LAWVERE THEORIES AND C-SYSTEMS, REPLY TO THE REFEREE COMMENTS

VLADIMIR VOEVODSKY

General note: after the required changes were applied the text became longer. It is now at 18 pages. I recognize that it has to be shortened, but would like to ask for a referee's advice on how best to do it. I will keep the full length article on my website and can make it available at the arXiv if the journal polices allow it.

1. General Comments

(1) Done?
(2) Done.
(3) Done.

2. Comments on individual sections

2.1. **Section 1.**

(1) Done.
(2) Done.
(3) Done.
(4) New introduction does not mention the simplicial set model.
(5) It is not relevant for the introduction any more, but let me make a comment. The point of some of the work that I am doing is to formally verify the correctness of the construction of the simplicial set model and of the corresponding interpretation of the relevant variant of Martin-Lof type theory. My remark concerned this work - since we want to verify something on which the consistency of UniMath depends, we can not use UniMath for this verification. Indeed, if UniMath is inconsistent than it will be possible to use it to verify wrong constructions.
(6) Done? The point is to use only arguments that can be formalized both in the ZFC and in the UniMath. While I do not have a precise list of ZFC arguments that can not be translated into the UniMath (for example, unbounded quantification, a very important and very beautiful in its own way part of the ZFC can be translated into bounded quantification and then into the type theory if one introduces a universe) it certainly includes the excluded middle and the axiom of choice.
(7) I commented it in one place but decided not to comment it everywhere. After all the ZFC is the main meta-theory of the paper and UniMath is the secondary one.

Date: May, 2017.

(8) Done.

2.2. **Section 2.**

(1) Done.
(2) Done.
(3) The reason for this reminder is that there are two ways to define what a function is in set theory - one as in Bourbaki where a function is a triple (X, Y, G) and another one where a function is simply a set G that is a graph-like subset in a set of pairs. In the latter case a function has a well defined domain set, but the codomain set can be any set that contains the image of G. I am away from the library and can not easily find the book on foundations of mathematics where this particular choice is discussed and where the author takes as the main one the definition that is different from the Bourbaki one.

    I removed the reference, but left a sentence emphasizing that every function has a well defined domain and codomain. Please let me know if you find this change sufficient.
(4) I replaced F by $\mathbb{F}$. This notation is standard since the foundational paper by Fiore, Plotkin and Turi [2]. See also [3], [6], [1]. It is very important that this category *is not* the category of finite sets in a universe. There are many more finite sets in any universe than there are natural numbers. Since we work with categories up to isomorphisms and not simply up to equivalences this difference is crucial.
(5) Excuse me, but I do not want to define $\mathbb{F}$ through a category of sets. Any category of sets requires a universe as the set of objects (or a two-levels set theory with sets and classes) and as such is a much more complex object than the category $\mathbb{F}$.
(6) Done.
(7) About the font such as $\mathcal{C}$ for the categories in Lawvere theories and C-systems. I feel that it would be appropriate to use such a font, and use it in some constructions, when the category is considered up to an equivalence. In this work categories are considered up to an isomorphism, that is, as objects of algebra rather than objects of category theory. Because of this I prefer to use the straight font for them. Please let me know if you disagree.
(8) Concerning the relation of Def. 2.1 to the definition of Lawvere theories as categories with finite products having $\mathbb{N}$ as the set of objects. Could you please provide me with a reference to a paper where this or such a definition is given?

    Note that as stated it is clearly imprecise. For example such a category may have 23 as the final object. It may also have the product of objects unrelated to the $+$ on $\mathbb{N}$. There are also more complex details related to the choice of $\pi_0^{m,n}$ and $\pi_1^{m,n}$.
(9) Use a different diagram package - done.
(10) "Page 3, Line 5: replace "isomorphism" with "natural isomorphism"'. I am not sure if I understand correctly. If it is in the sentence "Note that here one uses the equality rather than isomorphism of functors." then "isomorphism of functors" means isomorphism in the category of functors, that is, an invertible

natural transformation. I don't know what "natural isomorphism" would refer to.

(11) I use consistently the definition of a category as a pair of sets together with the dom, codom, id and comp operations of which comp is defined on a subset specified by an equation between dom and codom. That is, a category is considered as an instance of a model of a quasi-algebraic theory.

In the definition of the category of Lawvere theories I emphasize the fact that many forget about, that it is rarely possible to define a category, in the above sense, where the set of morphisms between two objects is exactly equal to the pre-defined set, because the sets of morphisms between different pairs of objects in "quasi-algebraic categories" must have empty intersection.

(Note: the other definition with a set of objects and a family of sets of morphisms is often more convenient, but it unfortunately requires a reminder of how a family of sets is defined in set theory and also requires some set-theoretic results related to the manipulation with families. I know only one definition of what a family of sets is - it is a graph-like set of pairs. Such an object has a domain and does not have a specific codomain while mapping every element of its domain to some set. That is, a family is given by that definition of a function alternative to the one in Bourbaki. I honestly don't know where one can refer to for the results on families in set theory.)

I have removed the word "coercions" and expressed the relevant fact in terms of "abuse of notation".

(12) Done (forward reference).
(13) Done (the to then, typo).
(14) Done (number to the first display).
(15) Done (duality).
(16) Done (push-out to pushout, pull-back to pullback).

