REGULAR SUB-QUOTIENTS OF THE C-SYSTEMS $C(\mathbf{RR}, \mathbf{LM})$

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Abstract.

Contents

1. Introduction

1.1. **Regular sub-quotients of $CC(R, LM)$.** Let (R, LM) be as above and

$$Ceq \subset \coprod_{n\ge 0}(\prod_{i=0}^{n-1} LM(\widehat{i})) \times LM(stn(n))^2$$

$$\widetilde{Ceq} \subset \coprod_{n\ge 0}(\prod_{i=0}^{n} LM(\widehat{i})) \times R(stn(n))^2$$

be two subsets.

For $\Gamma = (T_1, \ldots, T_n) \in ob(CC(R, LM))$ and $S_1, S_2 \in LM(stn(n))$ we write $(\Gamma \vdash_{Ceq} S_1 = S_2)$ to signify that $(T_1, \ldots, T_n, S_1, S_2) \in Ceq$. Similarly for $T \in LM(stn(n))$ and $o, o' \in R(stn(n))$ we write $(\Gamma \vdash_{\widetilde{Ceq}} o = o' : S)$ to signify that $(T_1, \ldots, T_n, S, o, o') \in \widetilde{Ceq}$. When no confusion is possible we will omit the subscripts Ceq and $\widetilde{Ceq}$ at $\vdash$.

Similarly we will write $\rhd$ instead of $\rhd_C$ and $\vdash$ instead of $\vdash_{\widetilde{C}}$ if the subsets C and $\widetilde{C}$ are unambiguously determined by the context.

Definition 1.1. **[simandsimeq]** *Given subsets C, $\widetilde{C}$, Ceq, $\widetilde{Ceq}$ as above define relations $\sim$ on C and $\simeq$ on $\widetilde{C}$ as follows:*

(1) for $\Gamma = (T_1, \ldots, T_n)$, $\Gamma' = (T'_1, \ldots, T'_n)$ in C we set $\Gamma \sim \Gamma'$ iff $ft(\Gamma) \sim ft(\Gamma')$ and

$$T_1, \ldots, T_{n-1} \vdash T_n = T'_n,$$

(2) for $(\Gamma \vdash o : S)$, $(\Gamma' \vdash o' : S')$ in $\widetilde{C}$ we set $(\Gamma \vdash o : S) \simeq (\Gamma' \vdash o' : S')$ iff $(\Gamma, S) \sim (\Gamma', S')$ and

$$(\Gamma \vdash o = o' : S).$$

Proposition 1.2. **[2014.07.10.prop1]** *Let C, $\widetilde{C}$, Ceq, $\widetilde{Ceq}$ be as above and suppose in addition that one has:*

