that for any X such that $\ell(X) > 0$ and any $f : Y \to \mathsf{ft}(X)$ there is given an object $f^*(X)$ and a morphism $\mathsf{q}(f, X) : f^*(X) \to X$ such that

$$\mathsf{ft}(f^*(X)) = Y \tag{2}$$

and the square

$$
\begin{array}{ccc}
f^*(X) & \xrightarrow{\;\mathsf{q}(f,X)\;} & X \\
{\scriptstyle \mathsf{p}_{f^*(X)}}\downarrow & & \downarrow{\scriptstyle \mathsf{p}_X} \\
Y & \xrightarrow{\;f\;} & \mathsf{ft}(X)
\end{array}
\tag{3}
$$

is a pullback. Squares of this form will be called the *canonical squares* of a C-system.

Definition 3.1 *An ℓ-bijective C-system is a C-system such that the length function $\ell : Ob(CC) \to \mathbb{N}$ is a bijection.*

We let $Cs_{\mathbb{N}}(CC)$ denote the set of ℓ-bijective C-system structures on a category CC.

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Remark 3.2 The element pt of a C-system structure is determined by ℓ since $\ell^{-1}(0) = \{\mathsf{pt}\}$ for any C-system structure.

Remark 3.3 For an ℓ-bijective C-system structure $cs = (\ell, \mathsf{pt}, \mathsf{ft}, \mathsf{p}, \mathsf{q}, \mathsf{s})$ the ft function is determined by the ℓ-function, that is, for two such structures cs_1, cs_2 satisfying $\ell_1 = \ell_2$ one has $\mathsf{ft}_1 = \mathsf{ft}_2$. Indeed, by the axioms of a C-system $\ell(\mathsf{ft}(X)) = \ell(X) -_{\mathbb{N}} 1$ where $n -_{\mathbb{N}} 1 = n - 1$ if $n > 0$ and $0 -_{\mathbb{N}} 1 = 0$. Therefore, in an ℓ-bijective C-system one has $\mathsf{ft}(X) = \ell^{-1}(\ell(X) -_{\mathbb{N}} 1)$.

The following result shows that ℓ-bijective C-systems are abundant.

Problem 3.4 *Let $\mathcal{C}$ be a category with a final object pt, a distinguished object X and, for all $Y \in \mathcal{C}$, a choice of a pullback square of the form* $Y \times X \longrightarrow X$. *To construct an ℓ-bijective C-system*

$$
\begin{array}{ccc}
Y \times X & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & pt
\end{array}
$$

$CC(\mathcal{C}, \pi_X)$ *and a functor* $int : CC(\mathcal{C}, \pi_X) \to \mathcal{C}$.

Construction 3.5 Note that the data specified in the problem is exactly a universe structure, in the sense of [5, Def. 2.1] on the morphism $\pi_X : X \to pt$. Therefore $(\mathcal{C}, \pi_X)$ is a universe category and we can apply the construction [5, Constr. 2.12] to it to obtain a C-system $CC(\mathcal{C}, \pi_X)$ and a functor $int : CC(\mathcal{C}, \pi_X) \to \mathcal{C}$. Objects of this C-system of length $n + 1$ are pairs of the form $(A, \tau : int(A) \to pt)$ where A is an object of length n and τ a morphism in $\mathcal{C}$. Since pt is a final object a simple inductive argument shows that $CC(\mathcal{C}, \pi_X)$ is an ℓ-bijective C-system.

Problem 3.6 *Let U be a universe. To construct a category $\mathbf{CSys}(U)$ of C-systems in U.*

Construction 3.7 A morphism of C-systems is a functor between the underlying categories that is compatible with the corresponding C-system structures. In particular a morphism of C-systems should commute with the length functions. For a detailed definition see [5, Definition 3.1]. We let $Hom_{CS}(CC_1, CC_2)$ denote the set of homomorphisms from the C-system CC_1 to the C-system CC_2.

That the composition of functors that are homomorphisms is again a homomorphism is stated in [5, Lemma 3.2]. That the identity functor is a homomorphism is very easy to prove. The associativity and the left and right unit axioms for the composition of homomorphisms follow directly from the similar properties of the composition of functors.

Repeating the approach that we used with Lawvere theories we obtain the category $\mathbf{CSys}(U)$ of C-systems in U.

We let $\mathbf{CSys}_{\mathbb{N}}(U)$ denote the full subcategory in $\mathbf{CSys}(U)$ that consists of ℓ-bijective C-systems.

4 A functor from Lawvere theories to ℓ-bijective C-systems

The goal of this section is to construct, for a universe U, a functor $\mathsf{LtoC}_U : \mathbf{LW}(U) \to \mathbf{CSys}_{\mathbb{N}}(U)$. This is achieved in Construction 4.5. We start in Construction 4.2 by defining, for a category T, a function $LtoC$ from the set of Lawvere theory structures on T to the set of ℓ-bijective C-system structures on T^{op}. Then in Lemma 4.3 we show that for a morphism of Lawvere theories

$G : (T_1, L_1) \to (T_2, L_2)$ the functor G^{op} is a homomorphism of the C-systems $(T_1^{op}, LtoC(L_1)) \to (T_2^{op}, LtoC(L_2))$. Note that this is a lemma and not a construction because being a homomorphism of C-systems is a property of a functor and not a structure on it. Combining Construction 4.2 with Lemma 4.3 get the desired functor $\mathsf{LtoC}_U : \mathbf{LW}(U) \to \mathbf{CSys}_\mathbb{N}(U)$. At the end of the section we outline the comparison of our construction with another construction of a C-system from a Lawvere theory based on Construction 3.5. This construction can also be used to pass from Lawvere theories to C-systems, but its use will lead to an equivalence between categories $\mathbf{LW}(U)$ and $\mathbf{CSys}_\mathbb{N}(U)$ instead of an isomorphism.

