Products of families of types in C-systems defined by a universe category[1]

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Abstract

We introduce the notion of a (Π, λ)-structure on a C-system and show that C-systems with (Π, λ)-structures are constructively equivalent to contextual categories with products of families of types. We then show how to construct (Π, λ)-structures on C-systems of the form $CC(\mathcal{C}, p)$ defined by a universe p in a locally cartesian closed category $\mathcal{C}$ from a simple pullback square based on p. In the last section we prove a theorem that asserts that our construction is functorial.

1 Introduction

The concept of a C-system in its present form was introduced in [**?**]. The type of the C-systems is constructively equivalent to the type of contextual categories defined by Cartmell in [**?**] and [**?**] but the definition of a C-system is slightly different from the Cartmell's foundational definition.

In this paper we consider what might be the most important structure on C-systems - the structure that corresponds, for the syntactic C-systems, to the operations of dependent product, λ-abstraction and application. A C-system formulation of this structure was introduced by John Cartmell in [**?**, pp. 3.37 and 3.41] as a part of what he called a strong M.L. structure. It was studied further by Thomas Streicher in [**?**, p.71] who called a C-system (contextual category) together with such a structure a "contextual category with products of families of types".

In the second section show that the structure that Cartmell defined is equivalent to another structure, which we call a (Π, λ)-structure. The proof of this equivalence consists of Constructions 3.5 and 3.6 (of mappings in both directions) and Lemmas 3.7 and 3.8 showing that these mappings are mutually inverse. This is probably the most technical part of the paper.

In order to prove Lemmas 3.7 and 3.8 we need some results about C-systems that have been certainly known to Cartmell and some of which are explicitly stated in [**?**] and [**?**]. We recall these results and sketch their proofs in the first section of the paper.

In the third section we consider the case of C-systems of the form $CC(\mathcal{C}, p)$ introduced in [**?**]. They are defined, in a functorial way, by a category $\mathcal{C}$ with a final object and a morphism $p : \widetilde{U} \to U$ together with the choice of pullbacks of p along all morphisms in $\mathcal{C}$. A morphism with such choices is called a universe in $\mathcal{C}$. As a corollary of general functoriality we also

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obtain a construction of an isomorphism that connects the C-systems $CC(\mathcal{C}, p)$ corresponding to different choices of pullbacks and different choices of final objects. It makes it possible to say that $CC(\mathcal{C}, p)$ is defined by $\mathcal{C}$ and p.

We provide several intermediate results about $CC(\mathcal{C}, p)$ when $\mathcal{C}$ is a locally cartesian closed category leading to the main result of this paper - Construction 5.3 that produces a (Π, λ)-structure on $CC(\mathcal{C}, p)$ from a simple pullback square based on p. This construction was first announced in [**?**]. It and the ideas that it is based on are among the most important ingredients of the construction of the univalent model of the Martin-Lof type theory.

In the following sections we study the behavior of our construction with respect to universe category functors and prove that it is functorial with respect to functors equipped with an additional structure that reflects compatibility with the choice of the generating pull-back squares.

One may wonder how the construction of this paper relates to the earlier ideas of Seely [**?**] and their refinement by Clairambault and Dybjer [**?**]. This question requires further study.

The methods of this paper are fully constructive. It is also written in the formalization-ready style that is in such a way that no long arguments are hidden even when they are required only to substantiate an assertion that may feel obvious to readers who are closely associated with a particular tradition of mathematical thought.

The main result of this paper is not a theorem but a construction and so are many of the intermediate results. Because of the importance of constructions for this paper we use a special pair of names Problem-Construction for the specification of the goal of a construction and the description of the particular solution.

In the case of a Theorem-Proof pair one usually refers (by name or number) to the theorem when using the proof of this theorem. This is acceptable in the case of theorems because the future use of their proofs is such that only the fact that there is a proof but not the particulars of the proof matter.

In the case of a Problem-Construction pair the content of the construction often matters in the future use. Because of this we have to refer to the construction and not to the problem and we assign in this paper numbers both to Problems and to Constructions.

In this paper we continue to use the diagrammatic order of writing composition of morphisms, i.e., for $f : X \to Y$ and $g : Y \to Z$ the composition of f and g is denoted by $f \circ g$.

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2 Some general results about C-systems

Let us start by making some changes to the notations that were introduced in [**?**]. The new notations that we introduce are consistent with the notations introduced in [**?**, pp.239-240].

Let CC be a C-system. We will say that an object X is over an object Y if $l(X) \geq l(Y)$

and $Y = ft^{l(X)-l(Y)}(X)$. Note that "is over" and "is above" are well-defined relations on $Ob(CC)$ with "is over" being reflexive and transitive and "is above" being transitive.

If X is over Γ we will write $p(X, \Gamma)$ for the composition of the p-morphisms going from X to Γ that was previously denoted $p_{X,n}$ where $n = l(X) - l(\Gamma)$. In particular, if $l(X) = l(\Gamma)$ then $X = \Gamma$ and $p(X, X) = Id_X$.

If X is over Γ and $f : \Gamma' \to \Gamma$ is a morphism we will write $f^*(X)$ for what previously was denoted $f^*(X, n)$ where $n = l(X) - l(\Gamma)$ and

$$q(f, X) : f^*(X) \to X$$

for what was previously denoted by $q(f, X, n)$.

Lemma 2.1 *For any X and f as above $f^*(X)$ is an object over Γ' and that the square*

$$\begin{array}{ccc} f^*(X) & \xrightarrow{q(f,X)} & X \\ {\scriptstyle p(f^*(X),\Gamma')}\downarrow & & {\scriptstyle p(X,\Gamma)}\downarrow \\ \Gamma' & \xrightarrow{f} & \Gamma \end{array} \tag{1}$$

is a pullback square.

Proof: It is proved easily by induction on $n = l(X) - l(\Gamma)$ applying the fact that the vertical composition of two pullback squares is a pullback square.

Lemma 2.2 *Let X be an object over Γ then one has:*

1. $Id_\Gamma^*(X) = X$ *and* $q(Id_\Gamma, X) = Id_X$,
2. *if* $f : \Gamma' \to \Gamma$, $g : \Gamma'' \to \Gamma'$ *are two morphisms then* $(g \circ f)^*(X) = g^*(f^*(X))$ *and*

$$q(g \circ f, X) = q(g, f^*(X)) \circ q(f, X)$$

Proof: The proof is by induction on $l(X) - l(\Gamma)$ using the axioms of a C-system.

If X, Y are objects over Γ and $f : X \to Y$ is a morphism we will say that f is a morphism over Γ if $f \circ p(Y, \Gamma) = p(X, \Gamma)$. Compositions of morphisms over Γ is easily seen to be a morphism over Γ.

If X, Y are objects over Γ, $a : X \to Y$ is a morphism over Γ and $f : \Gamma' \to \Gamma$ is a morphism then we let

$$f^*(a) : f^*(X) \to f^*(Y)$$

denote the unique morphism over Γ' such that

$$f^*(a) \circ q(f, Y) = q(f, X) \circ a \tag{2}$$

One verifies easily that one has

$$f^*(Id_X) = Id_{f^*(X)} \tag{3}$$

$$f^*(a \circ b) = f^*(a) \circ f^*(b) \tag{4}$$

and for $g : \Gamma'' \to \Gamma'$ one has

$$g^*(f^*(a)) = (g \circ f)^*(a) \tag{5}$$

One also has that for X over Γ, $p(X, \Gamma)$ is a morphism over Γ and that

$$f^*(p(X, \Gamma)) = p(f^*(X), \Gamma'). \tag{6}$$

If X is an object over Y and Y is an object over Γ then X is an object over Γ and one has

$$p(X, \Gamma) = p(X, Y) \circ p(Y, \Gamma) \tag{7}$$

If further $f : \Gamma' \to \Gamma$ is a morphism then $f^*(X)$ is over $f^*(Y)$ and one has

$$f^*(p_{X,Y}) = p_{f^*(X), f^*(Y)} \tag{8}$$

$$f^*(X) = q(f, Y)^*(X) \tag{9}$$

and

$$q(f, X) = q(q(f, Y), X) \tag{10}$$

The proofs of all of these equations are by induction on $l(X) - l(Y)$.

Lemma 2.3 *Let X, Z be objects over Γ, Y an object over Z and $g : X \to Z$ a morphism over Γ. Then one has:*

$$f^*(g^*(Y)) = (f^*(g))^*(f^*(Y)) \tag{11}$$

and

$$f^*(q(g, Y)) = q(f^*(g), f^*(Y)) \tag{12}$$

Proof: We have

$$f^*(g^*(Y)) = q(f, X)^*(g^*(Y)) = (q(f, X) \circ g)^*(Y) = (f^*(g) \circ q(f, Z))^*(Y) =$$

$$(f^*(g))^*(q(f, Z)^*(Y)) = (f^*(g))^*(f^*(Y))$$

where the first and the fifth equations are by (9), the second and the fourth are by (4) and the third equation is by (2). This proves (11) and also proves that the morphisms on the left and the right hand side of (12) have the same domain. The codomain of the morphisms on both sides of (12) is $f^*(Y)$ that is a pullback with projections $p(f^*(Y), f^*(Z))$ and $q(f, Y)$. It is, therefore sufficient to verify that the compositions of the right and the left hand side morphisms with each of the projections coincide. We have (where we leave matching of the steps with the previously established equations to the reader):

$$f^*(q(g, Y)) \circ p(f^*(Y), f^*(Z)) = f^*(q(g, Y)) \circ f^*(p(Y, Z)) =$$

$$f^*(q(g,Y)\circ p(Y,Z))$$

$$q(f^*(g),f^*(Y))\circ p(f^*(Y),f^*(Z)) = p((f^*(g))^*(f^*(Y)),f^*(X))\circ f(g) =$$
$$p(f^*(g^*(Y)),f^*(X))\circ f^*(g) = f^*(p(g^*(Y),X))\circ f^*(g) = f^*(p(g^*(Y),X)\circ g) =$$
$$f^*(q(g,Y)\circ p(Y,Z))$$

and

$$f^*(q(g,Y))\circ q(f,Y) = q(f,g^*(Y))\circ q(g,Y) = q(q(f,X),g^*(X))\circ q(g,Y) =$$
$$q(q(f,X)\circ g,Y)$$

$$q(f^*(g),f^*(Y))\circ q(f,Y) = q(f^*(g),f^*(Y))\circ q(q(f,Z),Y) = q(f^*(g)\circ q(f,Z),Y) =$$
$$q(q(f,X)\circ g,Y)$$

Lemma is proved.

Some of the previous results can be combined into the following theorem that was mentioned in [**?**, pp. 240-241] but without a proof.

Given a C-system CC and an object Γ of CC the set $CC(\Gamma)$ of objects X such that $X \geq \Gamma$ equipped with the length function given by $l_\Gamma(X) = l(X) - l(\Gamma)$ and the set $Mor(\Gamma)$ of morphisms over Γ with all of the other structures of a C-system restricted in the obvious way from CC form a new C-system which we will denote by $CC(\Gamma)$. Note that the sets of objects and morphisms of $CC(\Gamma)$ are subsets of the sets of objects and morphisms of CC respectively.

A detailed definition of a homomorphism of C-systems is given in [**?**, Definition 3.1].

Theorem 2.4 *The functions*

$$f^* : CC(\Gamma) \to CC(\Gamma')$$
$$f^* : Mor(CC(\Gamma)) \to Mor(CC(\Gamma'))$$

corresponding to a morphism $f : \Gamma' \to \Gamma$ *define a homomorphism of C-systems.*

Proof: The commutation with the length function is easy to prove. The commutation with the ft map is easy to prove. The commutation with the domain and codomain maps are automatic. The commutation with the identities is (3) and the commutation with compositions is (4). The commutation with the p-morphisms is a particular case of (8). The the commutation with q-morphisms is shown in Lemma 2.3. One now applies [**?**, Lemma 3.4] that shows that the commutation with the s-morphisms holds automatically.

3 Products of families of types and (Π, λ)-structures

Let CC be a C-system. For $\Gamma \in Ob(CC)$, let $Ob_n(\Gamma)$ be the set of elements Δ in Ob such that $\Delta \geq \Gamma$ and $l(\Delta) - l(\Gamma) = n$ and let $\widetilde{Ob}_n(\Gamma)$ be the set of elements $s \in \widetilde{Ob}(CC)$ such that $ft(\partial(s)) \geq \Gamma$ and $l(\partial(s)) - l(\Gamma) = n$.

The subset $Ob_n(\Gamma)$ in $Ob(CC)$ can also be seen as the set of objects of length n in $CC(\Gamma)$.

Note that since for any $s \in \widetilde{Ob}(CC)$ we have $l(\partial(s)) > 0$ we have $\widetilde{Ob}_0(\Gamma) = \emptyset$.

We let $\widetilde{Ob}(\Gamma)$ denote the set of $s \in \widetilde{Ob}$ such that $\partial(s) = \Gamma$.

Any element of $\widetilde{Ob}_n(\Gamma)$ is a morphism over Γ and therefore for $f : \Gamma' \to \Gamma$ the functions f^* on objects and morphisms restrict to functions

$$Ob_n(\Gamma) \to Ob_n(\Gamma')$$

$$\widetilde{Ob}_n(\Gamma) \to \widetilde{Ob}_n(\Gamma')$$

which we continue to write as f^*.