1. *C and $\widetilde{C}$ satisfy conditions (1)-(6) of Proposition **??** which are referred to below as conditions (1.1)-(1.6) of the present proposition,*
2.
$$\begin{array}{ll} (a) & (\Gamma \vdash T = T') \Rightarrow (\Gamma, T \rhd) \\ (b) & (\Gamma, T \rhd) \Rightarrow (\Gamma \vdash T = T) \\ (c) & (\Gamma \vdash T = T') \Rightarrow (\Gamma \vdash T' = T) \\ (d) & (\Gamma \vdash T = T') \wedge (\Gamma \vdash T' = T'') \Rightarrow (\Gamma \vdash T = T'') \end{array}$$
3.
$$\begin{array}{ll} (a) & (\Gamma \vdash o = o' : T) \Rightarrow (\Gamma \vdash o : T) \\ (b) & (\Gamma \vdash o : T) \Rightarrow (\Gamma \vdash o = o : T) \\ (c) & (\Gamma \vdash o = o' : T) \Rightarrow (\Gamma \vdash o' = o : T) \\ (d) & (\Gamma \vdash o = o' : T) \wedge (\Gamma \vdash o' = o'' : T) \Rightarrow (\Gamma \vdash o = o'' : T) \end{array}$$
4.
$$\begin{array}{ll} (a) & (\Gamma_1 \vdash T = T') \wedge (\Gamma_1, T, \Gamma_2 \vdash S = S') \Rightarrow (\Gamma_1, T', \Gamma_2 \vdash S = S') \\ (b) & (\Gamma_1 \vdash T = T') \wedge (\Gamma_1, T, \Gamma_2 \vdash o = o' : S) \Rightarrow (\Gamma_1, T', \Gamma_2' \vdash o = o' : S) \\ (c) & (\Gamma \vdash S = S') \wedge (\Gamma \vdash o = o' : S) \Rightarrow (\Gamma \vdash o = o' : S') \end{array}$$
5.
$$\begin{array}{lll} (a) & (\Gamma_1, T \rhd) \wedge (\Gamma_1, \Gamma_2 \vdash S = S') \Rightarrow (\Gamma_1, T, t_{i+1}\Gamma_2 \vdash t_{i+1}S = t_{i+1}S') & i = l(\Gamma) \\ (b) & (\Gamma_1, T \rhd) \wedge (\Gamma_1, \Gamma_2 \vdash o = o' : S) \Rightarrow (\Gamma_1, T, t_{i+1}\Gamma_2 \vdash t_{i+1}o = t_{i+1}o' : t_{i+1}S) & i = l(\Gamma) \end{array}$$
6.
$$\begin{array}{ll} (a) \quad (\Gamma_1, T, \Gamma_2 \vdash S = S') \wedge (\Gamma_1 \vdash r : T) \Rightarrow & \\ (\Gamma_1, s_{i+1}(\Gamma_2[r/i+1]) \vdash s_{i+1}(S[r/i+1]) = s_{i+1}(S'[r/i+1])) & i = l(\Gamma_1) \\ (b) \quad (\Gamma_1, T, \Gamma_2 \vdash o = o' : S) \wedge (\Gamma_1 \vdash r : T) \Rightarrow & \\ (\Gamma_1, s_{i+1}(\Gamma_2[r/i+1]) \vdash s_{i+1}(o[r/i+1]) = s_{i+1}(o'[r/i+1]) : s_{i+1}(S[r/i+1])) & i = l(\Gamma_1) \end{array}$$
7.
$$\begin{array}{ll} (a) \quad (\Gamma_1, T, \Gamma_2, S \rhd) \wedge (\Gamma_1 \vdash r = r' : T) \Rightarrow & \\ (\Gamma_1, s_{i+1}(\Gamma_2[r/i+1]) \vdash s_{i+1}(S[r/i+1]) = s_{i+1}(S[r'/i+1])) & i = l(\Gamma_1) \\ (b) \quad (\Gamma_1, T, \Gamma_2 \vdash o : S) \wedge (\Gamma_1 \vdash r = r' : T) \Rightarrow & \\ (\Gamma_1, s_{i+1}(\Gamma_2[r/i+1]) \vdash s_{i+1}(o[r/i+1]) = s_{i+1}(o[r'/i+1]) : s_{i+1}(S[r/i+1])) & i = l(\Gamma_1) \end{array}$$

Then the relations $\sim$ and $\simeq$ are equivalence relations on C and $\widetilde{C}$ which satisfy the conditions of [?, Proposition 5.4] *and therefore they correspond to a regular congruence relation on the C-system defined by $(C, \widetilde{C})$.*

Lemma 1.3. **[iseqrelsiml1]** *One has:*

1. *If conditions (1.2), (4a) of the proposition hold then $(\Gamma \vdash S = S') \wedge (\Gamma \sim \Gamma') \Rightarrow (\Gamma' \vdash S = S')$.*
2. *If conditions (1.2), (1.3), (4a), (4b), (4c) hold then $(\Gamma \vdash o = o' : S) \wedge ((\Gamma, S) \sim (\Gamma', S')) \Rightarrow (\Gamma' \vdash o = o' : S')$.*

Proof. By induction on $n = l(\Gamma) = l(\Gamma')$.