Problem 4.1 *For a category T to construct a function*

$$LtoC : Lw(T) \to Cs_\mathbb{N}(T^{op})$$

from the Lawvere theory structures on T to the ℓ-bijective C-system structures on T^{op}.

Construction 4.2 Let $CC = T^{op}$. We need to construct an ℓ-bijective C-system structure on CC. We set:

The length function $\ell = L^{-1}$.

The distinguished final object pt is $L(0)$.

The map $\mathsf{ft} : Ob(CC) \to Ob(CC)$ maps pt to pt and any object X such that $\ell(X) > 0$ to $L(\ell(X)-1)$. Since the equality on $\mathbb{N}$ is decidable we can distinguish objects such that $\ell(X) = 0$ from objects such that $\ell(X) \ne 0$ without the use of the excluded middle.

For pt the morphism p_{pt} is the identity. For X such that $\ell(X) > 0$ the morphism $\mathsf{p}_X : X \to \mathsf{ft}(X)$ is $L(\iota_1^{\ell(X)-1,1})$.

To define $\mathsf{q}(f, X)$ observe first that for any X such that $\ell(X) > 0$ we have a pullback square in CC of the form

$$\begin{array}{ccc} X & \xrightarrow{L(\iota_2^{\ell(X)-1,1})} & L(1) \\ {\scriptstyle \mathsf{p}_X} \downarrow & & \downarrow \\ \mathsf{ft}(X) & \longrightarrow & L(0) \end{array} \qquad (4)$$

Given $f : Y \to X$ we set

$$f^*(X) = L(\ell(Y) + 1)$$

As required by (2) we have

$$\mathsf{ft}(f^*(X)) = L(\ell(L(\ell(Y) + 1)) - 1) = L(\ell(Y)) = Y$$

Since (4) is a pullback square and $L(0)$ is a final object there is a unique morphism $\mathsf{q}(f, X) : f^*(X) \to X$ such that

$$\mathsf{q}(f, X) \circ \mathsf{p}_X = \mathsf{p}_{f^*(X)} \circ f \qquad (5)$$

and

$$\mathsf{q}(f, X) \circ L(\iota_2^{\ell(X)-1,1}) = L(\iota_2^{\ell(Y),1}) \qquad (6)$$

Let us check the conditions of [7, Definition 2.1] that will show that we obtained a C0-system. We have $\ell^{-1}(0) = L(0) = \{\mathsf{pt}\}$. For X such that $\ell(X) > 0$ we have $\ell(\mathsf{ft}(X)) = \ell(L(\ell(X)-1)) = \ell(X)-1$. We also have $\mathsf{ft}(\mathsf{pt}) = \mathsf{pt}$. The object $\mathsf{pt} = L(0)$ is final.

The square

$$f^*(X) \xrightarrow{\mathsf{q}(f,X)} X \qquad (7)$$

$$\mathsf{p}_{f^*(X)} \downarrow \qquad \qquad \downarrow \mathsf{p}_X$$

$$Y \xrightarrow{\quad f \quad} \mathsf{ft}(X)$$

commutes by (5).

If $f = Id_{\mathsf{ft}(X)}$ then $Y = \mathsf{ft}(X)$, $\ell(Y) = \ell(X) - 1$ and therefore $f^*(X) = X$. Therefore $\mathsf{q}(f, X) \circ \mathsf{p}_X = \mathsf{p}_X$ and $\mathsf{q}(f, X) \circ L(\iota_2^{\ell(X)-1,1}) = L(\iota_2^{\ell(X)-1,1})$ which proves that $\mathsf{q}(f, X) = Id_X$.

Given $g : Z \to Y$ we have to verify that $\mathsf{q}(g \circ f, X) = \mathsf{q}(g, f^*(X)) \circ \mathsf{q}(f, X)$. We have

$$(g \circ f)^*(X) = L(\ell(Z) + 1) = g^*(f^*(X)) \qquad (8)$$

Taking into account that (4) is a pullback square, it remains to verify two equalities

$$\mathsf{q}(g \circ f, X) \circ \mathsf{p}_X = \mathsf{q}(g, f^*(X)) \circ \mathsf{q}(f, X) \circ \mathsf{p}_X \qquad (9)$$

and

$$\mathsf{q}(g \circ f, X) \circ L(\iota_2^{\ell(X)-1,1}) = \mathsf{q}(g, f^*(X)) \circ \mathsf{q}(f, X) \circ L(\iota_2^{\ell(X)-1,1}) \qquad (10)$$

For (9) we have

$$\mathsf{q}(g \circ f, X) \circ \mathsf{p}_X = \mathsf{p}_{(g \circ f)^*(X)} \circ g \circ f = \mathsf{p}_{g^*(f^*(X))} \circ g \circ f =$$

$$\mathsf{q}(g, f^*(X)) \circ \mathsf{p}_{f^*(X)} \circ f = \mathsf{q}(g, f^*(X)) \circ \mathsf{q}(f, X) \circ \mathsf{p}_X$$

where the first equality is by the commutativity of the squares (7), the second one by (8) and the third and the fourth ones again by (7). For (10) we have

$$\mathsf{q}(g \circ f, X) \circ L(\iota_2^{\ell(X)-1,1}) = L(\iota_2^{\ell(Z),1}) = \mathsf{q}(g, f^*(X)) \circ L(\iota_2^{\ell(Y),1}) = \mathsf{q}(g, f^*(X)) \circ \mathsf{q}(f, X) \circ L(\iota_2^{\ell(X)-1,1})$$

Where the first equality is by (10), the second also by (10) since $l(Y) = l(f^*(X)) - 1$ and the third is by (10) as well.