The structure of "products of families of types" is defined in [**?**, pp.3.37 and 3.41] and also considered in [**?**, p.71]. Let us recall this definition here.

Definition 3.1 *The structure of products of families of types on a C-system CC is a collection of data of the form:*

1. *for every $\Gamma \in Ob$ a function $\Pi_\Gamma : Ob_2(\Gamma) \to Ob_1(\Gamma)$, which we write simply as Π,*
2. *for every Γ and $B \in Ob_2(\Gamma)$ a morphism $Ap_B : p_A^*(\Pi(B)) \to B$ over A, where $A = ft(B)$,*

such that:

1. *for any Γ and $B \in Ob_2(\Gamma)$ the map $\lambda inv_{Ap} : \widetilde{Ob}(\Pi(B)) \to \widetilde{Ob}(B)$ defined as:*

   $$s \mapsto p_A^*(s) \circ Ap_B$$

   is a bijection,
2. *for any $f : \Gamma' \to \Gamma$ the square*

   $$\begin{array}{ccc} Ob_2(\Gamma) & \xrightarrow{\Pi_\Gamma} & Ob_1(\Gamma) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ Ob_2(\Gamma') & \xrightarrow{\Pi_{\Gamma'}} & Ob_1(\Gamma') \end{array}$$

   commutes,
3. *for any Γ, $B \in Ob_2(\Gamma)$ and $f : \Gamma \to \Gamma'$ one has $f^*(Ap_B) = Ap_{f^*(B)}$.*

We will show in the next section how to construct products of families of types on C-systems of the form $CC(\mathcal{C}, p)$. For this construction we first need to introduce another structure on C-systems and construct a bijection between the set of products of families of types structures and the set of these new structures.

Definition 3.2 *Let CC be a C-system. A pre-(Π, λ)-structure on CC is a pair of families of functions*

$$\Pi_\Gamma : Ob_2(\Gamma) \to Ob_1(\Gamma)$$

$$\lambda_\Gamma : \widetilde{Ob}_2(\Gamma) \to \widetilde{Ob}_1(\Gamma)$$

such that $\partial(\lambda(s)) = \Pi(\partial(s))$ and one has:

1. *for any $f : \Gamma' \to \Gamma$ the square*

$$\begin{array}{ccc} Ob_2(\Gamma) & \xrightarrow{\Pi_\Gamma} & Ob_1(\Gamma) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ Ob_2(\Gamma') & \xrightarrow{\Pi_{\Gamma'}} & Ob_1(\Gamma') \end{array} \tag{13}$$

commutes,

2. *for any $f : \Gamma' \to \Gamma$ the square*

$$\begin{array}{ccc} \widetilde{Ob}_2(\Gamma) & \xrightarrow{\lambda_\Gamma} & \widetilde{Ob}_1(\Gamma) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \widetilde{Ob}_2(\Gamma') & \xrightarrow{\lambda_{\Gamma'}} & \widetilde{Ob}_1(\Gamma') \end{array} \tag{14}$$

commutes.

The condition that $\partial(\lambda(s)) = \Pi(\partial(s))$ can also be seen as the assertion that the square:

$$\begin{array}{ccc} \widetilde{Ob}_2(\Gamma) & \xrightarrow{\lambda_\Gamma} & \widetilde{Ob}_1(\Gamma) \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial} \\ Ob_2(\Gamma) & \xrightarrow{\Pi_\Gamma} & Ob_1(\Gamma) \end{array} \tag{15}$$

commutes.

Definition 3.3 *A pre-(Π, λ)-structure is called a (Π, λ)-structure if for any $\Gamma \in Ob_{\geq 2}$ the square (15) is a pullback square or, equivalently, if the functions*

$$\lambda'_\Gamma : \widetilde{Ob}(\Gamma) \to \widetilde{Ob}(\Pi(\Gamma))$$

defined by λ_Γ are bijections.

We are going to construct, for a given family of functions Π_Γ a bijection between the set of (Π,λ)-structures over Π_Γ and the set of products of families of types over the same Π_Γ.

We first reformulate the structure of products of families slightly. Instead of considering $p_A^*(\Pi(B))$ we will consider an object that is isomorphic (but not equal!) to it, namely $p_{\Pi(B)}^*(A)$. Our structure will then be a family of maps Π as before together with, for every Γ and $B \in Ob_2(\Gamma)$, a morphism

$$Ap'_B : p_{\Pi(B)}^*(ft(B)) \to B$$

over $ft(B)$ such that the function

$$\lambda inv'_B : \widetilde{Ob}(\Pi(B)) \to \widetilde{Ob}(B)$$

defined as:

$$\lambda inv'_B(s) = q(s, p_{\Pi(B)}^*(ft(B))) \circ Ap'_B$$

is a bijection (see the diagram below).

The bijection from the set of products of families of types structures as defined by Cartmell and Streicher and the modified products of families of types structures described above is given by the pre-composition of Ap_B, for each Γ, B, with the evident isomorphism $p_A^*(\Pi(B)) \to p_{\Pi(B)}^*(A)$.

We now state the problem which we will provide a construction for:

Problem 3.4 *Let CC be a C-system and let Π be a family of functions*

$$\Pi_\Gamma : Ob_2(\Gamma) \to Ob_1(\Gamma)$$

given for all $\Gamma \in Ob$ such that the corresponding squares of the form (13) commute.

To construct a bijection between the following two sets of structure:

1. *for every Γ and $B \in Ob_2(\Gamma)$ a bijection*

$$\lambda'_B : \widetilde{Ob}(B) \to \widetilde{Ob}(\Pi(B))$$

such that for every morphism $f : \Gamma' \to \Gamma$ the square

$$\begin{array}{ccc} \widetilde{Ob}(B) & \xrightarrow{\lambda'_B} & \widetilde{Ob}(\Pi(B)) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \widetilde{Ob}(f^*(B)) & \xrightarrow{\lambda'_{f^*(B)}} & \widetilde{Ob}(\Pi(f^*(B))) \end{array} \qquad (16)$$

defined by f, commutes.

2. *for every $\Gamma \in Ob$ and $B \in Ob_2(\Gamma)$ a morphism*

$$Ap'_B : p_{\Pi(B)}^*(ft(B)) \to B$$

over $ft(B)$ such that the function

$$\lambda inv'_B : \widetilde{Ob}(\Pi(B)) \to \widetilde{Ob}(B)$$

defined as:

$$\lambda inv'_B(s) = q(s, p^*_{\Pi(B)}(ft(B))) \circ Ap'_B$$

is a bijection and such that for every morphism $f : \Gamma' \to \Gamma$ and $B \in Ob_2(\Gamma)$ one has $f^*(Ap'_B) = Ap'_{f^*(B)}$.

We will construct the solution in four steps - first a function from structures of the first kind to structures of the second, then a function in the opposite direction and the two lemmas proving that the first function is a left and a right inverse to the second.

The computations one has to do show how daunting computations in C-systems may appear, however the C-systems in these proofs appear as purely essentially-algebraic objects and all the proofs are by rewriting without the induction by length.

Construction 3.5 Let us show how to construct a structure of the second kind from a structure of the first kind.

Since Π is stable under pullbacks we have

$$\Pi(p^*_{\Pi(B)}(B)) = p^*_{\Pi(B)}(\Pi(B))$$

and therefore $s_{Id_{\Pi(B)}}$ (which is the diagonal) gives us an element in $\widetilde{Ob}(\Pi(p^*_{\Pi(B)}(B)))$. Applying to it the inverse of our λ' we get an element

$$ap_B = (\lambda'_{p^*_{\Pi(B)}(B)})^{-1}(s_{Id_{\Pi(B)}})$$

in $\widetilde{Ob}(p^*_{\Pi(B)}(B))$:

$$\begin{array}{ccccc}
B & \xrightarrow{q(s,p^*_{\Pi(B)}(B))} & p^*_{\Pi(B)}(B) & \xrightarrow{q(p_{\Pi(B)},B)} & B \\
{\scriptstyle s^*(ap_B)\uparrow}\downarrow & & {\scriptstyle ap_B\uparrow}\downarrow & & \downarrow \\
ft(B) & \xrightarrow{q(s,p^*_{\Pi(B)}(ft(B)))} & p^*_{\Pi(B)}(ft(B)) & \xrightarrow{q(p_{\Pi(B)},ft(B))} & ft(B) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma & \xrightarrow{s} & \Pi(B) & \xrightarrow{p_{\Pi(B)}} & \Gamma
\end{array}$$

Define:

$$Ap'_B = ap_B \circ q(p_{\Pi(B)}, B)$$

One verifies easily that Ap'_B is a morphism over $ft(B)$.

Let us prove that these morphisms satisfy the conditions of bijectivity and the stability under pullbacks. We need to show that the mappings $\lambda inv'_B : \widetilde{Ob}(\Pi(B)) \to \widetilde{Ob}(B)$ defined as:

$$s \mapsto q(s, p^*_{\Pi(B)}(ft(B))) \circ Ap'_B$$

are bijective. It is sufficient to show that the mappings $\lambda inv'_B$ are inverse to the ones given by λ'_B from at least one side as any inverse to a bijection is a bijection.

We will show that

$$\lambda inv'_B \circ \lambda'_B = Id_{\widetilde{Ob}(B)}$$

We do it in two steps. First let

$$\lambda inv''_B(s) = s^*(ap) = q(s, p^*_{\Pi(B)}(ft(B)))^*(ap)$$

Let us show that $\lambda inv''_B = \lambda inv'_B$. Indeed:

$$q(s, p^*_{\Pi(B)}(ft(B)))^*(ap) = q(s, p^*_{\Pi(B)}(ft(B)))^*(ap) \circ q(s, p^*_{\Pi(B)}(B)) \circ q(p_{\Pi(B)}, B) =$$

$$q(s, p^*_{\Pi(B)}(ft(B))) \circ ap \circ q(p_{\Pi(B)}, B) = q(s, p^*_{\Pi(B)}(ft(B))) \circ Ap'_B$$

Where the first equality follows from Lemma 2.2 and the assumption that $s \circ \pi_{\Pi(B)} = Id_\Gamma$ and the second equality follows from (2).

Now we have:

$$\lambda'_B(\lambda inv''_B(s)) = \lambda'_B(s^*(ap)) = s^*(\lambda'_{p^*_{\Pi(B)}(B)}(ap)) = s^*(s_{Id_{\Pi(B)}}) = s_{s\circ Id_{\Pi(B)}} = s_s = s$$

where the second equality follows from (16), the third from the formula $s_{f\circ g} = f^*(s_g)$ and the sixth from the formula $s = s_s \circ q(ft(s), X)$ (see [**?**, Def. 2.3]) since $ft(s) = Id$.

It remains to check that the mappings Ap' are stable under the base change along morphisms $f : \Gamma' \to \Gamma$. Let us show first that $f^*(ap_B) = ap_{f^*(B)}$. We have:

$$f^*(ap_B) = f^*((\lambda')^{-1}(s_{Id_{\Pi(B)}})) = (\lambda')^{-1}(f^*(s_{Id_{\Pi(B)}})) = (\lambda')^{-1}(s_{f^*(Id_{\Pi(B)})}) =$$

$$(\lambda')^{-1}(s_{Id_{f^*(\Pi(B))}}) = (\lambda')^{-1}(s_{Id_{\Pi(f^*(B))}}) = ap_{f^*(B)} \tag{17}$$

where the second equality follows from (16), the third from (3) and the fourth from the commutativity of squares (14).

Now we have:

$$f^*(Ap'_B) = f^*(ap_B \circ q(p_{\Pi(B)}, B)) = f^*(ap_B) \circ f^*(q(p_{\Pi(B)}, B)) = ap_{f^*(B)} \circ f^*(q(p_{\Pi(B)}, B)) =$$

$$ap_{f^*(B)} \circ q(f^*(p_{\Pi(B)}), f^*(B)) = ap_{f^*(B)} \circ q(p_{f^*(\Pi(B))}, f^*(B)) = ap_{f^*(B)} \circ q(p_{\Pi(f^*(B))}, f^*(B))$$

where the second equality follows from (4), the third equality from (17), the fourth equality from (12), the fifth equality from (8) and the sixth equality from the commutativity of squares (14).

Construction 3.6 Let us now construct a structure of the first kind from a structure of the second. This is straightforward since a construction of the second kind gives us bijections $\lambda inv'_B$ and the inverse to these bijections are bijections required for the structure of the first kind. We set:

$$\lambda'_B = (\lambda inv'_B)^{-1}$$

To show that the bijections that we obtain in this way are stable under the pullbacks it is sufficient to show that the bijections $\lambda inv'_B$ are stable under pullbacks. Consider $f : \Gamma' \rightarrow \Gamma$. Then we have

$$f^*(\lambda inv'_B(s)) = f^*(q(s, p^*_{\Pi(B)}(ft(B))) \circ Ap'_B) = f^*(q(s, p^*_{\Pi(B)}(ft(B)))) \circ f^*(Ap'_B) =$$

$$q(f^*(s), f^*(p^*_{\Pi(B)}(ft(B)))) \circ Ap'_{f^*(B)} = q(f^*(s), (f^*(p_{\Pi(B)}))^*(f^*(ft(B)))) \circ Ap'_{f^*(B)} =$$

$$q(f^*(s), p^*_{f^*(\Pi(B))}(ft(f^*(B)))) \circ Ap'_{f^*(B)} = q(f^*(s), p^*_{\Pi(f^*(B))}(ft(f^*(B)))) \circ Ap'_{f^*(B)} =$$

$$\lambda inv'_{f^*(B)}(f^*(s))$$

where the second equality is by (4), the third equality is by (12) and our assumption on Ap', the fourth equality is by (11), the fifth is by (8) and the fact that f^* commutes with ft and finally the sixth is by our assumption on Π.