(1) For $n = 0$ the assertion is obvious. Therefore by induction we may assume that $(\Gamma \vdash S = S') \wedge (\Gamma \sim \Gamma') \Rightarrow (\Gamma' \vdash S = S')$ for all $i < n$ and all appropriate Γ,Γ', S and S' and that $(T_1, \dots, T_n \vdash S = S') \wedge (T_1, \dots, T_n \sim T'_1, \dots, T'_n)$ holds and we need to show that $(T'_1, \dots, T'_n \vdash S = S')$ holds. Let us show by induction on j that $(T'_1, \dots, T'_j, T_{j+1}, \dots, T_n \vdash S = S')$ for all $j = 0, \dots, n$. For $j = 0$ it is a part of our assumptions. By induction we may assume that $(T'_1, \dots, T'_j, T_{j+1}, \dots, T_n \vdash S = S')$. By definition of $\sim$ we have $(T_1, \dots, T_j \vdash T_{j+1} = T'_{j+1})$. By the inductive assumption we have $(T'_1, \dots, T'_j \vdash T_{j+1} = T'_{j+1})$. Applying (4a) with $\Gamma_1 = (T'_1, \dots T'_j)$, $T = T_{j+1}$, $T' = T'_{j+1}$ and $\Gamma_2 = (T_{j+2}, \dots, T_n)$ we conclude that $(T'_1, \dots, T'_{j+1}, T_{j+2}, \dots, T_n \vdash S = S')$.

(2) By the first part of the lemma we have $\Gamma' \vdash S = S'$. Therefore by (4c) it is sufficient to show that $(\Gamma \vdash o = o' : S) \wedge (\Gamma \sim \Gamma') \Rightarrow (\Gamma' \vdash o = o' : S)$. The proof of this fact is similar to the proof of the first part of the lemma using (4b) instead of (4a). □

Lemma 1.4. **/iseqrelsim/** *One has:*

1. *Assume that conditions (1.2), (2b), (2c), (2d) and (4a) hold. Then $\sim$ is an equivalence relation.*
2. *Assume that conditions of the previous part of the lemma as well as conditions (1.3), (3b), (3c), (3d), (4b) and (4c) hold. Then $\simeq$ is an equivalence relation.*

Proof. By induction on $n = l(\Gamma) = l(\Gamma')$.

(1) Reflexivity follows directly from (1.2) and (2b). For $n = 0$ the symmetry is obvious. Let $(\Gamma, T) \sim (\Gamma', T')$. By induction we may assume that $\Gamma' \sim \Gamma$. By Lemma 1.3(a) we have $(\Gamma' \vdash T = T')$ and by (2c) we have $(\Gamma' \vdash T' = T)$. We conclude that $(\Gamma', T') \sim (\Gamma, T)$. The proof of transitivity is by a similar induction.

(2) Reflexivity follows directly from reflexivity of $\sim$, (1.3) and (3b). Symmetry and transitivity are also easy using Lemma 1.3. □

From this point on we assume that all conditions of Proposition 1.2 hold. Let $C' = C/\sim$ and $\widetilde{C}' = \widetilde{C}/\simeq$. It follows immediately from our definitions that the functions $ft : C \to C$ and $\partial : \widetilde{C} \to C$ define functions $ft' : C' \to C'$ and $\partial' : \widetilde{C}' \to C'$.

Lemma 1.5. **/surjl1/** *The conditions (3) and (4) of* [?, Proposition 5.4] *hold for $\sim$ and $\simeq$.*

Proof. 1. We need to show that for $(\Gamma, T\rhd)$, and $\Gamma \sim \Gamma'$ there exists $(\Gamma', T'\rhd)$ such that $(\Gamma, T) \sim (\Gamma', T')$. It is sufficient to take $T = T'$. Indeed by (2b) we have $\Gamma \vdash T = T$, by Lemma 1.3(1) we conclude that $\Gamma' \vdash T = T$ and by (1a) that $\Gamma', T\rhd$.