According to [7, Proposition 2.4] it remains to show that the squares (7) are pullback squares.

Consider the diagram

$$f^*(X) \xrightarrow{\mathsf{q}(f,X)} X \xrightarrow{L(\iota_2^{l(X)-1,1})} L(1)$$

$$\mathsf{p}_{f^*(X)} \downarrow \qquad \qquad \mathsf{p}_X \downarrow \qquad \qquad \downarrow$$

$$Y \xrightarrow{\quad f \quad} \mathsf{ft}(X) \xrightarrow{\qquad\qquad} L(0)$$

where both the right hand side square and the outside square are of the form (4) and in particular are pullback squares and the left hand side square has been shown to be commutative. Therefore, the left hand side square is a pullback square. We conclude, by [7, Proposition 2.4], that there exists a unique s such that $(\ell, \mathsf{pt}, \mathsf{ft}, \mathsf{p}, \mathsf{q}, \mathsf{s})$ is a C-system structure.

Lemma 4.3 *Let $G : (T_1, L_1) \to (T_2, L_2)$ be a morphism of Lawvere theories. Then the functor G^{op} is a homomorphism of C-systems $(T_1^{op}, LtoC(L_1)) \to (T_2^{op}, LtoC(L_2))$.*

Proof: For convenience we will write H instead of G^{op}. In view of [5, Lemma 3.4] it is sufficient to verify that H is compatible with the length function, distinguished final object, ft function, p-morphisms and q-morphisms.

ℓ-**function.** The fact that $\ell_1 = H \circ \ell_2$ is equivalent to the fact that $L_1 \circ G = L_2$.

pt **object.** The fact that $H(L_1(0)) = L_2(0)$ follows from the same property of G.

ft **function.** The fact that $H(\mathsf{ft}(X)) = \mathsf{ft}(H(X))$ again follows from the same property of G.

p-**morphisms.** The fact that $H(\mathsf{p}_X) = \mathsf{p}_{H(X)}$ follows from the fact that $L_1 \circ G = L_2$ on objects and on morphisms of the form $\iota_1^{n,1}$.

q-**morphisms** It remains to verify that for X such that $\ell(X) > 0$ and $f : Y \to \mathsf{ft}(X)$ one has

$$H(\mathsf{q}(f, X)) = \mathsf{q}(H(f), H(X))$$

where the right hand side is defined because H is compatible with ℓ and ft. The morphism

$$\mathsf{q}(H(f), H(X)) : H(f)^*(H(X)) \to H(X)$$

as a q-morphism of the C-system structure $LtoC(L_2)$ is defined by equalities of the form (5) and (6) that for (T_2, L_2) take the form

$$\mathsf{q}(H(f), H(X)) \circ \mathsf{p}_{H(X)} = \mathsf{p}_{H(f)^*(H(X))} \circ H(f)$$

and

$$\mathsf{q}(H(f), H(X)) \circ L_2(\iota_2^{\ell_2(H(X))-1,1}) = L_2(\iota_2^{\ell_2(H(Y)),1})$$

Therefore we need to verify the same equalities for the morphism $H(\mathsf{q}(f, X))$. For the first equality we have

$$H(\mathsf{q}(f, X)) \circ \mathsf{p}_{H(X)} = H(\mathsf{q}(f, X)) \circ H(\mathsf{p}_X) = H(\mathsf{q}(f, X) \circ \mathsf{p}_X) = H(\mathsf{p}_{f^*(X)} \circ f) = \mathsf{p}_{H(f^*(X))} \circ H(f)$$

where the first equality follows from the case of p-morphisms, the second from the composition axiom of H, the third from the commutativity of the canonical squares (3) and the fourth by the composition axiom of H and the case of p-morphisms.

It remains to show that $H(f^*(X)) = H(f)^*(H(X))$. It follows from the fact that $f^*(X) = L_1(\ell_1(X) + 1)$ and $H(f)^*(H(X)) = L_2(\ell_2(H(X)) + 1)$.

For the second equality we have

$$H(\mathsf{q}(f, X)) \circ L_2(\iota_2^{\ell_2(H(X))-1,1}) = H(\mathsf{q}(f, X)) \circ H(L_1(\iota_2^{\ell_1(X)-1,1})) = H(\mathsf{q}(f, X) \circ L_1(\iota_2^{\ell_1(X)-1,1})) =$$

$$H(L_1(\iota_2^{\ell_1(Y),1})) = L_2(\iota_2^{\ell_2(H(Y)),1})$$

where the first equality follows from the fact that $\ell_2(H(X)) = \ell_1$ by the compatibility of H with the ℓ-functions and that $L_2(f) = H(L_1(f))$, the second equality follows from the composition axiom for H, the third from (6) and the fourth from the same argument as the first.

The lemma is now proved.

Problem 4.4 To construct, for any universe U, a functor $\mathsf{LtoC}_U : \mathbf{LW}(U) \to \mathbf{CSys}_{\mathbb{N}}(U)$.

Construction 4.5 We set LtoC_{Ob} to be the function that takes a Lawvere theory to the opposite category of its underlying category with the C-system structure defined by Construction 4.2. We set LtoC_{Mor} to be the function that takes a functor G that is a morphism of Lawvere theories to G^{op}. It is well defined by Lemma 4.3. That the functions $(\mathsf{LtoC}_{Ob}, \mathsf{LtoC}_{Mor})$ form a functor, i.e., commute with the identity morphisms and compositions is straightforward.