Let us denote the map of Construction 3.5 by $C1$ and the map of Construction 3.6 by $C2$.

Lemma 3.7 *For a structure of the first kind λ' one has $C2(C1(\lambda')) = \lambda'$.*

Proof: This is immediate since in Construction 3.5 we proved that the functions $\lambda inv'$ that we have constructed are bijections by showing that they are inverses to the λ''s that we started with and in Construction 3.6 we defined λ''s as inverses to $\lambda inv'$.

Lemma 3.8 *For a structure of the second kind Ap' one has $C1(C2(Ap')) = Ap'$.*

Proof: Let Ap' be a structure of the second kind. Then $\lambda' = C2(Ap')$ is the family of functions given for all $\Gamma \in CC$ and $B \in Ob_2(\Gamma)$ of the form

$$\lambda'_B : \widetilde{Ob}(B) \rightarrow \widetilde{Ob}(\Pi(B))$$

defined by the formula

$$\lambda'_B = (\lambda inv'_B)^{-1}$$

where

$$\lambda inv'_B(s) = q(s, p^*_{\Pi(B)}(ft(B))) \circ Ap'_B$$

Next we have

$$C1(\lambda')_B = ap_B \circ q(p_{\Pi(B)}, B) = (\lambda'_{p^*_{\Pi(B)}(B)})^{-1}(s_{Id_{\Pi(B)}}) \circ q(p_{\Pi(B)}, B) =$$

$$\lambda inv'_{p^*_{\Pi(B)}(B)}(s_{Id_{\Pi(B)}}) \circ q(p_{\Pi(B)}, B) =$$

$$q(s_{Id_{\Pi(B)}}, p^*_{\Pi(p^*_{\Pi(B)}(B))}(ft(p^*_{\Pi(B)}(B)))) \circ Ap'_{p^*_{\Pi(B)}(B)} \circ q(p_{\Pi(B)}, B)$$

Using that $\Pi(p^*_{\Pi(B)}(B)) = p^*_{\Pi(B)}(\Pi(B))$ and $ft(p^*_{\Pi(B)}(B)) = p^*_{\Pi(B)}(B)$ we obtain, as our goal, the equality:

$$q(s_{Id_{\Pi(B)}}, p^*_{p^*_{\Pi(B)}(\Pi(B))}(p^*_{\Pi(B)}(ft(B)))) \circ Ap'_{p^*_{\Pi(B)}(B)} \circ q(p_{\Pi(B)}, B) = Ap'_B \tag{18}$$

For any $f : \Gamma' \to \Gamma$ we have:

$$Ap'_{f*(B)} \circ q(f, B) = f^*(Ap'_B) \circ q(f, B) = q(f, p^*_{\Pi(B)}(ft(B))) \circ Ap'_B$$

where the first equality is by stability of Ap' under pullbacks and the second by (2). Applying it to (18) and $f = p_{\Pi(B)}$ we get:

$$q(s_{Id_{\Pi(B)}}, p^*_{p^*_{\Pi(B)}(\Pi(B))}(p^*_{\Pi(B)}(ft(B)))) \circ q(p_{\Pi(B)}, p^*_{\Pi(B)}(ft(B))) \circ Ap'_B = Ap'_B \qquad (19)$$

Next we have

$$p^*_{p^*_{\Pi(B)}(\Pi(B))}(p^*_{\Pi(B)}(ft(B))) = (p_{p^*_{\Pi(B)}(\Pi(B))} \circ p_{\Pi(B)})^*(ft(B)) =$$
$$(q(p_{\Pi(B)}, \Pi(B)) \circ p_{\Pi(B)})^*(ft(B)) = q(p_{\Pi(B)}, \Pi(B))^*(p^*_{\Pi(B)}(ft(B))) =$$
$$p^*_{\Pi(B)}(p^*_{\Pi(B)}(ft(B)))$$

as seen from the commutative diagram

$$\begin{array}{ccccc}
p^*_{\Pi(B)}(ft(B)) & \longrightarrow & p^*_{\Pi(B)}(p^*_{\Pi(B)}(ft(B))) & \xrightarrow{q(p_{\Pi(B)}, p^*_{\Pi(B)}(ft(B)))} & p^*_{\Pi(B)}(ft(B)) \\
\downarrow & & \downarrow & & \downarrow \\
\Pi(B) & \xrightarrow{s_{Id_{\Pi(B)}}} & p^*_{\Pi(B)}(\Pi(B)) & \xrightarrow{q(p_{\Pi(B)}, \Pi(B))} & \Pi(B) \\
 & & {\scriptstyle p_{p^*_{\Pi(B)}(\Pi(B))}} \downarrow & & \downarrow {\scriptstyle p_{\Pi(B)}} \\
 & & \Pi(B) & \xrightarrow{p_{\Pi(B)}} & \Gamma
\end{array}$$

and we can rewrite (19) as

$$q(s_{Id_{\Pi(B)}}, q(p_{\Pi(B)}, \Pi(B))^*(p^*_{\Pi(B)}(ft(B)))) \circ q(p_{\Pi(B)}, p^*_{\Pi(B)}(ft(B))) \circ Ap'_B = Ap'_B$$

From the definition of iterated q we have

$$q(p_{\Pi(B)}, p^*_{\Pi(B)}(ft(B))) = q(q(p_{\Pi(B)}, \Pi(B)), p^*_{\Pi(B)}(ft(B)))$$

and the goal becomes:

$$q(s_{Id_{\Pi(B)}}, q(p_{\Pi(B)}, \Pi(B))^*(p^*_{\Pi(B)}(ft(B)))) \circ q(q(p_{\Pi(B)}, \Pi(B)), p^*_{\Pi(B)}(ft(B))) \circ Ap'_B =$$
$$Ap'_B \qquad (20)$$

Applying the formula

$$q(f, g^*(X)) \circ q(g, X) = q(f \circ g, X)$$

for $f = s_{Id_{\Pi(B)}}$, $g = q(p_{\Pi(B)}, \Pi(B))$ and $X = p^*_{\Pi(B)}(ft(B))$ we get

$$q(s_{Id_{\Pi(B)}}, q(p_{\Pi(B)}, \Pi(B))^*(p^*_{\Pi(B)}(ft(B)))) \circ q(q(p_{\Pi(B)}, \Pi(B)), p^*_{\Pi(B)}(ft(B))) =$$
$$q(s_{Id_{\Pi(B)}} \circ q(p_{\Pi(B)}, \Pi(B)), p^*_{\Pi(B)}(ft(B))) = q(Id_{\Pi(B)}, p^*_{\Pi(B)}(ft(B)))$$
$$= Id_{p^*_{\Pi(B)}(ft(B))}$$

which implies (20).

This completes our construction for Problem 3.4.

4 More on the C-systems of the form $CC(\mathcal{C}, p)$

Let us start by considering a general (pre-)category $\mathcal{C}$. Let $p : \widetilde{U} \to U$ be a morphism in $\mathcal{C}$. Recall from [?] that a universe structure on p is a choice of pullback squares of the form

$$\begin{array}{ccc} (X;F) & \xrightarrow{Q(F)} & \widetilde{U} \\ {\scriptstyle p_{X,F}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{F} & U \end{array}$$

for all X and all morphisms $F : X \to U$. A universe in $\mathcal{C}$ is a morphism with a universe structure on it and a universe category is a category with a universe and a choice of a final object pt.

We may use the notation $(X; F_1, \ldots, F_n)$ for $(\ldots(X; F_1); \ldots F_n)$.

For $f : W \to X$ and $g : W \to \widetilde{U}$ we will denote by $f * g$ the unique morphism such that

$$(f * g) \circ p_{X,F} = f$$

$$(f * g) \circ Q(F) = g$$

For $X' \xrightarrow{f} X \xrightarrow{F} U$ we let $Q(f, F)$ denote the morphism

$$(p_{X',f\circ F} \circ f) * Q(f \circ F) : (X'; f \circ F) \to (X; F)$$

The construction of the C-system $CC(\mathcal{C}, p)$ presented in [?] can be described as follows. One defines first, by induction on n, pairs $(Ob_n, int_n : Ob_n \to \mathcal{C})$ where $Ob_n = Ob_n(\mathcal{C}, p)$ is a set and int_n is a function from Ob_n to objects of $\mathcal{C}$ as follows:

1. Ob_0 is the standard one point set *unit* whose element we denote by *tt*. The function int_0 maps *tt* to the final object *pt* of the universe category structure on $\mathcal{C}$,

2. $Ob_{n+1} = \amalg_{A \in Ob_n} Hom(int(A), U)$ and $int_{n+1}(A, F) = (int(A); F)$.

We then define $Ob(CC(\mathcal{C}, p))$ as $\amalg_{n \geq 0} Ob_n$ such that elements of $Ob(CC(\mathcal{C}, p))$ are pairs $\Gamma = (n, A)$ where $A \in Ob_n(\mathcal{C}, p)$. We define the function $int : Ob(CC(\mathcal{C}, p)) \to \mathcal{C}$ as the sum of functions int_n.

The morphisms in $CC(\mathcal{C}, p)$ are defined by

$$Mor_{CC(\mathcal{C},p)} = \amalg_{\Gamma,\Gamma' \in Ob(CC)} Hom_{\mathcal{C}}(int(\Gamma), int(\Gamma'))$$

and the function *int* on morphisms maps a triple $((\Gamma, \Gamma'), a)$ to a. Note that the subset in Mor that consists of f such that $dom(f) = \Gamma$ and $codom(f) = \Gamma'$ is not equal to the set $Hom_{\mathcal{C}}(int(\Gamma), int(\Gamma'))$ but instead to the set of triples of the form $f = ((\Gamma, \Gamma'), a)$ where $a \in Hom_{\mathcal{C}}(int(\Gamma), int(\Gamma'))$.

Problem 4.1 *To construct, for all* $\Gamma \in Ob(CC(\mathcal{C},p))$ *bijections*

$$u_{1,\Gamma} : Ob_1(\Gamma) \to Hom_{\mathcal{C}}(int(\Gamma), U)$$

$$\widetilde{u}_{1,\Gamma} : \widetilde{Ob}_1(\Gamma) \to Hom_{\mathcal{C}}(int(\Gamma), \widetilde{U})$$

such that:

1. *for* $(n, A) \in Ob(CC(\mathcal{C},p))$ *one has*

$$u_1(n+1,(A,F)) = F \tag{21}$$

   and if $l(\Gamma') = n > 0$ *then*

$$int(\Gamma') = (int(ft(\Gamma')); u_1(\Gamma')) \tag{22}$$

2. *for* $o \in \widetilde{Ob}_1(\Gamma)$ *one has*

$$\widetilde{u}_1(o) = int(o) \circ Q(u_1(\partial(o))) \tag{23}$$

   and

$$int(o) = Id_{ft(\partial(o))} * \widetilde{u}_1(o) \tag{24}$$

3. u_1 *and* $\widetilde{u}_1$ *are natural in* Γ *i.e. for any* $f : \Gamma' \to \Gamma$ *one has:*

$$u_1(f^*(T)) = f \circ u_1(T) \tag{25}$$

$$\widetilde{u}_1(f^*(o)) = f \circ \widetilde{u}_1(o) \tag{26}$$

4. *one has*

$$u_1(\partial(o)) = \widetilde{u}_1(o) \circ p \tag{27}$$

Remark 4.2 The families of sets $Ob_1(\Gamma)$ and $\widetilde{Ob}_1(\Gamma)$ together with the families of functions f^* satisfy the axioms of presheaves. To construct families of functions $u_{1,\Gamma}$ and $\widetilde{u}_{1,\Gamma}$ satisfying conditions (3) of the problem is the same as to construct presheaf isomorphisms $Ob_1 \to int_*(Yo(U))$ and $\widetilde{Ob}_1 \to int_*(Yo(\widetilde{U}))$ where Yo is the Yoneda embedding and

$$int_* : PreShv(\mathcal{C}) \to PreShv(CC)$$

is the functor given by $int_*(F)(X) = F(int(X))$. The fourth condition asserts that the square

$$\begin{array}{ccc} \widetilde{Ob}_1 & \xrightarrow{\widetilde{u}_1} & int_*(Yo(\widetilde{U})) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle Yo(p)} \\ Ob_1 & \xrightarrow{u_1} & int_*(Yo(U)) \end{array}$$

commutes.