2.3. **Section 3.**

(1) I have made the change to p instead of p etc.

As I already explained I prefer not to use the $\mathcal{C}$ font for the categories underlying C-systems and Lawvere theories because they are considered here as objects of algebra, that is, up to isomorphism, rather than as objects of category theory.

The order of components of the C-system data is determined by the dependency.

The components ℓ does not depend on anything.

The final object pt is not really a structure because it can be defined: $\mathsf{pt} = \ell^{-1}(0)$, but it can be introduced for convenience at any place after ℓ.

The function ft is required to satisfy $\ell(\mathsf{ft}(X)) = \mathsf{ft}(X) -_{\mathbf{N}} 1$. This suggests that ℓ should be introduced first and then ft.

The p-morphisms have the form $X \to \mathsf{ft}(X)$ so they should be introduced after ft.

The q-morphisms $\mathsf{q}(f, X)$ are only defined for X such that $\ell(X) > 0$ and f such that $codom(f) = \mathsf{ft}(X)$. Their basic axioms require p-morphisms, so it is reasonable to introduce them after p.

The s morphisms should be introduced last because the structure without the s morphisms is considered separately under the name C0-system.

(2) I included the square that you have suggested and also $\mathsf{ft}(f^*(X)) = Y$ as a separate condition to be able to refer to it later.

The reason for giving an explicit definition of a Lawvere theory and only a reference for the definition of a C-system is that, as you have noted above, there are different definitions of what a Lawvere theory is and their equivalence is not always clear. On the contrary, there is essentially only one definition of what a C-system is and it is easy to give a precise reference to it.

(3) Done.
(4) Theorem 3.4 is replaced by Construction 3.5. I think you will find this construction more satisfying than the theorem that it replaces.

2.4. **Section 4.**

(1) About the names of functors currently called LC and CL. These two functors are among the few main object constructed in the paper. They are, in a sense, the product of the paper if we look at the paper as at a something producing a useful for the future authors product. I can not call them F and G. If you find LC and CL confusing I suggest $LtoC$ and $CtoL$.
(2) Done.
(3) Done.
(4) the case $f = Id_{\mathsf{ft}X}$. Done.
(5) "Page 6, Line -5: you need to consider also $f : X \to Y$". Sorry, I do not understand the comment.
(6) That I presuppose that the two morphisms have the same domain and codomain. Actually, no, I do not have to presuppose it to write an equality between two morphism because the category is defined using the set of all morphisms with *dom* and *codom* as operations. Please correct me if I misunderstood the comment.
(7) Done.
(8) In `https://arxiv.org/pdf/1406.7413.pdf` the number of the proposition is 2.4. Under the 2.3 is the definition of C-systems as a C0-system with an additional operation.
(9) Done?
(10) Done in a slightly different way.

2.5. **Section 5.**

(1) Done.
(2) Done.
(3) Done?
(4) Done.
(5) Done (grammar corrected).
(6) Done.
(7) Done (proof of Lemma 5.8 expanded).
(8) Done.

(9) Done.
(10) Done (G replaced by H in Lemma 5.11).
(11) Done.

2.6. **Section 6.**

(1) Done.
(2) Done.
(3) Done ("this equality").
(4) Done.
(5) Done.
(6) The text is changed somewhat in this area making it hopefully more clear.
(7) Done ("a" replaced by "an").
(8) Done in a slightly different way.
(9) Done (remarks introduced in Section 3).
(10) Done.
(11) Done.
(12) Reference to the CMU explained.

References

[1] Marcelo Fiore and Chung-Kil Hur. On the construction of free algebras for equational systems. *Theoret. Comput. Sci.*, 410(18):1704–1729, 2009.

[2] Marcelo Fiore, Gordon Plotkin, and Daniele Turi. Abstract syntax and variable binding (extended abstract) (Warning: the paper uses the name "presheaves" for covariant functors to the category of sets). In *14th Symposium on Logic in Computer Science (Trento, 1999)*, pages 193–202. IEEE Computer Soc., Los Alamitos, CA, 1999.

[3] Makoto Hamana. Polymorphic abstract syntax via Grothendieck construction. In *Foundations of software science and computational structures*, volume 6604 of *Lecture Notes in Comput. Sci.*, pages 381–395. Springer, Heidelberg, 2011.

[4] Martin Hofmann. Syntax and semantics of dependent types. In *Semantics and logics of computation (Cambridge, 1995)*, volume 14 of *Publ. Newton Inst.*, pages 79–130. Cambridge Univ. Press, Cambridge, 1997.

[5] F. William Lawvere. Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories. *Repr. Theory Appl. Categ.*, (5):1–121, 2004. Reprinted from Proc. Nat. Acad. Sci. U.S.A. **5**0 (1963), 869–872 [MR0158921] and ıt Reports of the Midwest Category Seminar. II, 41–61, Springer, Berlin, 1968 [MR0231882].

[6] John Power and Miki Tanaka. Binding signatures for generic contexts. In *Typed lambda calculi and applications*, volume 3461 of *Lecture Notes in Comput. Sci.*, pages 308–323. Springer, Berlin, 2005.

School of Mathematics, Institute for Advanced Study, Princeton NJ, USA. e-mail: vladimir@ias.edu