Remark 4.6 Let (T, L) be a Lawvere theory. Consider T^{op}. The morphism $p : L(1) \to L(0)$, pullback squares (4) and the final object $L(0)$ make T^{op} into a universe category (T^{op}, p) (see [5]). It is easy to prove that the C-system of Construction 4.2 is isomorphic to the C-system $CC(T^{op}, p)$ of this universe category. However this isomorphism is not an equality since the set of objects of the category $CC(T^{op}, p)$ is not equal to the set of objects of T^{op}. Indeed, an object of $CC(T^{op}, p)$ is a sequence of the form

$$(\mathsf{pt}; \pi_{L(0)}, \ldots, \pi_{L(n)})$$

where $\pi_{L(i)}$ is the unique morphism $L(i) \to L(0)$.

5 A functor from ℓ-bijective C-systems to Lawvere theories

This is probably the most technical section of the paper. Our first goal is to construct, for a category CC, of a function $CtoL$ from the set of ℓ-bijective C-system structures on $CtoL$ to the set of Lawvere theory structures on CC^{op}. This is achieved in Construction 5.9. Since a Lawvere theory structure on CC^{op} is a functor $L : \mathbb{F} \to CC^{op}$ that maps the standard coproduct squares to coproduct squares what we need to construct is such a functor. While we do not use this terminology in the proof, the main idea is to show that in an ℓ-bijective C-system one has $\ell^{-1}(n) = (\ell^{-1}(1))^n$. This is completed with the proof of Lemma 5.7. The construction of a functor from $\mathbb{F}$ that maps the standard coproduct squares to coproduct squares that we provide can be considerably generalized since all that is required for it is a sequence of objects X_n, which in our case is the sequence $\ell^{-1}(n)$, a sequence of pushout squares of the form $X_0 \longrightarrow X_n$, which in our case is the sequence of

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X_n \\
\downarrow & & \downarrow \\
X_1 & \longrightarrow & X_{n+1}
\end{array}
$$

pullback squares (11) in the opposite category, and the condition that X_0 is an initial object. Of course, in general such a functor will not be a bijection on objects and therefore not a Lawvere theory structure.

After the function $CtoL$ is constructed we need, for a given universe U, to extend it to a functor $\mathsf{CtoL}_U : \mathbf{LW}(U) \to \mathbf{CSys}_{\mathbb{N}}(U)$. We choose to define this functor on morphisms such that, modulo the usual abuse of notation, for a C-system homomorphism $G : CC_1 \to CC_2$ we have $\mathsf{CtoL}_U(G) = G^{op}$. This construction requires G^{op} to commute with the functors L_1, L_2. It is shown in Lemma 5.13.

The definition of the functor CtoL_U is completed in Construction 5.15.

Problem 5.1 *For a category CC to construct a function*

$$CtoL : Cs_{\mathbb{N}}(CC) \to Lw(CC^{op})$$

10

To perform a construction we will need a number of lemmas and intermediate constructions. Let us fix a category CC and a ℓ-bijective C-system structure $cs = (\ell, \mathsf{pt}, \mathsf{ft}, \mathsf{p}, \mathsf{q}, \mathsf{s})$ on CC. We will often write CC both for the category and for the C-system (CC, cs).

We need to construct a Lawvere theory $CtoL(CC, cs)$. A Lawvere theory is a pair (T, L) satisfying certain conditions. Since out construction should be inverse to the construction of the previous section, we known that $T = CC^{op}$. We also know that $L(n) = \ell^{-1}(n)$. What remains is to construct L on morphisms and to prove all the necessary conditions.

Problem 5.2 *For $n \in \mathbb{N}$ and $i = 0, \ldots, m - 1$ to construct a morphism $\pi_i^m : \ell^{-1}(m) \to \ell^{-1}(1)$ in CC.*

The morphisms π_i^m make $\ell^{-1}(m)$ into the n-fold product of $\ell^{-1}(1)$ with itself. However, we do not provide an exact definition of an "n-fold product" and do not prove or use this fact below.

Construction 5.3 By induction on $m \in \mathbb{N}$.

For $m = 0$ there are no morphisms to construct.

For $m = 1$ we set $\pi_0^1 = Id_{\ell^{-1}(1)}$.

For the successor consider the canonical pullback square:

$$\ell^{-1}(m + 1) \xrightarrow{\ \mathsf{q}(\pi, \ell^{-1}(1))\ } \ell^{-1}(1) \tag{11}$$

with vertical maps $\mathsf{p}_{\ell^{-1}(m+1)}$ and $\mathsf{p}_{\ell^{-1}(1)}$, and bottom $\ell^{-1}(m) \xrightarrow{\ \pi\ } \ell^{-1}(0)$

where we use π to denote the unique morphisms from objects of CC to the final object $\ell^{-1}(0)$. We set

$$\pi_i^{m+1} = \begin{cases} \mathsf{p}_{\ell^{-1}(m+1)} \circ \pi_i^m & \text{for } i < m \\ \mathsf{q}(\pi, \ell^{-1}(1)) & \text{for } i = m \end{cases}$$

The construction to the following problem will provide us with L on morphisms.

Problem 5.4 *For any $m, n \in \mathbb{N}$ and a function $f : \mathsf{stn}(m) \to \mathsf{stn}(n)$ to construct a morphism $L_f : \ell^{-1}(n) \to \ell^{-1}(m)$ in CC.*

Construction 5.5 By induction on $m \in \mathbb{N}$.

For $m = 0$ we set $L_f = \pi$.

For $m = 1$ we set $L_f = \pi_{f(0)}^n$.