Construction 4.3 For $\Gamma = (n, A)$ where $A \in Ob_n(\mathcal{C}, p)$, an element Γ' in $Ob_1(\Gamma)$ is a triple $(n+1, (A, F))$ where $F : int(A) \to U$. Mapping such a triple to F we obtain a bijection

$$u_{1,\Gamma} : Ob_1(\Gamma) \to Hom_{\mathcal{C}}(int(\Gamma), U)$$

For Γ' such that $l(\Gamma') = n + 1 > 0$ we have $\Gamma' = (n + 1, (A, F))$ where $ft(\Gamma') = (n, A)$ and

$$int(\Gamma') = int((A, F)) = (int(A); F) = (int(ft(\Gamma)); u_1(\Gamma'))$$

An element o in $\widetilde{Ob}_1(\Gamma)$ is a triple $((\Gamma, \Gamma'), s)$ where

$$s = int(o) \in Hom_{\mathcal{C}}(int(\Gamma), int(\Gamma')),$$

$\Gamma' = \partial(o)$ is an object such that $ft(\Gamma') = \Gamma$, $s \circ int(p_{\Gamma'}) = Id_{int(\Gamma)}$ and $l(\Gamma') = n + 1 > 0$.

Define the function $\widetilde{u}_{1,\Gamma}$ by the formula

$$\widetilde{u}_{1,\Gamma}(o) = int(o) \circ Q(u_{1,\Gamma}(\partial(o)))$$

If $\Gamma = (n, A)$ then

$$\partial(o) = (n + 1, (A, F))$$

where $F = u_{1,\Gamma}(\partial(o)) : int(A) \to U$ and we have a canonical square

$$\begin{array}{ccc} int(\partial(o)) & \xrightarrow{Q(u_{1,\Gamma}(\partial(o)))} & \widetilde{U} \\ {\scriptstyle int(p_{\partial(o)})}\downarrow & & \downarrow{\scriptstyle p} \\ int(\Gamma) & \xrightarrow{u_{1,\Gamma}(\partial(o))} & U \end{array} \qquad (28)$$

which shows that the composition $s \circ Q(u_{1,\Gamma}(\Gamma'))$ is defined and is a morphism $int(\Gamma) \to \widetilde{U}$.

For the formula (24) we have

$$int(o) = Id_{ft(\partial(o))} * (int(o) \circ Q(u_1(\partial(o))))$$

because a morphism to a fiber product equals to the product of its composition with the projections and therefore

$$int(o) = Id_{ft(\partial(o))} * \widetilde{u}_1(o)$$

by definition of $\widetilde{u}_1(o)$.

To show $\widetilde{u}_{1,\Gamma}$ it is a bijection let us construct an inverse. For $f : int(\Gamma) \to \widetilde{U}$ let

$$\widetilde{u}^!_{1,\Gamma}(f) = (\Gamma, ((n + 1, (A, f \circ p)), s_f))$$

where $s_f : int(\Gamma) \to (A; f \circ p)$ is the unique section of $p_{A,f\circ p}$ such that $s_f \circ Q(f \circ p) = f$.

We have

$$\widetilde{u}^!(\widetilde{u}((\Gamma, \Gamma'), s)) = \widetilde{u}^!(s \circ Q(u(\Gamma'))) =$$

$$((\Gamma, (n + 1, (A, s \circ Q(u(\Gamma')) \circ p))), s') = ((\Gamma, (n + 1, (A, u(\Gamma')))), s')$$

where $s' = s_{s\circ Q(u(\Gamma'))} = s$ which proves that $\widetilde{u}^!$ is inverse to $\widetilde{u}$ from one side. In the opposite direction we have

$$\widetilde{u}(\widetilde{u}^!(f)) = \widetilde{u}((\Gamma, (n+1, (A, f\circ p))), s_f) = s_f \circ Q(u((n+1, (A, f\circ p)))) = s_f \circ Q(f\circ p) = f$$

The proofs of the naturality of u_1 and $\widetilde{u}_1$ with respect to morphisms in Γ follow easily from the definition of the canonical squares in $CC(\mathcal{C}, p)$.

Formula (27) is a corollary of the commutativity of the square (28).

We will now construct bijections $u_{2,\Gamma}$ and $\widetilde{u}_{2,\Gamma}$ similar to the bijections $u_{1,\Gamma}$ and $\widetilde{u}_{1,\Gamma}$ but having as sources $Ob_2(\Gamma)$ and $\widetilde{Ob}_2(\Gamma)$.

For any $V \in \mathcal{C}$ we define functor data $D_p(-, V)$ given on objects by

$$D_p(X, V) := \amalg_{F:X\to U} Hom((X; F), V)$$

and on morphisms by

$$D_p(f, V) : (F_1, F_2) \mapsto (f\circ F_1, Q(f, F_1)\circ F_2)$$

The sets $D_p(X, V)$ are also functorial in V according to the formula

$$D_p(X, g)(F_1, F_2) = (F_1, F_2 \circ g)$$

and for $f : X \to X'$, $g : V \to V'$ we have

$$D_p(f, V) \circ D_p(X, g) = D_p(X', g) \circ D_p(f, V')$$

Problem 4.4 *To construct for all $\Gamma \in Ob(CC(\mathcal{C}, p))$ bijections*

$$u_{2,\Gamma} : Ob_2(\Gamma) \to D_p(int(\Gamma), U)$$

$$\widetilde{u}_{2,\Gamma} : \widetilde{Ob}_2(\Gamma) \to D_p(int(\Gamma), \widetilde{U})$$

such that:

1. $u_{2,\Gamma}(T) = (u_{1,\Gamma}(ft(T)), u_{1,ft(T)}(T))$
2. $\widetilde{u}_{2,\Gamma}(o) = (u_{1,\Gamma}(ft(\partial(o))), \widetilde{u}_{1,ft(\partial(o))}(o))$
3. *for $f : \Gamma' \to \Gamma$ one has*
$$u_2(f^*(T)) = D_p(f, U)(u_2(T))$$
$$\widetilde{u}_2(f^*(o)) = D_p(f, \widetilde{U})(\widetilde{u}_2(o))$$
4. $u_2(\partial(o)) = D_p(int(\Gamma), p)(\widetilde{u}_2(o))$

Construction 4.5 By (22) we have

$$int(ft(T)) = (int(\Gamma); u_{1,\Gamma}(ft(T)))$$

and therefore $(u_{1,\Gamma}(ft(T)), u_{1,ft(T)}(T))$ is a well defined element of $D_p(int(\Gamma), U)$ for all $T \in Ob_2(\Gamma)$. Let us define the function $u_{2,\Gamma}$ by the formula

$$u_{2,\Gamma}(T) = (u_{1,\Gamma}(ft(T)), u_{1,ft(T)}(T))$$

We can write this function as a composition of the bijection

$$Ob_2(\Gamma) \to \amalg_{\Gamma' \in Ob_1(\Gamma)} Ob_1(\Gamma')$$

that sends T to $(ft(T), T)$ with the function

$$\amalg_{\Gamma' \in Ob_1(\Gamma)} Ob_1(\Gamma') \to \amalg_{F \in Hom(int(\Gamma), U)} Hom((int(\Gamma); F), U)$$

that is the total function of the function $u_{1,\Gamma}$ and the family of functions $u_{1,\Gamma'}$ given for all $\Gamma' \in Ob_1(\Gamma)$. Since $u_{1,\Gamma}$ is a bijection and for each Γ', $u_{1,\Gamma'}$ is a bijection, the total function is a bijection.

Similarly, $(u_{1,\Gamma}(ft(\partial(o))), \widetilde{u}_{1,ft(\partial(o))}(o))$ is a well defined element of $D_p(int(\Gamma), \widetilde{U})$ since

$$int(ft(\partial(o))) = (int(\Gamma); u_{1,\Gamma}(ft(\partial(o))))$$

Define the function $\widetilde{u}_{2,\Gamma}$ be the formula

$$\widetilde{u}_{2,\Gamma}(o) = (u_{1,\Gamma}(ft(\partial(o))), \widetilde{u}_{1,ft(\partial(o))}(o))$$

We can write this function as the composition of the bijection

$$\widetilde{Ob}_2(\Gamma) \to \amalg_{\Gamma' \in Ob_1(\Gamma)} \widetilde{Ob}_1(\Gamma')$$

that sends o to $(ft(\partial(o)), o)$ with the function

$$\amalg_{\Gamma' \in Ob_1(\Gamma)} \widetilde{Ob}_1(\Gamma') \to \amalg_{F \in Hom(int(\Gamma), U)} Hom((int(\Gamma); F), \widetilde{U})$$

that is the total function of the function $u_{1,\Gamma}$ and the family of functions $\widetilde{u}_{1,\Gamma'}$ given for all $\Gamma' \in Ob_1(\Gamma)$. Since $u_{1,\Gamma}$ is a bijection and for each Γ', $\widetilde{u}_{1,\Gamma'}$ is a bijection, the total function is a bijection.

The verification of the third and the fourth conditions of the problem are easy from the definition of u_2 and $\widetilde{u}_2$.

Remark 4.6 The families of sets $D_p(X, V)$ together with the families of functions $D_p(f, V)$ and $D_p(X, g)$ define, as one can easily prove from definitions, a functor from $\mathcal{C}^{op} \times \mathcal{C}$ to $Sets$ or, if viewed as families $V \mapsto D_p(-, V)$, a functor

$$Yo_2 : \mathcal{C} \to PreShv(\mathcal{C})$$

If $Yo_1 = Yo$ is the Yoneda embedding then we can see u_i for $i = 1, 2$ as isomorphisms

$$Ob_i \to int_*(Yo_i(U))$$

and $\widetilde{u}_i$ as isomorphisms

$$\widetilde{Ob}_i \to int_*(Yo_i(\widetilde{U}))$$

These isomorphisms generalize easily to all $i > 0$ if one defines, inductively,

$$Yo_{n+1}(V)(X) = \amalg_{F:X\to U} Yo_n(V)((X;F))$$

Moreover, if we define $Hom_n(X,Y)$ as $Yo_n(Y)(X)$ then there are composition functions

$$Hom_n(X,Y) \times Hom_m(Y,Z) \to Hom_{n+m}(X,Z)$$

that are likely to satisfy the unity and associativity axioms such that one obtains, from any universe category $(\mathcal{C}, p)$, a new category $(\mathcal{C}, p)_*$ with the same collection of objects and morphisms between two objects given by

$$Hom_{(\mathcal{C},p)_*}(X,Y) = \amalg_{n\geq 1} Hom_m(X,Y)$$

In this paper we will not need Yo_n for $n > 2$ and we defer the study of this structure until the future papers.

When $\mathcal{C}$ is a locally cartesian closed category (see appendix), the functors $D_p(-,V)$ become representable providing us with a way to describe operations such as Π and λ on $CC(\mathcal{C}, p)$ in terms of morphisms between objects in $\mathcal{C}$.

For a morphism $p : \widetilde{U} \to U$ in a locally cartesian closed category and an object V of this category let

$$I_p(V) := \underline{Hom}_U((\widetilde{U}, p), (U \times V, pr_1))$$

and let

$$prI_p(V) = p\triangle pr_1 : I_p(V) \to U$$

be the morphism that defines $I_p(V)$ as an object over U.

Remark 4.7 In [**?**] generalized polynomial functors are defined as functors isomorphic to functors of the form I_p.

Note that I_p depends on the choice of a locally cartesian closed structure on $\mathcal{C}$. On the other hand, the construction of the functors $D_p(X,V)$ requires a universe structure on p but do not require a locally cartesian closed structure on $\mathcal{C}$.

The computations below are required in order to establish the connections between the constructions that use the locally cartesian closed structure and the constructions that use universe structures.

Let $p : \widetilde{U} \to U$ be a universe and V an object of $\mathcal{C}$. We assume that $\mathcal{C}$ is equipped with a locally cartesian closed structure. For $F : X \to U$ there is a unique morphism

$$\iota_F : (X;F) \to (X,f) \times_U (\widetilde{U}, p)$$

such that $\iota_F \circ pr_1 = p_{X,F}$ and $\iota_F \circ pr_2 = Q(F)$ which is a particular case of the morphisms ι, ι' of Lemma 9.1.

The evaluation morphism in the case of $I_p(V)$ is of the form

$$evI_p : (I_p(V), prI_p(V)) \times_U (U \times V, pr_1) \to U \times V$$

Define a morphism

$$st_p(V) : (I_p(V); prI_p(V)) \to V$$

as the composition:

$$st_p(V) := \iota_{prI_p(V)} \circ evI_p(V) \circ pr_2$$

We will need to use some properties of these morphisms.

