A C-system defined by a universe in a category[1]

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Abstract

This is the third paper in a series started in [7]. In it we construct, in a functorial way, a C-system $CC(\mathcal{C}, p)$ starting from a category $\mathcal{C}$ together with a morphism $p : \widetilde{U} \to U$ assuming that $\mathcal{C}$ has a final object and pull-backs of p. The choice of pull-back is not required for the resulting C-system $CC(\mathcal{C}, p)$ to be well-defined up to a canonical isomorphism.

1 Introduction

The concept of a C-system was introduced in [7]. The type of the C-systems is constructively equivalent to the type of contextual categories defined by Cartmell in [3] and [2] but the definition of a C-system is slightly different from the Cartmell's foundational definition.

In [6] we constructed for any pair (R, LM) where R is a monad on $Sets$ and LM a left R-module with values in $Sets$ a C-system $CC(R, LM)$. In the particular case of pairs (R, LM) corresponding to signatures as in [4, p.228] or to nominal signatures the regular sub-quotients of $CC(R, LM)$ are the C-systems corresponding to the dependent type theories.

A signature is an object of a type of h-level 3 since one of the components of a signature is an abstract set of operations. However, if we fix this set then a signature over this set together with the sets of judgements of four Martin-Lof kinds over this signature that specify a regular sub-quotient of $CC(M, LM)$ is an object of a type of h-level 2 i.e. an element of a set.

A C-system is an object of a type of h-level 3. The constructions of [7] and [6] provide a function from the first type to the second - from a type of h-level 2 to a type of h-level 3.

In this short paper we describe another construction that generates C-systems. This time the input data is a pair that consists of a category $\mathcal{C}$ with a final object and a morphism $p : \mathcal{U} \to U$ in this category that satisfy a certain property. For any such $(\mathcal{C}, p)$ we construct a C-system $CC(\mathcal{C}, p)$ and then show that this construction is functorial.

Pairs of the form $(\mathcal{C}, p)$ form a type of h-level 4. The construction of this paper is a function from this type to the type of C-systems that is of h-level 3.

Taken together these two constructions connect a type of h-level 2 that contains syntactic data with a type of h-level 4 that contains category-level data.

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The content of the present paper is two constructions. The usual Definition-Theorem-Proof style of writing mathematics is not convenient for presenting constructions. In this paper we use the pair of names Problem-Construction for the specification of the goal of a construction and the description of the particular solution.

In the case of a Theorem-Proof pair one usually refers (by name or number) to the theorem when using the proof of this theorem. This is acceptable in the case of theorems because the future use of their proofs is such that only the fact that there is a proof but not the particulars of the proof matter.

In the case of a Problem-Construction pair the content of the construction often matters in the future use. Because of this we have to refer to the construction and not to the problem and we assign in this paper numbers both to Problems and to the Constructions.

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2 Construction of $CC(\mathcal{C}, p)$.

Definition 2.1 *Let $\mathcal{C}$ be a category. A universe on $\mathcal{C}$ is a morphism $p : \widetilde{U} \to U$ together with a mapping which assigns to any morphism $f : X \to U$ in $\mathcal{C}$ a pull-back square*

$$\begin{array}{ccc} (X, f) & \xrightarrow{Q(f)} & \widetilde{U} \\ {\scriptstyle p_{(X,f)}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & U \end{array}$$

In what follows we will write $(X, f_1, \dots, f_n)$ for $(\dots((X, f_1), f_2)\dots, f_n)$.

Problem 2.2 *Let $\mathcal{C}$ be a category, p a universe on $\mathcal{C}$ and pt a final object of $\mathcal{C}$. For such a triple define a C-system $CC = CC(\mathcal{C}, p)$.*

Construction 2.3 Objects of CC are sequences of the form $(F_1, \dots, F_n)$ where

$F_1 \in Hom_{\mathcal{C}}(pt, U)$ and

$F_{i+1} \in Hom_{\mathcal{C}}((pt, F_1, \dots, F_i), U)$.

Morphisms from $(G_1, \dots, G_n)$ to $(F_1, \dots, F_m)$ are given by

$$Hom_{CC}((G_1, \dots, G_n), (F_1, \dots, F_m)) = Hom_{\mathcal{C}}((pt, G_1, \dots, G_n), (pt, F_1, \dots, F_m))$$

and units and compositions are defined as units and compositions in $\mathcal{C}$ such that the mapping $(F_1, \dots, F_n) \to (pt, F_1, \dots, F_n)$ is a full embedding of the underlying category of CC to $\mathcal{C}$. The image of this embedding consists of objects X for which the canonical morphism $X \to pt$

is a composition of morphisms which are (canonical) pull-backs of p. We will denote this embedding by int.