For the successor consider $f : \mathsf{stn}(m + 1) \to \mathsf{stn}(n)$ and the square (11). We define L_f as the unique morphism such that:

$$L_f \circ \mathsf{p}_{\ell^{-1}(m+1)} = L_{\iota_1^{m,1} \circ f} \tag{12}$$

and

$$L_f \circ \mathsf{q}(\pi, \ell^{-1}(1)) = L_{\iota_2^{m,1} \circ f} \tag{13}$$

where, let us recall,

$$\iota_1^{m,1} : \mathsf{stn}(m) \to \mathsf{stn}(m + 1)$$
$$\iota_2^{m,1} : \mathsf{stn}(1) \to \mathsf{stn}(m + 1)$$

are the inclusion of the initial segment and the morphism corresponding to the last element.

Lemma 5.6 *Let $m, n \in \mathbb{N}$ and $f : \mathsf{stn}(m) \to \mathsf{stn}(n)$. Then for any $i = 0, \ldots, m-1$ one has*

$$L_f \circ \pi_i^m = \pi_{f(i)}^n$$

Proof: By induction on $m \in \mathbb{N}$.

For $m = 0$ there is nothing to prove.

For $m = 1$ we need to prove that $L_f \circ \pi_0^n = \pi_{f(0)}^n$. By construction, $\pi_0^n = Id_{\ell^{-1}(1)}$ and $L_f = \pi_{f(0)}^n$ which implies the goal.

For the successor consider $f : \mathsf{stn}(m+1) \to \mathsf{stn}(n)$ and the square (11) for $m+1$.

If $i = m$ then $\pi_i^{m+1} = \mathsf{q}(\pi, \ell^{-1}(1))$ and by the construction of L_f we have $L_f \circ \mathsf{q}(\pi, \ell^{-1}(1)) = \pi_{f(m)}^n$.

If $i < m$ then $\pi_i^{m+1} = \mathsf{p}_{\ell^{-1}(m+1)} \circ \pi_i^m$. Therefore

$$L_f \circ \pi_i^{m+1} = L_f \circ \mathsf{p}_{\ell^{-1}(m+1)} \circ \pi_i^m = L_{\iota_1^{m,1} \circ f} \circ \pi_i^m$$

and by the inductive assumption

$$L_{\iota_1^{m,1} \circ f} \circ \pi_i^m = \pi_{f(\iota_1^{m,1})}^n = \pi_{f(i)}^n$$

The lemma is now proved.

Lemma 5.7 *Let $m, n \in \mathbb{N}$ and let $f, g : \ell^{-1}(n) \to \ell^{-1}(m)$ be two morphisms such that for all $i = 0, \ldots, m-1$ one has*

$$f \circ \pi_i^m = g \circ \pi_i^m$$

Then $f = g$.

Proof: By induction on $m \in \mathbb{N}$.

For $m = 0$, $\ell^{-1}(m)$ is a final object and $f = g$.

For $m = 1$, $\pi_0^1 = Id_{\ell^{-1}(1)}$ and $f = f \circ Id = g \circ Id = g$.

For the successor the square (11) for $m+1$. Since the square is a pullback square it is sufficient to show that

$$f \circ \mathsf{p}_{\ell^{-1}(m+1)} = g \circ \mathsf{p}_{\ell^{-1}(m+1)}$$

and

$$f \circ \mathsf{q}(\pi, \ell^{-1}(1)) = g \circ \mathsf{q}(\pi, \ell^{-1}(1))$$

The second equality follows from the fact that $\mathsf{q}(\pi, \ell^{-1}(1)) = \pi_m^{m+1}$.

The first equality follows by the inductive assumption since for $i = 0, \ldots, m-1$ we have

$$\left(f \circ \mathsf{p}_{\ell^{-1}(m+1)} \right) \circ \pi_i^m = f \circ \pi_i^{m+1}$$

and

$$\left(g \circ \mathsf{p}_{\ell^{-1}(m+1)} \right) \circ \pi_i^m = g \circ \pi_i^{m+1}$$

The lemma is now proved.

Lemma 5.8   *1. For any $m \in \mathbb{N}$ one has $L_{Id_{\mathsf{stn}(m)}} = Id_{\ell^{-1}(m)}$,*

2. *For $k, m, n \in \mathbb{N}$ and $f : \mathsf{stn}(k) \to \mathsf{stn}(m)$, $g : \mathsf{stn}(m) \to \mathsf{stn}(n)$ one has*

$$L_{f \circ g} = L_g \circ L_f$$

Proof:

1. By Lemma 5.6 we have $L_{Id_{\mathsf{stn}(m)}} \circ \pi_i^m = \pi_i^m$ for all $i \in \mathsf{stn}(m)$. Therefore, by Lemma 5.7 we have $L_{Id_{\mathsf{stn}(m)}} = Id_{\ell^{-1}(m)}$.

2. By Lemma 5.6 we have, for all $i \in \mathsf{stn}(k)$, $L_{f \circ g} \circ \pi_i^k = \pi_{g(f(i))}^n$ and

$$L_g \circ L_f \circ \pi_i^k = L_g \circ \pi_{f(i)}^m = \pi_{g(f(i))}^n$$

Therefore, by Lemma 5.7 we have $L_{f \circ g} = L_g \circ L_f$.

The lemma is now proved.

We can now provide a construction for Problem 5.1.

Construction 5.9 We need to construct a Lawvere theory structure on CC^{op}, i.e., a functor $L : \mathbb{F} \to CC^{op}$ satisfying the conditions of Definition 2.1. We define the object part of L as ℓ^{-1}. We define the morphism part of L as $L_{Mor}(f) = L_f$ using the identification of the sets of morphisms of CC and CC^{op}. Lemma 5.8 shows that L is a covariant functor to CC^{op}.