Lemma 4.8 *Let $f : V \to V'$ be a morphism, then one has*

$$Q(I_p(f), prI_p(V')) \circ st_p(V') = st_p(V) \circ f$$

Proof: Let $pr = prI_p(V)$, $pr' = prI_p(V')$, $\iota = \iota_{pr}$, $\iota' = \iota_{pr'}$, $ev = evI_p(V)$ and $ev' = evI_p(V')$. Then we have to verify that the outer square of the following diagram commutes:

$$\begin{array}{ccccccc}
(I_p(V); pr) & \xrightarrow{\iota} & (I_p(V), pr) \times_U (\widetilde{U}, p) & \xrightarrow{ev} & U \times V & \xrightarrow{pr_2} & V \\
{\scriptstyle Q(I_p(f), pr')}\downarrow & & {\scriptstyle I_p(f) \times Id_{\widetilde{U}}}\downarrow & & {\scriptstyle Id_U \times f}\downarrow & & \downarrow{\scriptstyle f} \\
(I_p(V'); pr') & \xrightarrow{\iota'} & (I_p(V'), pr') \times_U (\widetilde{U}, p) & \xrightarrow{ev'} & U \times V' & \xrightarrow{pr_2} & V'
\end{array}$$

The commutativity of the left square is a particular case of Lemma 9.1. The commutativity of the right square is an immediate corollary of the definition of $Id_U \times f$. The commutativity of the middle square is a particular case of the axiom of locally cartesian closed structure that says that morphisms ev_Y^X are natural in Y.

Problem 4.9 *Let $(\mathcal{C}, p, pt)$ be a locally cartesian closed universe category. To construct, for all $X, V \in \mathcal{C}$, bijections*

$$\eta_{X,V} : D_p(X, V) \to Hom(X, I_p(V))$$

that are natural in X and V, i.e., such that for any $d \in D_p(X, V)$ one has

1. *for all $f : V \to V'$ one has $\eta(d) \circ I_p(f) = \eta(D_p(X, f)(d))$,*
2. *for all $f : X' \to X$ one has $f \circ \eta(d) = \eta(D_p(f, V)(d))$.*

Construction 4.10 We will construct bijections

$$\eta^!_{X,V} : Hom(X, I_p(V)) \to D_p(X, V)$$

such that for any $g : X \to I_p(V)$ one has:

1. for all $f : V \to V'$ one has $D_p(X, f)(\eta^!(g)) = \eta^!(g \circ I_p(f))$,

2. for all $f : X' \to X$ one has $D_p(f, V)(\eta^!(g)) = \eta^!(f \circ g)$.

and then define $\eta_{X,V}$ as the inverse to $\eta^!_{X,V}$.

For $g : X \to I_p(V)$ we set

$$\eta^!_{X,V}(g) := (g \circ prI_p(V), Q(g, prI_p(V)) \circ st_p(V))$$

To see that this is a bijection observe first that it equals to the composition

$$Hom(X, I_p(V)) \to \amalg_{F:X\to U} Hom_U((X, F), (I_p(V), prI_p(V))) \to \amalg_{F:X\to U} Hom((X; F), V)$$

where the first map is of the form $g \mapsto (g \circ prI_p(V), g)$ and the second is the sum over all $F : X \to U$ of maps $g \mapsto Q(g, prI_p(V)) \circ st_p(V)$. The first of these two maps is a bijection. It remains to show that the second one is a bijection for every F.

By definition of the $\underline{Hom}$ structure we know that for each F the map

$$Hom_U((X, F), (I_p(V), prI_p(V))) \to Hom_U(((X, F) \times_U (\widetilde{U}, p), -), (U \times V, pr_1))$$

given by $g \mapsto (g \times Id_{\widetilde{U}}) \circ evI_p(V)$ is a bijection. We also know that the map

$$Hom_U(((X, F) \times_U (\widetilde{U}, p), F \diamond p), (U \times V, pr_1)) \to Hom((X, F) \times_U (\widetilde{U}, p), V)$$

is a bijection. Since ι_F is an isomorphism the composition with it is a bijection. Now we have two maps

$$Hom_U((X, F), (I_p(V), prI_p(V))) \to Hom((X; F), V)$$

given by $g \mapsto \iota_F \circ (g \times Id_{\widetilde{U}}) \circ evI_p(V) \circ p_V$ and $g \mapsto Q(g, prI_p(V)) \circ st_p(V)$ of which the first one is the bijection. It remains to show that these maps are equal. For this it is sufficient to show that

$$Q(g, prI_p(V)) \circ \iota_{prI_p(V)} = \iota_F \circ (g \times Id_{\widetilde{U}})$$

which follows easily from computing compositions with the projections pr_1 to $I_p(V)$ and pr_2 to $\widetilde{U}$.

We now have to check the behavior of $\eta^!$ with respect to morphisms in X and V.

Let $pr = prI_p(V)$ and $pr' = prI_p(V')$. For $f : V' \to V$ and $f : X \to I_p(V)$ we have

$$D_p(X, f)(\eta^!(g)) = D_p(X, f)(g \circ pr, Q(g, pr) \circ st_p(V)) = (g \circ pr, Q(g, pr) \circ st_p(V) \circ f)$$

and

$$\eta^!(g \circ I_p(f)) = (g \circ I_p(f) \circ pr', Q(g \circ I_p(f), pr') \circ st_p(V'))$$

We have $pr = I_p(f) \circ pr'$ because $I_p(f)$ is a morphism over U. It remains to check that

$$Q(g, pr) \circ st_p(V) \circ f = Q(g \circ I_p(f), pr') \circ st_p(V')$$

By [**?**, Lemma 2.5] we have

$$Q(g \circ I_p(f), pr') = Q(g, pr) \circ Q(I_p(f), pr')$$

and the remaining equality

$$Q(g, pr) \circ st_p(V) \circ f = Q(g, pr) \circ Q(I_p(f), pr') \circ st_p(V')$$

follows from Lemma 4.8.

Consider now $f : X' \to X$. Then

$$D_p(f, V)(\eta^!(g)) = D_p(f, V)(g \circ pr, Q(g, pr) \circ st_p(V)) = (f \circ g \circ pr, Q(f, g \circ pr) \circ Q(g, pr) \circ st_p(V))$$

$$\eta^!(f \circ g) = (f \circ g \circ pr, Q(f \circ g, pr) \circ st_p(V))$$

and the required equality follows from [**?**, Lemma 2.5].

Problem 4.11 *For a locally cartesian closed closed $\mathcal{C}$ and a universe $p : \widetilde{U} \to U$ in $\mathcal{C}$ to construct for any $\Gamma \in Ob(CC(\mathcal{C}, p))$ bijections*

$$\mu_{2,\Gamma} : Ob_2(\Gamma) \to Hom_{\mathcal{C}}(int(\Gamma), I_p(U))$$

and

$$\widetilde{\mu}_{2,\Gamma} : \widetilde{Ob}_2(\Gamma) \to Hom_{\mathcal{C}}(int(\Gamma), I_p(\widetilde{U}))$$

that are natural in Γ and such that with respect to these bijections ∂ corresponds to composition with $I_p(p)$.

Construction 4.12 Compose bijections u_2 and $\widetilde{u}_2$ with the bijection η of Construction 4.10 in the case $V = U$ and $V = \widetilde{U}$ respectively.

Remark 4.13 The previous constructions related to Ob_2 and $\widetilde{Ob}_2$ can be easily generalized to Ob_n and $\widetilde{Ob}_n$ for all $n > 0$. For example there are natural bijections

$$\mu_{n+1} : Ob_{n+1}(\Gamma) \to Hom(int(\Gamma), I_p^n(U))$$

$$\widetilde{\mu}_{n+1} : \widetilde{Ob}_{n+1}(\Gamma) \to Hom(int(\Gamma), I_p^n(\widetilde{U}))$$

where I_p^n is the n-th iteration of the functor I_p and $\mu_1 = u_1$ and $\widetilde{\mu}_1 = \widetilde{u}_1$. More generally, the functors $Yo_n(V)$ of Remark 4.6 in the case of a locally cartesian closed universe category $(\mathcal{C}, p)$ are representable by objects $I_p^n(V)$.

5 (Π, λ)-structures on the C-systems $CC(\mathcal{C}, p)$

We will show now how to construct (Π, λ)-structures on C-systems of the form $CC(\mathcal{C}, p)$ for locally cartesian closed (pre-)categories[4] $\mathcal{C}$.

Definition 5.1 *Let $\mathcal{C}$ be a locally cartesian closed category, pt be a final object in $\mathcal{C}$ and $p : \widetilde{U} \to U$ a universe. A Π-structure on p is a pair of morphisms*

$$\widetilde{P} : I_p(\widetilde{U}) \to \widetilde{U}$$

$$P : I_p(U) \to U$$

such that the square

$$\begin{array}{ccc} I_p(\widetilde{U}) & \xrightarrow{\widetilde{P}} & \widetilde{U} \\ \downarrow{\scriptstyle I_p(p)} & & \downarrow{\scriptstyle p} \\ I_p(U) & \xrightarrow{P} & U \end{array} \tag{29}$$

is a pullback square.

Problem 5.2 *Let $\mathcal{C}$ be a locally cartesian closed category, pt be a final object in $\mathcal{C}$ and $p : \widetilde{U} \to U$ a universe. Let $(\widetilde{P}, P)$ be a Π-structure on p. To construct a (Π, λ)-structure on $CC(\mathcal{C}, p)$.*

Construction 5.3 Let $\Gamma \in Ob(CC(\mathcal{C}, p))$. For $T \in Ob_2(\Gamma)$ set

$$\Pi_\Gamma(T) = u_1^{-1}(u(T) \circ P)$$

and for $s \in \widetilde{Ob}_2(\Gamma)$ set

$$\lambda_\Gamma(s) = \widetilde{u}_1^{-1}(\widetilde{u}_2(s) \circ \widetilde{P})$$

These gives us maps

$$\Pi_\Gamma : Ob_2(\Gamma) \to Ob_1(\Gamma)$$

$$\lambda_\Gamma : \widetilde{Ob}_2(\Gamma) \to \widetilde{Ob}_1(\Gamma)$$

The naturality of u and $\widetilde{u}_2$ relative to morphisms $f : \Gamma' \to \Gamma$ implies that these maps are natural with respect to such morphisms i.e. the squares (13) and (14) of Definition 3.2 commute. One also verifies easily that $\partial(\lambda_\Gamma(s)) = \Pi_\Gamma(\partial(s))$.

To verify that this pre-(Π, λ)-structure satisfies the Definition 3.3 of (Π, λ)-structure one verifies that the bijections $\widetilde{u}_2$, u_2, $\widetilde{u}_1$ and u_1 define an isomorphism from the square (15) to the square obtained from (29) by taking Hom-sets $Hom(int(\Gamma), -)$. Since the later square is pullback and a square isomorphic to a pullback square is a pullback square the square (15) is a pullback square and (Π, λ) is a (Π, λ)-structure.

[4]For the discussion of the difference between a category and a pre-category see the introduction to [**?**] and [**?**].

6 More on universe category functors I

Let $(\mathcal{C}, p, pt)$ and $(\mathcal{C}, p', pt')$ be two universe (pre-)categories. Recall from [**?**] that a functor of universe categories from $(\mathcal{C}, p, pt)$ to $(\mathcal{C}, p', pt')$ is a triple $\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi})$ where Φ is a functor $\mathcal{C} \to \mathcal{C}'$ and $\phi : \Phi(U) \to U'$, $\widetilde{\phi} : \Phi(\widetilde{U}) \to \widetilde{U}'$ are two morphisms such that Φ takes the final object to a final object, pullback squares based on p to pullback squares and such that the square

$$\begin{array}{ccc} \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(U) & \xrightarrow{\phi} & U' \end{array} \tag{30}$$

is a pullback square.

For X, V in $\mathcal{C}$ we have the functoriality map

$$\Phi : Hom(X, V) \to Hom(\Phi(X), \Phi(V))$$

Problem 6.1 *For a universe category functor* $\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi})$*, to define, for all* $X, V \in \mathcal{C}$*, functions*

$$\mathbf{\Phi}^2 : D_p(X, V) \to D_{p'}(\Phi(X), \Phi(V))$$

Construction 6.2 Let $(F_1 : X \to U, F_2 : (X; F_1) \to V)$ be an element in $D_p(X, V)$. Consider $(\Phi(X); \Phi(F_1) \circ \phi)$. Since the square (30) is a pullback square there is a unique morphism q such that $q \circ \widetilde{\phi} = Q(\Phi(F_1) \circ \phi)$ and $q \circ \Phi(p) = p_{\Phi(X), \Phi(F_1) \circ \phi} \circ \Phi(F_1)$ and then the left hand side square in the diagram

$$\begin{array}{ccccc} (\Phi(X); \Phi(F_1) \circ \phi) & \xrightarrow{q} & \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ \downarrow{\scriptstyle p_{\Phi(X), \Phi(F_1) \circ \phi}} & & {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(X) & \xrightarrow{\Phi(F_1)} & \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

is a pullback square. Together with the fact that Φ takes pullback squares based on p to pullback squares we obtain a unique morphism, which is an isomorphism,

$$\iota : (\Phi(X); \Phi(F_1) \circ \phi) \to \Phi(X; F_1)$$

such that

$$\iota \circ \Phi(p_{X, F_1}) = p_{\Phi(X), \Phi(F_1) \circ \phi} \tag{31}$$

$$\iota \circ \Phi(Q(F_1)) \circ \widetilde{\phi} = Q(\Phi(F_1) \circ \phi) \tag{32}$$

and we define:

$$\mathbf{\Phi}^2(F_1, F_2) := (\Phi(F_1) \circ \phi, \iota \circ \Phi(F_2))$$

We will need the following properties of the maps below.