The final object of CC is the empty sequence $()$. The map ft sends $(F_1, \ldots, F_n)$ to $(F_1, \ldots, F_{n-1})$. The canonical morphism $p_{(F_1,\ldots,F_n)}$ is the projection

$$p_{((pt,F_1,\ldots,F_{n-1}),F_n)} : ((pt, F_1, \ldots, F_{n-1}), F_n) \to (pt, F_1, \ldots, F_{n-1})$$

For an object $(F_1, \ldots, F_{m+1})$ and a morphism $f : (G_1, \ldots, G_n) \to (F_1, \ldots, F_m)$ the canonical pull-back square is of the form

$$\begin{array}{ccc} (G_1, \ldots, G_n, F_{m+1}f) & \xrightarrow{q(f)} & (F_1, \ldots, F_{m+1}) \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_F} \\ (G_1, \ldots, G_n) & \xrightarrow{f} & (F_1, \ldots, F_m) \end{array} \qquad (1)$$

where $int(p_F) = p((pt, F_1, \ldots, F_{n-1}), F_n)$, $int(p_G) = p((pt, G_1, \ldots, G_{n-1}), F_{m+1} \circ f)$ and $q(f)$ is the morphism such that $p_F q(f) = f p_G$ and $Q(F_{m+1}) int(q(f)) = Q(F_{m+1} f)$. The unity and composition axioms for the canonical squares follow immediately from the unity and associativity axioms for compositions of morphisms in C.

3 Functoriality of $CC(\mathcal{C}, p)$.

Problem 3.1 *Let $(\mathcal{C}, p, pt)$ and $(\mathcal{C}', p', pt')$ be two sets of data as above. Let $\Phi : \mathcal{C} \to \mathcal{C}'$ be a functor which takes distinguished squares in $\mathcal{C}$ to pull-back squares in $\mathcal{C}'$ and such that $\Phi(pt)$ is a final object of $\mathcal{C}'$. Let further $\phi : \Phi(U) \to U'$, $\widetilde{\phi} : \Phi(\widetilde{U}) \to \widetilde{U}'$ be two morphisms such that*

$$\begin{array}{ccc} \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

is a pull-back square. Define a functor $H = H(\Phi, \phi, \widetilde{\phi})$ from $CC(\mathcal{C}, p)$ to $CC(\mathcal{C}', p')$.

Construction 3.2 Denote by ψ the isomorphism $\psi : pt' \to \Phi(pt)$. We define by induction on n objects $H(F_1, \ldots, F_n) \in CC(\mathcal{C}', p')$ and isomorphisms

$$\psi_{(F_1,\ldots,F_n)} : int'(H(F_1, \ldots, F_n)) \to \Phi(int(F_1, \ldots F_n))$$

where int and int' are the canonical functors $CC(\mathcal{C}, p) \to \mathcal{C}$ and $CC(\mathcal{C}', p') \to \mathcal{C}'$ respectively.

For $n = 0$ we set $H(()) = ()$ and $\psi_{()} = \psi$. For $n > 0$ let

$$(F'_1, \ldots, F'_{n-1}) = H(F_1, \ldots, F_{n-1})$$

and let $F_n : int(F_1, \ldots, F_{n-1}) \to U$. Define F'_n as the composition

$$F'_n : int'(F'_1, \ldots, F'_{n-1}) \stackrel{\psi_{(F_1,\ldots,F_{n-1})}}{\longrightarrow} \Phi(int(F_1, \ldots, F_{n-1})) \stackrel{\Phi(F_n)}{\to} \Phi(U) \stackrel{\phi}{\to} U' \qquad (2)$$

and let $H(F_1, \ldots, F_n) = (F'_1, \ldots, F'_{n-1}, F'_n)$. Then

$$int'(H(F_1, \ldots, F_n)) = (int'(H(F_1, \ldots, F_n)), F'_n)$$

To define

$$\psi_{(F_1,\ldots,F_n)} : int'(H(F_1, \ldots, F_n)) \to \Phi(int(F_1, \ldots, F_n))$$

observe that by our conditions on $\phi, \widetilde{\phi}$ and Φ the squares of the diagram

$$\begin{array}{ccccc} \Phi(int(F_1, \ldots, F_n)) & \xrightarrow{\Phi(Q(F_n))} & \Phi(\widetilde{U}) & \longrightarrow & \widetilde{U}' \\ \downarrow & & \downarrow & & \downarrow \\ \Phi(int(F_1, \ldots, F_{n-1})) & \xrightarrow{\Phi(F_n)} & \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