2. We need to show that for $(\Gamma \vdash o : S)$ and $(\Gamma, S) \sim (\Gamma', S')$ there exists $(\Gamma' \vdash o' : S')$ such that $(\Gamma' \vdash o' : S') \simeq (\Gamma \vdash o : S)$. It is sufficient to take $o' = o$. Indeed, by (3b) we have $(\Gamma \vdash o = o : S)$, by Lemma 1.3(2) we conclude that $(\Gamma' \vdash o = o : S')$ and by (2a) that $(\Gamma' \vdash o : S')$. □

Lemma 1.6. *[***TSetc***] The equivalence relations $\sim$ and $\simeq$ are compatible with the operations $T, \widetilde{T}, S, \widetilde{S}$ and δ.*

Proof. (1) Given $(\Gamma_1, T\rhd) \sim (\Gamma'_1, T'\rhd)$ and $(\Gamma_1, \Gamma_2\rhd) \sim (\Gamma'_1, \Gamma'_2\rhd)$ we have to show that

$$(\Gamma_1, T, t_{n+1}\Gamma_2) \sim (\Gamma'_1, T', t_{n+1}\Gamma'_2).$$

where $n = l(\Gamma_1) = l(\Gamma'_1)$.

Proceed by induction on $l(\Gamma_2)$. For $l(\Gamma_2) = 0$ the assertion is obvious. Let $(\Gamma_1, T\rhd) \sim (\Gamma'_1, T'\rhd)$ and $(\Gamma_1, \Gamma_2, S\rhd) \sim (\Gamma'_1, \Gamma'_2, S'\rhd)$. The later condition is equivalent to $(\Gamma_1, \Gamma_2\rhd) \sim (\Gamma'_1, \Gamma'_2\rhd)$ and $(\Gamma_1, \Gamma_2 \vdash S = S')$. By the inductive assumption we have $(\Gamma_1, T, t_{n+1}\Gamma_2) \sim (\Gamma'_1, T', t_{n+1}\Gamma'_2)$. By (5a) we conclude that $(\Gamma_1, T, t_{n+1}\Gamma_2 \vdash t_{n+1}S = t_{n+1}S')$. Therefore by definition of $\sim$ we have $(\Gamma_1, T, t_{n+1}\Gamma_2, t_{n+1}S) \sim (\Gamma'_1, T', t_{n+1}\Gamma'_2, t_{n+1}S')$.

(2) Given $(\Gamma_1, T\rhd) \sim (\Gamma'_1, T'\rhd)$ and $(\Gamma_1, \Gamma_2 \vdash o : S) \simeq (\Gamma'_1, \Gamma'_2 \vdash o' : S')$ we have to show that $(\Gamma_1, T, t_{n+1}\Gamma_2 \vdash t_{n+1}o : t_{n+1}S) \simeq (\Gamma'_1, T', t_{n+1}\Gamma'_2 \vdash t_{n+1}o' : t_{n+1}S')$ where $n = l(\Gamma_1) = l(\Gamma'_1)$. We have $(\Gamma_1, \Gamma_2, S) \sim (\Gamma'_1, \Gamma'_2, S')$ and $(\Gamma_1, \Gamma_2 \vdash o = o' : S)$. By (5b) we get $(\Gamma_1, T, t_{n+1}\Gamma_2 \vdash t_{n+1}o = t_{n+1}o' : t_{n+1}S)$. By (1) of this lemma we get $(\Gamma_1, T, t_{n+1}\Gamma_2, t_{n+1}S) \sim (\Gamma'_1, T', t_{n+1}\Gamma'_2, t_{n+1}S')$ and therefore by definition of $\simeq$ we get $(\Gamma_1, T, t_{n+1}\Gamma_2 \vdash t_{n+1}o : t_{n+1}S) \simeq (\Gamma'_1, T', t_{n+1}\Gamma'_2 \vdash t_{n+1}o' : t_{n+1}S')$.