We now verify the conditions of Definition 2.1. Condition 2.1(1) is obvious. Condition 2.1(2) is obvious as well (it follows from the axioms of a C-system). To prove Condition 2.1(3) we first apply Lemma 2.5. It remains to prove that squares of the form (1) are pushout squares in CC^{op} or, equivalently, that squares of the form

$$
\begin{array}{ccc}
\ell^{-1}(m+1) & \xrightarrow{L_{\iota_2^{m,1}}} & \ell^{-1}(1) \\
{\scriptstyle L_{\iota_1^{m,1}}} \downarrow & & \downarrow \\
\ell^{-1}(m) & \longrightarrow & \ell^{-1}(0)
\end{array}
\tag{14}
$$

in CC are pullback squares. We will do it by showing that the square (14) equals to the square of the form (3) for the pair $X = \ell^{-1}(1)$ and $f = \pi_{\ell^{-1}(m)}$. The right hand side vertical morphism, $\pi_{\ell^{-1}(1)}$, is a unique morphism from $\ell^{-1}(1)$ to $\ell^{-1}(0)$ and since $\mathsf{ft}(\ell^{-1}(1)) = \ell^{-1}(0)$ it equals $\mathsf{p}_{\ell^{-1}(1)}$. The same argument shows that the lower horizontal morphism is $\pi_{\ell^{-1}(m)}$.

It remains to show that

$$L_{\iota_1^{1,m}} = \mathsf{p}_{\ell^{-1}(m+1)} \tag{15}$$

and

$$L_{\iota_2^{m,1}} = \mathsf{q}(\pi_{\ell^{-1}(m)}, \ell^{-1}(1)) \tag{16}$$

These equalities follow from the equalities (12) and (13) for $f = Id_{\ell^{-1}(m+1)}$ because of Lemma 5.8(1). The construction is completed.

Remark 5.10 Construction 3.5 composed with Construction 5.9 can be used to construct Lawvere theories. Indeed, this composition defines a Lawvere theory every time that we are given a category $\mathcal{C}$, a final object pt in $\mathcal{C}$, and object X in $\mathcal{C}$ and for any $Y \in \mathcal{C}$, a binary product diagram for Y and X.

Example 5.11 Let U be a universe and Gr be the category of groups in U. The standard construction of the free product $G_1 * G_2$ defines a binary coproduct structure on Gr. The standard one point set $stn(1)$ with its unique group structure is an initial object in Gr. Let $\mathbb{Z}$ be the additive group of integers. Applying to these data considered in Gr^{op} Construction 3.5 composed with Construction 5.9 we obtain a Lawvere theory where $Mor_T(L(1), L(n)) = Hom_{Gr}(\mathbb{Z}, \mathbb{Z}^{*n})$. This Lawvere theory is called the Lawvere theory of groups. Similarly one obtains Lawvere theories corresponding to other classes of the algebraic hierarchy.

Next we will show that our function on objects extends to a functor from the category of ℓ-bijective C-systems to the category of Lawvere theories. First we need the following lemma.

Lemma 5.12 Let $H : CC_1 \to CC_2$ be a homomorphism of ℓ-bijective C-systems. Then for any $n \in \mathbb{N}$ and $i = 0, \ldots, n-1$ one has

$$H(\pi_i^n) = \pi_i^n \tag{17}$$

Proof: Note that since we have $\ell_2(H(X)) = \ell_1(X)$ both sides of (17) are morphisms from $\ell_2^{-1}(n)$ to $\ell_2^{-1}(1)$.

The proof is by induction on $n \in \mathbb{N}$.

For $n = 0$ there are no equations to prove.

For $n = 1$ we have $\pi_0^1 = Id_{\ell^{-1}(1)}$ and the statement of the lemma follows from the identity axiom of the definition of a functor.

For the successor we have two cases. For $i < n$ we have

$$H(\pi_i^{n+1}) = H(\mathsf{p}_{\ell^{-1}(n+1)} \circ \pi_i^n) = H(\mathsf{p}_{\ell^{-1}(n+1)}) \circ H(\pi_i^n) = \mathsf{p}_{\ell^{-1}(n+1)} \circ \pi_i^n = \pi_i^{n+1}$$

where the third equality uses the inductive assumption. For $i = n$ we have

$$H(\pi_n^{n+1}) = H(\mathsf{q}(\pi, \ell^{-1}(1))) = \mathsf{q}(\pi, \ell^{-1}(1)) = \pi_n^{n+1}$$

The lemma is now proved.

Lemma 5.13 Let $H : (CC_1, cs_1) \to (CC_2, cs_2)$ be a homomorphism of C-systems. Then the functor $H^{op} : CC_1^{op} \to CC_2^{op}$ is a morphism of Lawvere theories $(CC_1^{op}, CtoL(cs_1)) \to (CC_2^{op}, CtoL(cs_2))$.