Lemma 6.3 *Let Φ be as above, $f : X' \to X$ be a morphism and V be an object of $\mathcal{C}$. Then the square*

$$\begin{array}{ccc} D_p(X,V) & \xrightarrow{D_p(f,V)} & D_p(X',V) \\ \mathbf{\Phi}^2 \downarrow & & \mathbf{\Phi}^2 \downarrow \\ D_{p'}(\Phi(X),\Phi(V)) & \xrightarrow{D_{p'}(\Phi(f),\Phi(V))} & D_{p'}(\Phi(X'),\Phi(V)) \end{array}$$

commutes.

Proof: We have to show that for any $d \in D_p(X,V)$ one has

$$D_{p'}(\Phi(f),\Phi(V))(\mathbf{\Phi}^2(d)) = \mathbf{\Phi}^2(D_p(f,V)(d))$$

Let $d = (F_1, F_2)$. Then

$$D_{p'}(\Phi(f),\Phi(V))(\mathbf{\Phi}^2(d)) = D_{p'}(\Phi(f),\Phi(V))(\Phi(F_1)\circ\phi, \iota\circ\Phi(F_2)) =$$
$$(\Phi(f)\circ\Phi(F_1)\circ\phi, q'\circ\iota\circ\Phi(F_2))$$

and

$$\mathbf{\Phi}^2(D_p(f,V)(F_1,F_2)) = \mathbf{\Phi}^2(f\circ F_1, q\circ F_2) =$$
$$(\Phi(f\circ F_1)\circ\phi, \iota'\circ\Phi(q\circ F_2))$$

where

$$\iota : (\Phi(X);\Phi(F_1)\circ\phi) \to \Phi(X;F_1) \qquad \iota' : (\Phi(X');\Phi(f\circ F_1)\circ\phi) \to \Phi(X';f\circ F_1)$$
$$q : (X'; f\circ F_1) \to (X;F_1) \qquad q' : (\Phi(X');\Phi(f)\circ\Phi(F_1)\circ\phi) \to (\Phi(X);\Phi(F_1)\circ\phi)$$

are the morphisms defined in Construction 6.2. We have

$$\Phi(f)\circ\Phi(F_1)\circ\phi = \Phi(f\circ F_1)\circ\phi$$

and it remains to check that

$$q'\circ\iota\circ\Phi(F_2) = \iota'\circ\Phi(q\circ F_2)$$

or that $q'\circ\iota = \iota'\circ\Phi(q)$. The codomain of both morphisms is $\Phi(X;F_1)$ that by our assumption on Φ is a pullback of p' and $\Phi(F_1)\circ\phi$. Therefore it is sufficient to verify that the compositions of these two morphisms with the projections to $\widetilde{U}'$ and $\Phi(X)$ coincide.

This is done by a direct computation from definitions.

Lemma 6.4 *Let $\mathbf{\Phi}$ be as above, X an object of $\mathcal{C}$ and $f : V \to V'$ a morphism. Then the square*

$$\begin{array}{ccc} D_p(X,V) & \xrightarrow{D_p(X,f)} & D_p(X,V') \\ \mathbf{\Phi}^2 \downarrow & & \downarrow \mathbf{\Phi}^2 \\ D_{p'}(\Phi(X),\Phi(V)) & \xrightarrow{D_p(\Phi(X),\Phi(f))} & D_{p'}(\Phi(X),\Phi(V')) \end{array}$$

commutes.

Proof: Let $d = (F_1, F_2) \in D_p(X, V)$. We have to show that

$$\mathbf{\Phi}^2(D_p(X, f)(F_1, F_2)) = D_p(\Phi(X), \Phi(f))(\mathbf{\Phi}^2(F_1, F_2))$$

We have:

$$\mathbf{\Phi}^2(D_p(X, f)(F_1, F_2)) = \mathbf{\Phi}^2((F_1, F_2 \circ f)) = (\Phi(F_1) \circ \phi, \iota \circ \Phi(F_2 \circ f)) =$$

$$(\Phi(F_1) \circ \phi, \iota \circ \Phi(F_2) \circ \Phi(f)) = D_p(\Phi(X), \Phi(f))(\mathbf{\Phi}^2(F_1, F_2))$$

Note that in the problem below no assumption is made about the compatibility of Φ with the locally cartesian closed structures on $\mathcal{C}$ and $\mathcal{C}'$.

Problem 6.5 *Assume that $\mathcal{C}$ and $\mathcal{C}'$ are locally cartesian closed universe categories. For $\mathbf{\Phi}$ as above and $V \in \mathcal{C}$ to construct a morphism*

$$\chi_{\mathbf{\Phi}}(V) : \Phi(I_p(V)) \to I_{p'}(\Phi(V))$$

Construction 6.6 Let

$$\eta : D_p(X, V) \to Hom(X, I_p(V))$$

$$\eta' : D_{p'}(X', V') \to Hom(X', I_{p'}(V'))$$

be bijections from Construction 4.10. We define:

$$\chi_{\mathbf{\Phi}}(V) := \eta'(\mathbf{\Phi}^2(\eta^!(Id_{I_p(V)})))$$

for $X = I_p(V)$ and $X' = \Phi(I_p(V))$.

Let us show that $\chi_{\mathbf{\Phi}}$ are natural in V.

Lemma 6.7 *For $\mathbf{\Phi}$ as above let $f : V_1 \to V_2$ be a morphism. Then the square*

$$\begin{array}{ccc} \Phi(I_p(V_1)) & \xrightarrow{\chi(V_1)} & I_{p'}(\Phi(V_1)) \\ {\scriptstyle \Phi(I_p(f))}\downarrow & & \downarrow{\scriptstyle I_{p'}(\Phi(f))} \\ \Phi(I_p(V_2)) & \xrightarrow{\chi(V_2)} & I_{p'}(\Phi(V_2)) \end{array}$$

commutes.

Proof: We have:

$$\chi(V_1) \circ I_{p'}(\Phi(V_1)) = \eta'(\mathbf{\Phi}^2(\eta^!(Id_{X_1}))) \circ I_{p'}(\Phi(f)) = \eta'(D_p(X_1, \Phi(f))(\mathbf{\Phi}^2(\eta^!(Id_{X_1}))))$$

where $X = I_p(V_1)$, by naturality of η'. Then

$$\eta'(D_p(X_1, \Phi(f))(\mathbf{\Phi}^2(\eta^!(Id_{X_1})))) = \eta'(\mathbf{\Phi}^2(D_p(X_1, f)(\eta^!(Id_{X_1})))) =$$

$$\eta'(\mathbf{\Phi}^2(\eta^!(Id_{X_1} \circ I_p(f))) = \eta'(\mathbf{\Phi}^2(\eta^!(I_p(f))))$$

where the first equality holds by Lemma 6.4 and the second by Problem 4.9(1).

On the other hand:

$$\Phi(I_p(f)) \circ \chi(V_2) = \Phi(I_p(f)) \circ \eta'(\mathbf{\Phi}^2(\eta^!(Id_{X_2}))) =$$
$$\eta'(D_{p'}(\Phi(I_p(f)), \Phi(X_2))(\mathbf{\Phi}^2(\eta^!(Id_{X_2}))))$$

by naturality of η'. Then

$$\eta'(D_{p'}(\Phi(I_p(f)), \Phi(X_2))(\mathbf{\Phi}^2(\eta^!(Id_{X_2})))) = \eta'(\mathbf{\Phi}^2(D_p(I_p(f), X_2)(\eta^!(Id_{X_2})))) =$$
$$\eta'(\mathbf{\Phi}^2(\eta^!(I_p(f) \circ Id_{X_2}))) = \eta'(\mathbf{\Phi}^2(\eta^!(I_p(f))))$$

where the first equality holds by Lemma 6.4 and the second by Problem 4.9(2). This finishes the proof of Lemma 6.7.

Lemma 6.8 *For all $X, V \in \mathcal{C}$ and $a \in D_p(X, V)$ one has*

$$\Phi(\eta(a)) \circ \chi_{\mathbf{\Phi}}(V) = \eta'(\mathbf{\Phi}^2(a))$$

Proof: By definition of $\chi_{\mathbf{\Phi}}$ and contravariant functoriality of η' we have

$$\Phi(\eta(a)) \circ \chi_{\mathbf{\Phi}}(V) = \Phi(\eta(a)) \circ \eta'(\mathbf{\Phi}^2(\eta^!(Id))) = \eta'(D_{p'}(\Phi(\eta(a)), \Phi(V))(\mathbf{\Phi}^2(\eta^!(Id_{I_p(V)}))))$$

By Lemma 6.3 we further have:

$$\eta'(D_{p'}(\Phi(\eta(a)), \Phi(V))(\mathbf{\Phi}^2(\eta^!(Id)))) = \eta'(\mathbf{\Phi}^2(D_p(\eta(a), V)(\eta^!(Id))))$$

It remains to show that $D_p(\eta(a), V)(\eta^!(Id)) = f$. Since η is a bijection we may apply it on both sides and by functoriality of η we get

$$\eta(D_p(\eta(a), V)(\eta^!(Id))) = \eta(f) \circ \eta(\eta^!(Id)) = \eta(f) \circ Id = \eta(f)$$

7 More on universe category functors II

By [**?**, Construction 4.7] any universe category functor $\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi})$ defines a homomorphism of C-systems

$$H : CC(\mathcal{C}, p) \to CC(\mathcal{C}', p')$$

Let $\psi : pt' \to \Phi(pt)$ be the unique morphism. To define H on objects, one uses the fact that

$$Ob(CC(\mathcal{C}, p)) = \amalg_{n \geq 0} Ob_n(\mathcal{C}, p)$$

and defines $H(n, A)$ as $(n, H_n(A))$ where

$$H_n : Ob_n(\mathcal{C}, p) \to Ob_n(\mathcal{C}', p')$$

To obtain H_n one defines by induction on n, pairs (H_n, ψ_n) where H_n is as above and ψ_n is a family of isomorphisms

$$\psi_n(A) : int'(H_n(A)) \to \Phi(int(A))$$

as follows:

1. for $n = 0$, H_0 is the unique map from one point set to one point set and $\psi_0(A) = \psi$,
2. for the successor of n one has

$$H_{n+1}(A, F) = (H_n(A), \psi_n(A) \circ \Phi(F) \circ \phi)$$

and $\psi_{n+1}A, F$ is the unique morphism $int'(H(A, F)) \to \Phi(int(A, F))$ such that

$$\psi(A, F) \circ \Phi(Q(F)) \circ \widetilde{\phi} = Q'(\psi(A) \circ \Phi(F) \circ \phi)$$

and

$$\psi(A, F) \circ \Phi(p_(A, F)) = p_{H(A,F)} \circ \psi(A)$$

The action of H on morphisms is given, for $f : (m, A) \to (n, B)$, by

$$H(f) = \psi(A) \circ \Phi(int(f)) \circ \psi(B)^{-1}$$

We will often write H also for the functions H_n and ψ for the functions ψ_n.

Let $\Gamma \in Ob(CC(\mathcal{C}, p))$ and consider the bijections of Constructions 4.3 and 4.5.

In order to prove our main functoriality Theorem 8.1 we need describe in more detail the maps

$$Ob_1(\Gamma) \to Ob_1(H(\Gamma))$$
$$Ob_2(\Gamma) \to Ob_2(H(\Gamma))$$

and the similar maps on $\widetilde{Ob}_1$ and $\widetilde{Ob}_2$ that are defined by H.

Lemma 7.1 *Let $(\Phi, \phi, \widetilde{\phi})$ be universe category functor. Then:*

1. *for $T \in Ob_1(\Gamma)$ one has*

$$u_{1,H(\Gamma)}(H(T)) = \psi(\Gamma) \circ \Phi(u_{1,\Gamma}(T)) \circ \phi$$

2. *for $o \in \widetilde{Ob}_1(\Gamma)$ one has*

$$\widetilde{u}_{1,H(\Gamma)}(H(o)) = \psi(\Gamma) \circ \Phi(\widetilde{u}_{1,\Gamma}(o)) \circ \widetilde{\phi}$$

3. *for $T \in Ob_2(\Gamma)$ one has*

$$u_{2,H(\Gamma)}(H(T)) = D_{p'}(\psi(\Gamma), U')(D_{p'}(int'(H(\Gamma)), \phi)(\mathbf{\Phi}^2(u_{2,\Gamma}(T))))$$

4. *for $o \in \widetilde{Ob}_2(\Gamma)$ one has*

$$\widetilde{u}_{2,H(\Gamma)}(H(o)) = D_{p'}(\psi(\Gamma), \widetilde{U}')(D_{p'}(int'(H(\Gamma)), \widetilde{\phi})(\mathbf{\Phi}^2(\widetilde{u}_{2,\Gamma}(o))))$$

Proof: Let $\Gamma = (n, A)$.