(3) Given $(\Gamma_1 \vdash r : T) \simeq (\Gamma'_1 \vdash r' : T')$ and $(\Gamma_1, T, \Gamma_2\rhd) \sim (\Gamma'_1, T', \Gamma'_2\rhd)$ we have to show that

$$(\Gamma_1, s_{n+1}(\Gamma_2[r/n+1])) \sim (\Gamma'_1, s_{n+1}(\Gamma'_2[r'/n+1])).$$

where $n = l(\Gamma_1) = l(\Gamma'_1)$. Proceed by induction on $l(\Gamma_2)$. For $l(\Gamma_2) = 0$ the assertion follows directly from the definitions. Let $(\Gamma_1 \vdash r : T) \simeq (\Gamma'_1 \vdash r' : T')$ and $(\Gamma_1, T, \Gamma_2, S\rhd) \sim (\Gamma'_1, T', \Gamma'_2, S'\rhd)$. The later condition is equivalent to $(\Gamma_1, T, \Gamma_2\rhd) \sim (\Gamma'_1, T', \Gamma'_2\rhd)$ and $(\Gamma_1, T, \Gamma_2 \vdash S = S')$. By the inductive assumption we have $(\Gamma_1, s_{n+1}(\Gamma_2[r/n+1])) \sim (\Gamma'_1, s_{n+1}(\Gamma'_2[r'/n+1]))$. It remains to show that $(\Gamma_1, s_{n+1}(\Gamma_2[r/n+1]) \vdash s_{n+1}(S[r/n+1]) = s_{n+1}(S'[r'/n+1]))$. By (2d) it is sufficient to show that $(\Gamma_1, s_{n+1}(\Gamma_2[r/n+1]) \vdash s_{n+1}(S[r/n+1]) = s_{n+1}(S'[r/n+1]))$ and $(\Gamma_1, s_{n+1}(\Gamma_2[r/n+1]) \vdash s_{n+1}(S'[r/n+1]) = s_{n+1}(S'[r'/n+1]))$. The first relation follows directly from (6a). To prove the second one it is sufficient by (7a) to show that $(\Gamma_1, T, \Gamma_2, S'\rhd)$ which follows from our assumption through (2c) and (2a).

(4) Given $(\Gamma_1 \vdash r : T) \simeq (\Gamma'_1 \vdash r' : T')$ and $(\Gamma_1, T, \Gamma_2 \vdash o : S) \simeq (\Gamma'_1, T', \Gamma'_2 \vdash o' : S')$ we have to show that

$$(\Gamma_1, s_{n+1}(\Gamma_2[r/n+1]) \vdash s_{n+1}(o[r/n+1]) : s_{n+1}(S[r/n+1])) \simeq$$
$$(\Gamma'_1, s_{n+1}(\Gamma'_2[r'/n+1]) \vdash s_{n+1}(o'[r'/n+1]) : s_{n+1}(S'[r'/n+1])).$$

where $n = l(\Gamma_1) = l(\Gamma'_1)$ or equivalently that

$$(\Gamma_1, s_{n+1}(\Gamma_2[r/n+1]), s_{n+1}(S[r/n+1])) \sim (\Gamma'_1, s_{n+1}(\Gamma'_2[r'/n+1]), s_{n+1}(S'[r'/n+1]))$$

and $(\Gamma_1, s_{n+1}(\Gamma_2[r/n+1]) \vdash s_{n+1}(o[r/n+1]) = s_{n+1}(o'[r'/n+1]) : s_{n+1}(S[r/n+1]))$. The first statement follows from part (3) of the lemma. To prove the second statement it is sufficient by (3d) to show that $(\Gamma_1, s_{n+1}(\Gamma_2[r/n+1]) \vdash s_{n+1}(o[r/n+1]) = s_{n+1}(o'[r/n+1]) : s_{n+1}(S[r/n+1]))$ and $(\Gamma_1, s_{n+1}(\Gamma_2[r/n+1]) \vdash s_{n+1}(o'[r/n+1]) = s_{n+1}(o'[r'/n+1]) : s_{n+1}(S[r/n+1]))$. The first assertion follows directly from (6b).