are pull-back. Therefore there is a unique morphism $\psi_{(F_1,\ldots,F_n)}$ such that the diagram

$$\begin{array}{ccccc} int'(H(F_1, \ldots, F_n)) & \xrightarrow{\psi_{(F_1,\ldots,F_n)}} & \Phi(int(F_1, \ldots, F_n)) & \xrightarrow{\widetilde{\phi}\Phi(Q(F_n))} & \widetilde{U}' \\ \downarrow & & \downarrow & & \downarrow \\ int'(H(F_1, \ldots, F_{n-1})) & \xrightarrow{\psi_{(F_1,\ldots,F_{n-1})}} & \Phi(int(F_1, \ldots, F_{n-1})) & \xrightarrow{\phi\Phi(F_n)} & U' \end{array} \quad (3)$$

commutes and

$$\widetilde{\phi}\Phi(Q(F_n))\psi_{(F_1,\ldots,F_n)} = Q(\phi\Phi(F_n)\psi_{(F_1,\ldots,F_{n-1})}) \quad (4)$$

and this morphism is an isomorphism.

To define H on morphism we use the fact that morphisms $\psi_{(F_1,\ldots,F_n)}$ are isomorphisms and for $f : (F_1, \ldots, F_n) \to (G_1, \ldots, G_m)$ we set

$$H(f) = \psi^{-1}_{(G_1,\ldots,G_m)}\Phi(f)\psi_{(F_1,\ldots,F_n)} \quad (5)$$

The fact that this construction gives a functor i.e. satisfies the unity and composition axioms is straightforward.

It remains to verify that this morphism respects the rest of the C-system. It is clear that it respects the length function and the ft maps. The fact that it takes the canonical projections to canonical projections is equivalent to the commutativity of the left hand side square in (3).

Consider a canonical square of the form (1). Its image is a square of the form

$$\begin{array}{ccc} (G'_1, \ldots, G'_n, G'_{n+1}) & \xrightarrow{H(q(f))} & (F'_1, \ldots, F'_{m+1}) \\ {\scriptstyle H(p_G)}\downarrow & & \downarrow{\scriptstyle H(p_F)} \\ (G'_1, \ldots, G'_n) & \xrightarrow{H(f)} & (F'_1, \ldots, F'_m) \end{array} \quad (6)$$

We already know that the vertical arrows are canonical projections. Therefore, in order to prove that (6) is a canonical square in $CC(\mathcal{C}', p')$ we have to show that $G'_{n+1} = F'_{m+1}int(H(f))$ and

$$Q(F'_{m+1})int(H(q(f))) = Q(F'_{m+1}int(H(f))) \quad (7)$$

By (2) we have

$$G'_{n+1} = \phi\Phi(F_{m+1}f)\psi_{(G_1,\ldots,G_n)}$$
$$F'_{m+1} = \phi\Phi(F_{m+1})\psi_{(F_1,\ldots,F_m)}$$

and by (5)

$$int(H(f)) = \psi^{-1}_{(F_1,\ldots,F_m)}\Phi(f)\psi_{(G_1,\ldots,G_n)}$$
$$int(H(q(f))) = \psi^{-1}_{(F_1,\ldots,F_{m+1})}\Phi(q(f))\psi_{(G_1,\ldots,G_n,F_{m+1}f)}$$

Therefore the relation $G'_{n+1} = F'_{m+1}int(H(f))$ follows immediately and the relation (7) follows by application of (4).

Lemma 3.3 *Let $(\Phi.\phi,\widetilde{\phi})$ be as in Problem 3.1 and let H be the corresponding solution of Construction 3.2. Then if Φ is a full embedding and ϕ and $\widetilde{\phi}$ are isomorphisms then H is an isomorphism of C-systems.*

Proof: Straightforward.

Lemma 3.3 implies in particular that considered up to a canonical isomorphism $CC(\mathcal{C},p)$ depends only on the equivalence class of the pair $(\mathcal{C},p)$ i.e. that our construction maps the type of pairs $(\mathcal{C},p)$ to the type of C-systems.

Remark 3.4 As far as I know this is the only known construction that generates a model of one of the essentially-algebraic theories that are connected to the syntax of dependent type theories from a category-level data in a functorial way. The use of representable morphisms of presheaves in [1] does not provide set level objects defined up to an isomorphism. Even when a particular representability structure is chosen as is done in the original definition of categories with families one still does not obtain an object defined up to an isomorphism when one considers the underlying category up to an equivalence.

Let us describe now an inverse construction which shows that any C-system is isomorphic to a C-system of the form $CC(\mathcal{C},p)$.