In the case of $T \in Ob_1(\Gamma)$, if $T = (n+1, (A, F))$ then

$$u_1(H(T)) = u_1(n+1, H(A,F)) = u_1(n+1, (H(A), \psi(\Gamma)\circ\Phi(F)\circ\phi)) = \psi(\Gamma)\circ\Phi(F)\circ\phi$$

In the case of $s \in \widetilde{Ob}_1(\Gamma)$, if $F = u_1(\partial(s))$ then

$$\widetilde{u}_1(H(s)) = H(s)\circ Q'(u_1(n+1, H(A,F))) = \psi(A)\circ\Phi(s)\circ\psi(A,F)^{-1}\circ Q'(\psi(A)\circ\Phi(F)\circ\phi) =$$

$$\psi(A)\circ\Phi(s)\circ\Phi(Q(F))\circ\widetilde{\phi} = \psi(A)\circ\Phi(s\circ Q(F))\circ\widetilde{\phi} = \psi(A)\circ\Phi(\widetilde{u}_1(s))\circ\widetilde{\phi}$$

In the case $T \in Ob_2(\Gamma)$, if $T = (n+2, ((A, F_1), F_2))$ then

$$u_2(H(T)) = u_2(n+2, H(((A,F_1),F_2))) = u_2(n+2, (H(A,F_1), \psi(A,F_1)\circ\Phi(F_2)\circ\phi)) =$$

$$u_2(n+2, (H(A), \psi(A)\circ\Phi(F_1)\circ\phi, \psi(A,F_1)\circ\Phi(F_2)\circ\phi)) =$$

$$(\psi(A)\circ\Phi(F_1)\circ\phi, \psi(A,F_1)\circ\Phi(F_2)\circ\phi)$$

On the other hand

$$D_{p'}(\psi(A),-)D_{p'}(-,\phi)(\mathbf{\Phi}^2(u_2(T))) = D_{p'}(\psi(A),-)D_{p'}(-,\phi)(\mathbf{\Phi}^2(u_2(n+2,((A,F_1),F_2)))) =$$

$$D_{p'}(\psi(A),-)D_{p'}(-,\phi)(\mathbf{\Phi}^2(F_1,F_2)) = D_{p'}(\psi(A),-)D_{p'}(-,\phi)(\Phi(F_1)\circ\phi, \iota\circ\Phi(F_2)) =$$

$$D_{p'}(\psi(A),-)(\Phi(F_1)\circ\phi, \iota\circ\Phi(F_2)\circ\phi) = (\psi(A)\circ\Phi(F_1)\circ\phi, Q'(\psi(A),\Phi(F_1)\circ\phi)\circ\iota\circ\Phi(F_2)\circ\phi)$$

therefore we need to show that

$$\psi(A,F_1)\circ\Phi(F_2)\circ\phi = Q'(\psi(A),\Phi(F_1)\circ\phi)\circ\iota\circ\Phi(F_2)\circ\phi \quad (33)$$

Using the fact that the external square of the diagram

$$\begin{array}{ccccc} \Phi(int(A,F_1)) & \xrightarrow{\Phi(Q(F_1))} & \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ \downarrow{\scriptstyle \Phi(p_{(A,F_1)})} & & \downarrow{\scriptstyle \Phi(p)} & & \downarrow{\scriptstyle p'} \\ \Phi(int(A)) & \xrightarrow{\Phi(F_1)} & \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

is a pullback square we see that equality (33) would follow from the following two equalities:

$$\psi(A,F_1)\circ\Phi(Q(F_1))\circ\widetilde{\phi} = Q'(\psi(A),\Phi(F_1)\circ\phi)\circ\iota\circ\Phi(Q(F_1))\circ\widetilde{\phi}$$

and

$$\psi(A,F_1)\circ\Phi(p_{(A,F_1)}) = Q'(\psi(A),\Phi(F_1)\circ\phi)\circ\iota\circ\Phi(p_{(A,F_1)})$$

For the first equality we have

$$\psi(A,F_1)\circ\Phi(Q(F_1))\circ\widetilde{\phi} = Q'(\psi(A)\circ\Phi(F_1)\circ\phi)$$

by definition of $\psi(\Gamma, F_1)$ and

$$Q'(\psi(A),\Phi(F_1)\circ\phi)\circ\iota\circ\Phi(Q(F_1))\circ\widetilde{\phi} = Q'(\psi(A),\Phi(F_1)\circ\phi)\circ Q'(\Phi(F_1)\circ\phi) = Q'(\psi(A)\circ\Phi(F_1)\circ\phi)$$

where the first equality holds by definition of ι and second by the definition of $Q(-,-)$.
For the second equality we have

$$\psi(A,F_1)\circ\Phi(p_{(A,F_1)}) = p_{H(A,F_1)}\circ\psi(A)$$

by definition of $\psi(A,F_1)$ and

$$Q'(\psi(A),\Phi(F_1)\circ\phi)\circ\iota\circ\Phi(p_{(A,F_1)}) = Q'(\psi(A),\Phi(F_1)\circ\phi)\circ p_{\Phi(int(A)),\Phi(F_1)\circ\phi} = p_{H(A,F_1)}\circ\psi_\Gamma$$

by definitions of Q' and ι.
The case of $s\in\widetilde{Ob}_2(\Gamma)$ is strictly parallel to the case of $T\in Ob_2(\Gamma)$ with $\Phi(F_2)\circ\phi$ at the end of the formulas replaced by $\Phi(F_2')\circ\widetilde{\phi}$ where instead of $F_2: int(A,F_1)\rightarrow U$ one has $F_2': int(A,F_1)\rightarrow\widetilde{U}$.

For $(\Phi,\phi,\widetilde{\phi})$ as above let us denote by

$$\xi_{\mathbf{\Phi}}:\Phi(I_p(U))\rightarrow I_{p'}(U')$$

the composition $\chi_{\mathbf{\Phi}}(U)\circ I_{p'}(\phi)$ and by

$$\widetilde{\xi}_{\mathbf{\Phi}}:\Phi(I_p(\widetilde{U}))\rightarrow I_{p'}(\widetilde{U}')$$

the composition $\chi_{\mathbf{\Phi}}(\widetilde{U})\circ I_p(\widetilde{\phi})$.

Lemma 7.2 *Let $(\Phi,\phi,\widetilde{\phi})$ be a universe category functor and $\Gamma\in Ob(CC(\mathcal{C},p))$. Then one has:*

1. *for $T\in Ob_2(\Gamma)$*
$$\eta_{p'}(u_2'(H(T))) = \psi(\Gamma)\circ\Phi(\eta_p(u_2(T)))\circ\xi_{\mathbf{\Phi}}$$
2. *for $s\in\widetilde{Ob}_2(\Gamma)$*
$$\eta_{p'}(\widetilde{u}_2'(H(s))) = \psi(\Gamma)\circ\Phi(\eta_p(\widetilde{u}_2(s)))\circ\widetilde{\xi}_{\mathbf{\Phi}}$$

Proof: We have

$$\eta_{p'}(u_2'(H(T))) = \eta_{p'}(D_{p'}(\psi(\Gamma),\_)(D_{p'}(\_,\phi)(\mathbf{\Phi}^2(u_2(T))))) = \psi(\Gamma)\circ\eta_{p'}(\mathbf{\Phi}^2(u_2(T)))\circ I_{p'}(\phi)$$

where the first equality holds by Lemma 7.1(3) and the second by the naturality of $\eta_{p'}$. Next

$$\eta_{p'}(\mathbf{\Phi}^2(u_2(T)))\circ I_{p'}(\phi) = \Phi(\eta(u_2(T)))\circ\chi_\Phi(U)\circ I_{p'}(\phi) = \Phi(\eta(u_2(T)))\circ\xi_{\mathbf{\Phi}}$$

where the first equality holds by Lemma 6.8 and the second one by the definition of $\xi_{\mathbf{\Phi}}$.
The proof of the second part of the lemma is strictly parallel to the proof of the first part.

8 Functoriality properties of the (Π, λ)-structures arising from universes

Let us prove the functoriality properties of the (Π, λ) structures of Construction 5.3.

The notion of a homomorphism of C-systems with (Π, λ)-structures used in the theorem below is defined in the obvious way.

Theorem 8.1 *Let $(\Phi, \phi, \widetilde{\phi})$ be as above and let $(P, \widetilde{P})$, $(P', \widetilde{P}')$ be as in Problem 5.2 for $\mathcal{C}$ and $\mathcal{C}'$ respectively.*

Assume that the squares

$$\begin{array}{ccc} \Phi(I_p(U)) & \xrightarrow{\xi_\Phi} & I_{p'}(U') \\ {\scriptstyle \Phi(P)}\downarrow & & \downarrow{\scriptstyle P'} \\ \Phi(U) & \xrightarrow{\phi} & U \end{array} \tag{34}$$

and

$$\begin{array}{ccc} \Phi(I_p(\widetilde{U})) & \xrightarrow{\widetilde{\xi}_\Phi} & I_{p'}(\widetilde{U}') \\ {\scriptstyle \Phi(\widetilde{P})}\downarrow & & \downarrow{\scriptstyle \widetilde{P}'} \\ \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U} \end{array} \tag{35}$$

commute. Then the homomorphism

$$H(\Phi, \phi, \widetilde{\phi}) : CC(\mathcal{C}, p) \to CC(\mathcal{C}', p')$$

is a homomorphism of C-systems with (Π, λ)-structures.

Proof: We have to show that for all $\Gamma \in Ob(CC(\mathcal{C}, p))$ and $T \in Ob_2(\Gamma)$ we have

$$\Pi'(H(T)) = H(\Pi(T))$$

and for all $\Gamma \in Ob(CC(\mathcal{C}, p))$ and $s \in \widetilde{Ob}_2(\Gamma)$ we have

$$\lambda'(H(s)) = H(\lambda(s))$$

We will prove the first equality. The proof of the second is strictly parallel to the proof of the first.

By definition we have:

$$\Pi'(H(T)) = (u_1')^{-1}(u_2'(H(T)) \circ P') = (u_1')^{-1}(\eta'(u_2'(H(T))) \circ P')$$

and

$$H(\Pi(T)) = H(u_1^{-1}(\eta(u_2(T)) \circ P)) = (u_1')^{-1}(\psi(\Gamma) \circ \Phi(\eta(u_2(T)) \circ P) \circ \phi) =$$
$$(u_1')^{-1}(\psi(\Gamma) \circ \Phi(\eta(u_2(T))) \circ \Phi(P) \circ \phi)$$

where the second equality holds by Lemma 7.1(1). Let us show that

$$\eta'(u'_2(H(T))) \circ P' = \psi(\Gamma) \circ \Phi(\eta(u_2(T))) \circ \Phi(P) \circ \phi$$

By Lemma 7.2(1) we have

$$\eta'(u'_2(H(T))) \circ P' = \psi(\Gamma) \circ \Phi(\eta(u_2(T))) \circ \xi_\Phi \circ P'$$

It remains to show that

$$\xi_\Phi \circ P' = \Phi(P) \circ \phi$$

which is our assumption about the commutativity of the square (34).

9 Appendix: some constructions and theorems about categories

Lemma 9.1 *Let $\mathcal{C}$ be a category. Consider four fiber squares*

$$\begin{array}{ccccccc} pb_i & \xrightarrow{pr_{Y,i}} & Y & \quad & pb'_i & \xrightarrow{pr_{Y',i}} & Y' \\ {\scriptstyle pr_{X,i}}\downarrow & & \downarrow{\scriptstyle g} & & {\scriptstyle pr_{X,i}}\downarrow & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow{f} & Z & & X' & \xrightarrow{f'} & Z \end{array}$$

where $i = 1, 2$. Let $a : X' \to X$ and $b : Y' \to Y$ be such that $a \circ f = f'$ and $b \circ g = g'$. Let $\iota : pb_1 \to pb_2$ be the unique morphism such that $\iota \circ pr_{X_2} = pr_{X,1}$ and $\iota \circ pr_{Y,1} = pr_{Y,2}$ and similarly for $\iota' : pb'_1 \to pb'_2$. Let $pb_i(a, b) : pb'_i \to pb_i$ be the unique morphisms such that $pb_i(a, b) \circ pr_{X,i} = pr_{X',i} \circ a$ and $pb_i(a, b) \circ pr_{Y,i} = b \circ pr_{Y',i}$. Then the square

$$\begin{array}{ccc} pb'_1 & \xrightarrow{pb_1(a,b)} & pb_1 \\ {\scriptstyle \iota'}\downarrow & & \downarrow{\scriptstyle \iota} \\ pb'_2 & \xrightarrow{pb_2(a,b)} & pb_2 \end{array}$$

commutes, i.e., $pb_1(a, b) \circ \iota = \iota' \circ pb_2(a, b)$.

Proof: Since pb_2 is a fiber product it is sufficient to prove that

$$pb_1(a, b) \circ \iota \circ pr_{X,2} = \iota' \circ pb_2(a, b) \circ pr_{X,2}$$

and

$$pb_1(a, b) \circ \iota \circ pr_{Y,2} = \iota' \circ pb_2(a, b) \circ pr_{Y,2}$$

For the first one we have:

$$pb_1(a, b) \circ \iota \circ pr_{X,2} = pb_1(a, b) \circ pr_{X,1} = pr_{X',1} \circ a$$

and

$$\iota' \circ pb_2(a, b) \circ pr_{X,2} = \iota' \circ pr_{X',2} \circ a = pr_{X',1} \circ a$$

The verification of the second equality is similar.