To prove the second one it is sufficient in view of (7b) to show that $(\Gamma_1, T, \Gamma_2 \vdash o' : S)$ which follows conditions (3c) and (3a).

(5) Given $(\Gamma, T) \sim (\Gamma', T')$ we need to show that $(\Gamma, T \vdash (n+1) : T) \simeq (\Gamma', T' \vdash (n+1) : T')$ or equivalently that $(\Gamma, T, T) \sim (\Gamma, T', T')$ and $(\Gamma, T \vdash (n+1) = (n+1) : T)$. The second part follows from (3b). To prove the first part we need to show that $(\Gamma, T \vdash T = T')$. This follows from our assumption by (5a). ☐

Lemma 1.7. *[2014.07.12.l1] Let C be a subset of $Ob(CC(R, LM))$ which is closed under ft. Let $\leq$ be a transitive relation on C such that:*

(1) $\Gamma \leq \Gamma'$ implies $l(\Gamma) = l(\Gamma')$,
(2) $\Gamma \in C$ and $ft(\Gamma) \leq F$ implies $\sigma(\Gamma, F) \in C$ and $\Gamma \leq \sigma(\Gamma, F)$.

Then $\Gamma \in C$ and $ft^i(\Gamma) \leq F$ for some $i \geq 1$, implies that $\Gamma \leq \sigma(\Gamma, F)$.

Proof. Simple induction on i. ☐

Lemma 1.8. *[2014.07.12.l2] Let C and $\leq$ be as in Lemma 1.7. Then one has:*

(1) $(\Gamma, T) \leq (\Gamma, T')$ and $\Gamma \leq \Gamma'$ implies that $(\Gamma, T) \leq (\Gamma', T')$,
(2) if $\leq$ is ft-monotone (i.e. $\Gamma \leq \Gamma'$ implies $ft(\Gamma) \leq ft(\Gamma')$) and symmetric then $(\Gamma, T) \leq (\Gamma', T')$ implies that $(\Gamma, T) \leq (\Gamma, T')$.

Proof. The first assertion follows from

$$(\Gamma, T) \leq (\Gamma, T') \leq \sigma((\Gamma, T'), \Gamma') = (\Gamma', T')$$

The second assertion follows from

$$(\Gamma, T) \leq (\Gamma', T') \leq \sigma((\Gamma', T'), \Gamma) = (\Gamma, T')$$

where the second $\leq$ requires $\Gamma' \leq \Gamma$ which follows from ft-monotonicity and symmetry. ☐

Lemma 1.9. *[2014.07.12.l3] Let $C, \leq$ be as in Lemma 1.7, let $\widetilde{C}$ be a subset of $\widetilde{Ob}(CC(R, LM))$ and $\leq'$ a transitive relation on $\widetilde{C}$ such that:*

(1) $\mathcal{J} \leq' \mathcal{J}'$ implies $\partial(\mathcal{J}) \leq \partial(\mathcal{J}')$,
(2) $\mathcal{J} \in \widetilde{C}$ and $\partial(\mathcal{J}) \leq F$ implies $\widetilde{\sigma}(\mathcal{J}, F) \in \widetilde{C}$ and $\mathcal{J} \leq' \widetilde{\sigma}(\mathcal{J}, F)$.

Then $\mathcal{J} \in \widetilde{C}$ and $ft^i(\partial(\mathcal{J})) \leq F$ for some $i \geq 0$ implies $\mathcal{J} \leq \widetilde{\sigma}(\mathcal{J}, F)$.