Proof: Let $CtoL(cs_1) = L_1$ and $CtoL(cs_2) = L_2$. We need to show that $L_1 \circ H^{op} = L_2$. The equality between the object components of these functors follows from the fact that a homomorphism of C-systems is compatible with the length functions. For the morphism component it is more convenient to consider the equivalent equation

$$L_1' \circ H = L_2'$$

where $L_i' : \mathbb{F}^{op} \to CC_i$. Then we have to show that for any $f : \mathsf{stn}(m) \to \mathsf{stn}(n)$ one has $H(L_{1,f}) = L_{2,f}$. Both sides of this equality are morphisms $\ell_2^{-1}(n) \to \ell_2^{-1}(m)$. By Lemma 5.7 it is sufficient to show that $H(L_{1,f}) \circ \pi_i^n = L_{2,f} \circ \pi_i^n$ for all $i = 0, \ldots, n-1$. We have

$$H(L_{1,f}) \circ \pi_i^n = H(L_{1,f}) \circ H(\pi_i^n) = H(L_{1,f} \circ \pi_i^n) = H(\pi_{f(i)}^n) = \pi_{f(i)}^n$$

where we used Lemma 5.6 and twice Lemma 5.12. On the other hand

$$L_{2,f} \circ \pi_i^n = \pi_{f(i)}^n$$

again by Lemma 5.6. This completes the proof of Lemma 5.13.

Problem 5.14 *For any universe U to construct a functor $\mathsf{CtoL}_U : \mathbf{CSys}_{\mathbb{N}}(U) \to \mathbf{LW}(U)$.*

Construction 5.15 The object component of CtoL_U takes a C-system (CC, cs) to the Lawvere theory $(CC^{op}, CtoL(cs))$ where $CtoL(cs)$ is defined by Construction 5.9.

The morphism component takes a homomorphism $G : (CC_1, cs_1) \to (CC_2, cs_2)$ to G^{op}. It is well defined by Lemma 5.13.

The identity and composition axioms are straightforward from the corresponding properties of functor composition and its compatibility with the function $G \mapsto G^{op}$.

6 The isomorphism theorem

This is the final section of the paper. It contains the proof of a single theorem stating that, for any universe U, the functors LtoC_U and CtoL_U constructed previously are mutually inverse isomorphisms, that is, that

$$\mathsf{LtoC}_U \circ \mathsf{CtoL}_U = Id_{\mathbf{LW}(U)}$$

$$\mathsf{CtoL}_U \circ \mathsf{LtoC}_U = Id_{\mathbf{CSys}(U)}$$

The part that requires work is that they are mutually inverse functions between the sets of objects. After this fact is established the fact that they are mutually inverse on morphisms follows easily from their constructions. In the proof of the theorem we first prove both equalities for functions on objects and then conclude that they hold for functions on morphisms as well.

Theorem 6.1 *For any universe U, LtoC_U and CtoL_U are mutually inverse isomorphisms between the categories of Lawvere theories and ℓ-bijective C-systems in U.*

Proof: Let us show first that the object components of the functors LtoC_U and CtoL_U are mutually inverse bijections of sets.

To show that $(\mathsf{LtoC}_U \circ \mathsf{CtoL}_U)_{Ob}$ is identity we need to show that for all categories T in U the composition $LtoC \circ CtoL$ of the function of Construction 4.2 with the function of Construction 5.9 is identity on $Lw(T)$.

Let $L : \mathbb{F} \to T$ be a Lawvere theory structure on T and let

$$L' = (LtoC \circ CtoL)(L) = CtoL(LtoC(L))$$

We have to prove that $L'(n) = L(n)$ for $n \in \mathbb{N}$ and $L'(f) = L(f)$ for $f : \mathsf{stn}(m) \to \mathsf{stn}(n)$. Let $(\ell, \mathsf{pt}, \mathsf{ft}, \mathsf{p}, \mathsf{q}, \mathsf{s})$ be the components of $LtoC(L)$.

On objects we have

$$L'(n) = \ell^{-1}(n) = (L^{-1})^{-1}(n) = L(n)$$

Let $f : \mathsf{stn}(m) \to \mathsf{stn}(n)$ be a morphism in $\mathbb{F}$. By Construction 5.9 we have $L'(f) = L_f$ using identification of morphisms of CC and CC^{op}. To avoid confusion we will denote this identification by $(-)^{op}$ below. Note that L_f is defined using $LtoC(L)$.

Since L and L' coincide on objects both morphisms $L(f)$ and $L'(f)$ are of the form $L(m) \to L(n)$.

The proof of their equality is by induction on $m \in \mathbb{N}$.

If $m = 0$ then $L(0) = L'(0)$ is an initial object and any two morphisms with it as the domain and equal codomains are equal.

If $m = 1$ we have, by Constructions 5.9 and 5.5, $L_f = \pi^n_{f(0)}$. Therefore, we need to show that

$$L(f) = (\pi^n_{f(0)})^{op}$$

We prove this equality by induction on $n \in \mathbb{N}$.

If $n = 0$ then no $f : \mathsf{stn}(1) \to \mathsf{stn}(n)$ exists.

If $n = 1$ then, by Construction 5.3, we have $\pi^n_{f(0)} = \pi^n_0 = Id_{\ell^{-1}(1)}$ and since L is a functor and $f = Id_1$ we have $L(f) = Id_{L(1)} = (Id_{\ell^{-1}(1)})^{op}$.

For the successor we have $f : \mathsf{stn}(1) \to \mathsf{stn}(n+1)$. Consider diagram (11). We have that $\pi^{n+1}_{f(0)}$ is given by:

$$\pi^{n+1}_{f(0)} = \begin{cases} \mathsf{p}_{\ell^{-1}(n+1)} \circ \pi^n_{f(0)} & \text{for } f(0) < n \\ \mathsf{q}(\pi_{\ell^{-1}(n)}, \ell^{-1}(1)) & \text{for } f(0) = n \end{cases}$$

Assume that $f(0) < n$. By Construction 4.2 we have $\mathsf{p}_{\ell^{-1}(n+1)} = L(\iota^{n,1}_1)^{op}$. By the inductive assumption we have $\pi^n_{f(0)} = L(g)^{op}$ where $g : \mathsf{stn}(1) \to \mathsf{stn}(n)$ is such that $g(0) = f(0)$. Therefore,

$$\pi^{n+1}_{f(0)} = L(\iota^{n,1}_1)^{op} \circ L(g)^{op} = L(g \circ \iota^{n,1}_1)^{op} = L(f)^{op}$$

where the second equality is by the composition axiom from L.