Definition 9.2 *A category with fiber products is a category together with, for all pairs of morphisms of the form $f : X \to Z$, $g : Y \to Z$, fiber squares*

$$\begin{array}{ccc} (X,f)\times_Z (Y,g) & \xrightarrow{pr_2^{(X,f),(Y,g)}} & Y \\ {\scriptstyle pr_1^{(X,f),(Y,g)}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

We will often abbreviate these main notations in various ways. The morphism $pr_2 \circ g = pr_1 \circ f$ from $(X, f) \times (Y, g)$ to Z is denoted by $f \diamond g$.

Given a category with fiber products, morphisms $g_i : Y_i \to Z$, $i = 1, 2$ and morphisms $a : X_1 \to Y_1$, $b : X_2 \to Y_2$ denote by

$$(a \times b)^{g_1,g_2} : ((X_1, a \circ g_1) \times_Z (X_2, b \circ g_2), (a \circ g_1) \diamond (b \circ g_2)) \to ((Y_1, g_1) \times_Z (Y_2, g_2), g_1 \diamond g_2)$$

the unique morphism over Z such that

$$(a \times b)^{g_1,g_2} \circ pr_1 = pr_1 \circ a$$

and

$$(a \times b)^{g_1,g_2} \circ pr_2 = pr_2 \circ b$$

To show that $(a \times b)^{g_1,g_2}$ exists we need to check that

$$pr_1 \circ a \circ g_1 = pr_2 \circ b \circ g_2$$

which is immediate from the definition of the fiber product.

Lemma 9.3 *In the setting introduced above suppose that we have in addition $a' : X_1' \to X_1$ and $b' : X_2' \to X_2$. Then one has*

$$((a' \circ a) \times (b' \circ b))^{g_1,g_2} = (a' \times b')^{a \circ g_1, b \circ g_2} \circ (a \times b)^{g_1,g_2}$$

Proof: Straightforward rewriting to compute the compositions of both sides with $pr_1^{g_1,g_2}$ and $pr_2^{g_1,g_2}$.

Definition 9.4 *A locally cartesian closed structure on a (pre-)category $\mathcal{C}$ is a collection of data of the form:*

1. *A structure of a category with fiber products on $\mathcal{C}$.*
2. *For all f, g of the form $f : X \to Z$, $g : Y \to Z$, an object $\underline{Hom}_Z((X, f), (Y, g))$ and a morphism*

   $$f \triangle g : \underline{Hom}_Z((X, f), (Y, g)) \to Z$$

   together with morphisms of the form

   $$\underline{Hom}((X, f), a) : \underline{Hom}((X, f), (Y, g)) \to \underline{Hom}((X, f), (Y', g'))$$

   for all $a : (Y, g) \to (Y', g')$ over Z, that make $\underline{Hom}((X, f), -)$ into a functor from $\mathcal{C}/Z$ to $\mathcal{C}$.

3. For all f, g as above a morphism

$$ev^{(X,f)}_{(Y,g)} : (\underline{Hom}_Z((X,f),(Y,g)), f\triangle g) \times (X,f) \to (Y,g)$$

over Z such that for all $h : W \to Z$ the map

$$adj^{(W,h),(X,f)}_{(Y,g)} : Hom_Z((W,h),(\underline{Hom}_Z((X,f),(Y,g)), f\triangle g)) \to$$
$$Hom_Z(((W,h)\times(X,f), h\diamond f),(Y,g))$$

given by

$$u \mapsto (u \times Id_X)^{f\triangle g, f} \circ ev^{(X,f)}_{(Y,g)}$$

is a bijection and such that the morphisms $ev^{(X,f)}_{(Y,g)}$ are natural in Y.

A locally cartesian closed (pre-)category is a (pre-)category together with a locally cartesian closed structure on it.

If a locally cartesian closed category is given with a final object pt we will write $X \times Y$ for $(X, \pi_X) \times_{pt} (Y, \pi_Y)$ where π_X and π_Y are the unique morphisms from X and Y respectively to pt.

By definition the objects $(\underline{Hom}((X,f),(Y,g)), f\triangle g)$ of $\mathcal{C}/Z$ are functorial only in (Y,g). Their functoriality in (X,f) is a consequence of a lemma. For $f : X \to Z$, $f' : X' \to Z$, $g : Y \to Z$ and $h : X' \to X$ such that $h \circ f = f'$ let

$$\underline{Hom}_Z(h,(Y,g)) : \underline{Hom}_Z((X,f),(Y,g)) \to \underline{Hom}_Z((X',f'),(Y,g))$$

be the unique map whose adjoint

$$adj(\underline{Hom}_Z(h,(Y,g))) : (\underline{Hom}_Z((X,f),(Y,g)), f\triangle g) \times_Z (X',f') \to (Y,g)$$

equals $(Id_{\underline{Hom}_Z((X,f),(Y,g))} \times h)^{f\triangle g, f} \circ ev^X_Y$. Then one has:

Lemma 9.5 *The morphisms $\underline{Hom}_Z(h,(Y,g))$ satisfy the equations*

$$\underline{Hom}_Z(h,(Y,g)) \circ (f'\triangle g) = f\triangle g$$

and the equations

$$\underline{Hom}_Z(h_1 \circ h_2,(Y,g)) = \underline{Hom}(h_2,(Y,g)) \circ \underline{Hom}(h_1,(Y,g))$$
$$\underline{Hom}_Z(Id,(Y,g)) = Id$$

making $\underline{Hom}_Z(-,(Y,g))$ into a contravariant functor from $\mathcal{C}/Z$ to itself. In addition, for each $h' : (Y,g) \to (Y,g')$ the square

$$\begin{array}{ccc}
\underline{Hom}_Z((X',f'),(Y,g)) & \xrightarrow{\underline{Hom}_Z((X',f'),h')} & \underline{Hom}_Z((X',f'),(Y',g')) \\
{\scriptstyle \underline{Hom}_Z(h,(Y,g))}\downarrow & & \downarrow{\scriptstyle \underline{Hom}_Z(h,(Y',g'))} \\
\underline{Hom}_Z((X,f),(Y,g)) & \xrightarrow{\underline{Hom}_Z((X,f),h')} & \underline{Hom}_Z((X,f),(Y',g'))
\end{array}$$

commutes.

Proof: It is a particular case of [**?**, Theorem 3, p.100]. The commutativity of the square is a part of the "bifunctor" claim of the theorem.

Lemma 9.6 *In a locally cartesian closed category let $f : X \to Z$, $f' : X' \to Z$, $g : Y \to Z$ be objects over Z and let $a : X' \to X$ be a morphism over Z. Then the square*

$$\begin{array}{ccc}
(\underline{Hom}((X,f),(Y,g)), f\triangle g) \times_Z (X',f') & \xrightarrow{1} & (\underline{Hom}((X,f),(Y,g)), f\triangle g) \times_Z (X,f) \\
{\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle ev} \\
(\underline{Hom}_Z((X',f'),(Y,g)), f'\triangle g) \times_Z (X',f') & \xrightarrow{ev'} & Y
\end{array}$$

where 1 is $(Id_{\underline{Hom}((X,f),(Y,g))} \times a)^{f\triangle g, f}$ and 2 is $(\underline{Hom}(a,(Y,g)) \times Id_{X'})^{f'\triangle g, f'}$, commutes.

Proof: Let us show that both paths in the square are adjoints to $\underline{Hom}(a,(Y,g))$. For the path that goes through the upper right corner it follows from the definition of $\underline{Hom}(a,(Y,g))$ as the morphism whose adjoint is $(Id \times a) \circ ev$. For the path that goes through the lower left corner it follows from the definition of adjoint applied to $\underline{Hom}(a,(Y,g))$. Indeed, the adjoint to this morphism is

$$adj(\underline{Hom}(a,(Y,g))) = (\underline{Hom}(a,(Y,g)) \times Id_{X'}) \circ ev'$$

Lemma 9.7 *Let $\mathcal{C}$ be a locally cartesian closed category. Let Z, $(X,f),(Y,g),(W,h)$ be as above.*

1. *Let (Y',g') be an object over Z and $a : (Y,g) \to (Y',g')$ a morphism over Z. Then for any $b \in Hom_Z((W,h), \underline{Hom}_U((X,f),(Y,g)))$ one has*

$$adj(b \circ \underline{Hom}_Z((X,f),a)) = adj(b) \circ a$$

2. *Let (X',f') be an object over Z and $a : (X',f') \to (X,f)$ a morphism over Z. Then for any $b \in Hom_Z((W,h), \underline{Hom}_U((X,f),(Y,g)))$ one has*

$$adj(b \circ \underline{Hom}_Z(a,(Y,g))) = (Id_W \times a)^{h,f} \circ adj(b)$$

3. *Let (W',h') be an object over Z and $a : (W',h') \to (W,h)$ a morphism over Z. Then for any $b \in Hom_Z((W,h), \underline{Hom}_U((X,f),(Y,g)))$ one has*

$$adj(a \circ b) = (a \times Id_X)^{h,f} \circ adj(b)$$

Proof: The proof of the first case is given by

$$adj(b \circ \underline{Hom}_Z((X,f),a)) = ((b \circ \underline{Hom}_Z((X,f),a)) \times Id_X)^{f\triangle g', f} \circ ev^{(X,f)}_{(Y',g')} =$$

$$(b \times Id_X)^{f\triangle g, f} \circ (\underline{Hom}_Z((X,f),a)) \times Id_X)^{f\triangle g', f} \circ ev^{(X,f)}_{(Y',g')} =$$

$$(b \times Id_X)^{f\triangle g,f} \circ ev^{(X,f)}_{(Y,g)} \circ a = adj(b) \circ a$$

where the second equality holds by Lemma 9.3 and the third equality by the naturality axiom for morphisms $ev^{(X,f)}_{(Y,g)}$ in (Y,g).

The proof of the second case is given by the following sequence of equalities where we use the notation Hm for $\underline{Hom}_Z(a,(Y,g))$ as well as a number of other abbreviations:

$$adj(b \circ Hm) = ((b \circ Hm) \times Id) \circ ev = (b \times Id) \circ (Hm \times Id) \circ ev = (b \times Id) \circ adj(Hm) =$$

$$(b \times Id) \circ (Id \times a) \circ ev = (b \times a) \circ ev = (Id \times a) \circ (b \times Id) \circ ev = (Id \times a) \circ adj(b)$$

The proof of the third case is given by

$$adj(a \circ b) = ((a \circ b) \times Id_X) \circ ev^{(X,f)}_{(Y,g)} = (a \times Id_X) \circ (b \times Id_X) \circ ev^{(X,f)}_{(Y,g)} =$$

$$(a \times Id_X) \circ adj(b)$$

where the second equality holds by Lemma 9.3.

Lemma is proved.

Example 9.8 The following example shows that there can be many different structures of a category with fiber products on a (pre-)category and also many locally cartesian closed structures.

Let us take as our (pre-)category the (pre-)category *preStn* whose objects are natural numbers and $Hom(n,m) = Hom(\{0,\dots,n-1\},\{0,\dots,m-1\})$.

Since every isomorphism class contains exactly one object every auto-equivalence of this category is an automorphism. Let Φ be such an automorphism. It is easy to see that it must be identity on the set of objects. Let $X = \{0,1\}$. Consider Φ on $End(X)$. Since Φ must respect identities and compositions, Φ must take $Aut(X)$ to itself and must act on it by identity. If 1 and σ are the two elements of $Aut(X)$ we conclude that $\Phi(1) = 1$ and $\Phi(\sigma) = \sigma$.

Let us choose now any structure str_0 of a category with fiber products on *preStn* and let us consider two structures str_1 and str_σ that are obtained by choosing all the fiber squares as in str_0 except for the square for the pair (Id_X, Id_X) which we choose to be, correspondingly, as follows:

$$\begin{array}{ccc} X & \xrightarrow{Id_X} & X \\ {\scriptstyle Id_X}\downarrow & & \downarrow{\scriptstyle Id_X} \\ X & \xrightarrow{Id_X} & X \end{array} \quad \text{for } str_1 \text{ and } \quad \begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle Id_X} \\ X & \xrightarrow{Id_X} & X \end{array} \quad \text{for } str_\sigma. \qquad (36)$$

The preceding discussion of the auto-equivalences of *preStn* shows that there is no auto-equivalence which would transform str_1 into str_σ.

The (pre-)category *preStn* also has a locally cartesian closed structure that can be modified so that its underlying fiber product structures are str_1 and str_σ. This shows that *preStn* has at least two locally cartesian closed structures that are not interchanged by auto-equivalences of *preStn*.

Remark 9.9 The previous example has a continuation in the univalent foundations where there is a notion of a category and pre-category. There one expects it to be true that the type of fiber square structures and the type of locally cartesian closed structures on a category (as opposed to those on a general pre-category) are of h-level 1, i.e., classically speaking are either empty or contain only one element.

In addition any such structure on a pre-category should define a structure of the same kind on the Rezk completion of this pre-category with all the different structures on the pre-category becoming equal on the Rezk completion. In the case of the previous example the Rezk completion of $preStn$ is the category $FSets$ of finite sets and in view of the univalence axiom for finite sets the two pullback squares of 36 will become equal in $FSets$.

References