Proof. Simple induction on i. ☐

Lemma 1.10. *[2014.07.12.l4] Let $C, \leq$ and $\widetilde{C}, \leq'$ be as in Lemma 1.9. Then one has:*

(1) $(\Gamma \vdash o : T) \leq' (\Gamma \vdash o' : T)$ and $(\Gamma, T) \leq (\Gamma', T')$ implies that $(\Gamma \vdash o : T) \leq' (\Gamma' \vdash o' : T')$,
(2) if $(\leq, \leq')$ is ∂-monotone (i.e. $\mathcal{J} \leq' \mathcal{J}'$ implies $\partial(\mathcal{J}) \leq \partial(\mathcal{J}')$) and $\leq$ is symmetric then $(\Gamma \vdash o : T) \leq' (\Gamma' \vdash o' : T')$ implies that $(\Gamma \vdash o : T) \leq' (\Gamma \vdash o' : T)$.

Proof. The first assertion follows from

$$(\Gamma\vdash o:T)\leq'(\Gamma\vdash o':T)\leq'\widetilde{\sigma}((\Gamma\vdash o':T),(\Gamma',T'))=(\Gamma'\vdash o':T')$$

The second assertion follows from

$$\Gamma\vdash o:T)\leq'(\Gamma'\vdash o':T')\leq'\sigma((\Gamma'\vdash o':T'),(\Gamma,T))=(\Gamma\vdash o':T)$$

where the second $\leq$ requires $\Gamma'\leq\Gamma$ which follows from ∂-monotonicity of $\leq'$ and symmetry of $\leq$. □

Proposition 1.11. *[***2014.07.10.prop2***] Let $(C,\widetilde{C})$ be subsets in $Ob(CC(R,LM))$ and $\widetilde{Ob}(CC(R,LM))$ respectively which correspond to a C-subsystem CC of $CC(R,LM)$. Then the constructions presented above establish a bijection between pairs of subsets $(Ceq,\widetilde{Ceq})$ which together with $(C,\widetilde{C})$ satisfy the conditions of Proposition 1.2 and pairs of equivalence relations $(\sim,\simeq)$ on $(C,\widetilde{C})$ such that:*

(1) $(\sim,\simeq)$ corresponds to a regular congruence relation on CC (i.e., satisfies the conditions of [**?**, Proposition 5.4]*),*
(2) $\Gamma\in C$ and $ft(\Gamma)\sim F$ implies $\Gamma\sim\sigma(\Gamma,F)$,
(3) $\mathcal{J}\in\widetilde{C}$ and $\partial(\mathcal{J})\sim F$ implies $\mathcal{J}\simeq\widetilde{\sigma}(\mathcal{J},F)$.

Proof. One constructs a pair $(\sim,\simeq)$ from $(Ceq,\widetilde{Ceq})$ as in Definition 1.1. This pair corresponds to a regular congruence relation by Proposition 1.2. Conditions (2),(3) follow from Lemma 1.3.

Let $(\sim,\simeq)$ be equivalence relations satisfying the conditions of the proposition. Define Ceq as the set of sequences (Γ,T,T') such that $(\Gamma,T),(\Gamma,T')\in C$ and $(\Gamma,T)\sim(\Gamma,T')$. Define $\widetilde{Ceq}$ as the set of sequences (Γ,T,o,o') such that $(\Gamma,T,o),(\Gamma,T,o')\in\widetilde{C}$ and $(\Gamma,T,o)\simeq(\Gamma,T,o')$.

Let us show that these subsets satisfy the conditions of Proposition 1.2. Conditions (2.a-2.d) and (3.a-3d) are obvious.

Condition (4a) follows from (2) by Lemma 1.7. Conditions (4b) and (4c) follow from (3) by Lemma 1.9.

Conditions (5a) and (5b) follow from the compatibility of $(\sim,\simeq)$ with T and $\widetilde{T}$.

Conditions (6a),(6b),(7a),(7b) follow from the compatibility of $(\sim,\simeq)$ with S and $\widetilde{S}$. □

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