For $f(0) = n$ we have $\pi^{n+1}_{f(0)} = \mathsf{q}(\pi_{\ell^{-1}(n)}, \ell^{-1}(1)) = L(\iota^{n,1}_2) = L(f)$ where the second equality follows from (6) since $\iota^{0,1}_2 = Id_1$.

We have to consider now the case of the successor of m.

The morphism L_f for $f : \mathsf{stn}(m+1) \to \mathsf{stn}(n)$ is defined in (12) and (13) as the unique morphism such that

$$L_f \circ \mathsf{p}_{\ell^{-1}(m+1)} = L_{\iota^{m,1}_1 \circ f} \tag{18}$$

and

$$L_f \circ \mathsf{q}(\pi_{\ell^{-1}(m)}, \ell^{-1}(1)) = L_{\iota^{m,1}_2 \circ f} \tag{19}$$

By the inductive assumption we have

$$L_{\iota^{m,1}_1 \circ f} = L(\iota^{m,1}_1 \circ f)$$

$$L_{\iota^{m,1}_2 \circ f} = L(\iota^{m,1}_2 \circ f)$$

It remains to prove that

$$L(f) \circ \mathsf{p}_{\ell^{-1}(m+1)} = L(\iota^{m,1}_1 \circ f) \tag{20}$$

and

$$L(f) \circ \mathsf{q}(\pi_{\ell^{-1}(m)}, \ell^{-1}(1)) = L(\iota^{m,1}_2 \circ f) \tag{21}$$

By (15) and (16) we have

$$\mathsf{p}_{\ell^{-1}(m+1)} = L_{\iota^{m,1}_1} = L(\iota^{m,1}_1)$$

$$\mathsf{q}(\pi_{\ell^{-1}(m)}, \ell^{-1}(1)) = L_{\iota^{m,1}_2} = L(\iota^{m,1}_2)$$

Where the second equality in each line is by the inductive assumption on m. This proves equalities (20) and (21) and completes the proof of the fact that $L' = L$ and, therefore, $LtoC \circ CtoL = Id_{Lw(T)}$ for any category T.

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To prove that $(\mathsf{CtoL}_U \circ \mathsf{LtoC}_U)_{Ob}$ is the identity we need to show that for any category CC in U the composition $CtoL \circ LtoC$ of the function of Construction 5.9 with the function of Construction 4.2 is identity on $Cs_\mathbb{N}(CC)$.

Let $cs = (\ell, \mathsf{pt}, \mathsf{ft}, \mathsf{p}, \mathsf{q}, \mathsf{s})$ be an ℓ-bijective C-system structure on CC. Let $CtoL(cs) = L$ and let $LtoC(L) = (\ell', \mathsf{pt}', \mathsf{ft}', \mathsf{p}', \mathsf{q}', \mathsf{s}')$.

Then $\ell' = (L_{Ob})^{-1} = (\ell^{-1})^{-1} = \ell$. Therefore, $\mathsf{pt} = \mathsf{pt}'$ and $\mathsf{ft} = \mathsf{ft}'$ by Remarks 3.2 and 3.3.

For X such that $\ell(X) > 0$ we have by Construction 4.2 that $\mathsf{p}'_X = L(\iota_1^{\ell(X)-1,1})$. Together with (15) we obtain
$$\mathsf{p}'_X = L(\iota_1^{\ell(X)-1,1}) = \mathsf{p}_{\ell^{-1}(\ell(X))}$$
since the C-system is ℓ-bijective we have $X = \ell^{-1}(\ell(X))$ and therefore $\mathsf{p}' = \mathsf{p}$.

The morphism q' is defined in Construction 4.2 as the unique morphism such that (5) and (6) hold. In our notation these equations take the form:
$$\mathsf{q}'(f, X) \circ \mathsf{p}'_X = \mathsf{p}'_{f^*(X)} \circ f \tag{22}$$
and
$$\mathsf{q}'(f, X) \circ L(\iota_2^{\ell(X)'-1,1}) = L(\iota_2^{\ell(Y)',1}) \tag{23}$$
We need to check that the same equations with q' replaced by q hold. For the first one it follows immediately from the fact that $\mathsf{p}' = \mathsf{p}$.

To prove the second one consider equation (16) for $m = \ell(X)' - 1 = \ell(X) - 1$. Applying this equation to (23) with q' replaced by q we get
$$\mathsf{q}(f, X) \circ \mathsf{q}(\pi_{\mathsf{ft}(X)}, \ell^{-1}(1)) = \mathsf{q}(\pi_Y, \ell^{-1}(1))$$
which a particular case of the composition axiom for q (see [7, Definition 2.1(7)]).

This proves that CtoL_U and LtoC_U are mutually inverse bijections on the sets of objects of our categories.

The fact that they give mutually inverse functions on morphisms between each pair of objects is straightforward. Indeed
$$\mathsf{CtoL}_{U,Mor}(G) = G^{op}$$
and
$$\mathsf{LtoC}_{U,Mor}(G) = G^{op}$$
as functors and since the mappings from morphisms of Lawvere theories and homomorphisms of C-systems to functors between the corresponding categories are injective we see that $\mathsf{CtoL}_{U,Mor}$ and $\mathsf{LtoC}_{U,Mor}$ are mutually inverse bijections.

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