Problem 3.5 *Let CC be a C-system. Construct a pair $(\mathcal{C},p)$ as above and an isomorphism $CC \cong CC(\mathcal{C},p)$.*

Construction 3.6 Denote by $PreShv(CC)$ the category of contravariant functors from the category underlying CC to $Sets$.

Let Ty be the functor which takes an object $\Gamma \in CC$ to the set

$$Ty(\Gamma) = \{\Gamma' \in CC \,|\, ft(\Gamma') = \Gamma\}$$

and a morphism $f : \Delta \to \Gamma$ to the map $\Gamma' \mapsto f^*\Gamma'$. It is a functor due to the composition and unity axioms for f^*. Let Tm be the functor which takes an object Γ to the set

$$Tm(\Gamma) = \{s \in \widetilde{CC} \,|\, ft\,\partial(s) = \Gamma\}$$

and a morphism $f : \Delta \to \Gamma$ to the map $s \mapsto f^*(s)$. Let further $p : Tm \to Ty$ be the morphism which takes s to $\partial(s)$. It is well defined as a morphisms of families of sets and forms a morphism of presheaves since $\partial(f^*(s)) = f^*(\partial(s))$.

Let us construct an isomorphism $CC \cong CC(PreShv(CC), p)$.

We start with the key lemma. (In what follows we identify objects of CC with the corresponding representable presheaves and, for a presheaf F and an object Γ, we identify morphisms $\Gamma \to F$ in $PreShv(CC)$ with $F(\Gamma)$).

Lemma 3.7 *Let $\Gamma' \in Ob(CC)$ and let $\Gamma = ft(\Gamma')$. Then the square*

$$\begin{array}{ccc} \Gamma' & \xrightarrow{\delta_{\Gamma'}} & Tm \\ {\scriptstyle p_{\Gamma'}}\downarrow & & \downarrow{\scriptstyle p} \\ \Gamma & \xrightarrow{\Gamma'} & Ty \end{array}$$

is a pull-back square.

Proof: We have to show that for any $\Delta \in CC$ the obvious map

$$Hom(\Delta, \Gamma') \to Hom(\Delta, \Gamma) \times_{Ty(\Delta)} Tm(\Delta) \tag{8}$$

is a bijection. Let $f_1, f_2 : \Delta \to \Gamma'$ be two morphisms such that their images under (8) coincide i.e. such that $p_{\Gamma'} f_1 = p_{\Gamma'} f_2$ and $f_1^*(\delta_{\Gamma'}) = f_2^*(\delta'_{\Gamma})$. These two conditions are equivalent to saying, in the notation introduced above, that $ft(f_1) = ft(f_2)$ and $s_{f_1} = s_{f_2}$. This implies that $f_1 = f_2$ i.e. that (8) is injective. Let $f : \Delta \to \Gamma$ be a morphism and $s \in Tm(\Delta)$ a section such that $ft(\partial(s)) = f^*(\Gamma')$. Then the composition $q(f, \Gamma')s$ is a morphism $f' : \Delta \to \Gamma'$ such that $p_{\Gamma'} f' = f$. We also have

$$(f')^*(\delta_{\Gamma'}) = s^* q(f, \Gamma')^*(\delta_{\Gamma'}) = s$$

which proves that (8) is surjective.

To construct the required isomorphism we now choose a universe structure on p such that the pull-back squares associated with morphisms from representable objects are squares (8). The isomorphism is now obvious.

Definition 3.8 *Let CC be a C-system. A closed model of CC is a collection of data of the following form:*

1. *A category $\mathcal{C}$,*

2. *a universe $p : \widetilde{U} \to U$ in $\mathcal{C}$ and a final object pt of $\mathcal{C}$,*

3. *a C-system morphism $CC \to CC(\mathcal{C}, p)$.*

Conjecture Let $\mathcal{C}$ be a category, CC be a C-system and $M : CC \to \mathcal{C}$ a functor such that $M(pt_{CC})$ is a final object of $\mathcal{C}$ and M maps distinguished squares of CC to pull-back squares of $\mathcal{C}$. Then there exists a universe $p_M : \widetilde{U}_M \to U_M$ in $PreShv(\mathcal{C})$ and a C-system morphism $M' : CC \to CC(PreShv(\mathcal{C}), p_M)$ such that the square

$$\begin{array}{ccc} CC & \xrightarrow{M} & \mathcal{C} \\ \downarrow{\scriptstyle M'} & & \downarrow \\ CC(PreShv(\mathcal{C}), p_M) & \xrightarrow{int} & PreShv(C) \end{array}$$

where the right hand side vertical arrow is the Yoneda embedding, commutes up to a natural isomorphism.

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