# A C-system defined by a universe in a category ${ }^{\text {W }}$ <br> Vladimir Voevodsky ${ }^{\text {and }}$ 

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#### Abstract

This is the third paper in a series started in $[8]$. In it we construct, in a functorial way, a C-system $C C(\mathcal{C}, p)$ starting from a category $\mathcal{C}$ together with a morphism $p$ : $\widetilde{U} \rightarrow U$ assuming that $\mathcal{C}$ has a final object and pull-backs of $p$. The choice of pull-back is not required for the resulting C-system $C C(\mathcal{C}, p)$ to be well-defined up to a canonical isomorphism.


## 1 Introduction

The concept of a C-system in its present form was introduced in [8]. The type of the Csystems is constructively equivalent to the type of contextual categories defined by Cartmell in [4] and [3] but the definition of a C-system is slightly different from the Cartmell's foundational definition.

In [7] we constructed for any pair $(R, L M)$ where $R$ is a monad on Sets and $L M$ a left $R$ module with values in Sets a C-system $C C(R, L M)$. In the particular case of pairs ( $R, L M$ ) corresponding to signatures as in [5, p.228] or to nominal signatures the regular sub-quotients of $C C(R, L M)$ are the C-systems corresponding to dependent type theories of the Martin-Lof genus.

In this paper we describe another construction that generates C-systems. This time the input data is a pair that consists of a category $\mathcal{C}$ with a final object and a morphism $p: \mathcal{U} \rightarrow U$ in this category that satisfy a certain property. For any such $(\mathcal{C}, p)$ we construct a C -system $C C(\mathcal{C}, p)$ and then show that this construction is functorial.

To the best of our knowledge it is the only known functorial construction of a C-system from a category level data. Because of this we find it important to present both the construction of the C-system and the construction of the homomorphisms defined by functors in detail.
The main result of the present paper is two constructions. To avoid the abuse of language inherent in the use of the Theorem-Proof style of presenting mathematics when dealing with constructions we use the pair of names Problem-Construction for the specification of the goal of a construction and the description of the particular solution.
In the case of a Theorem-Proof pair one usually refers (by name or number) to the theorem when using the proof of this theorem. This is usually acceptable in the case of theorems because the future use of their proofs is such that only the fact that there is a proof but not the particulars of the proof matter.

[^0]In the case of a Problem-Construction pair the content of the construction often matters in the future use. Because of this we often have to refer to the construction and not to the problem and we assign in this paper numbers both to Problems and to the Constructions.
Following the approach used in [8] we write the composition of morphisms in categories in the diagrammatic order, i.e., for $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ their composition is written as $f \circ g$. This makes it much easier to translate between diagrams and equations involving morphisms.
This paper is based almost entirely on the material of [6]. I am grateful to The Centre for Quantum Mathematics and Computation (QMAC), the Mathematical Institute of the University of Oxford for their hospitality during my work on the present version of the paper.

## 2 Construction of $C C(\mathcal{C}, p)$.

Definition 2.1 Let $\mathcal{C}$ be a (pre-)category $\Psi^{\boxplus \text {. A }}$ universe structure on a morphism $p: \widetilde{U} \rightarrow U$ in $\mathcal{C}$ is a mapping that assigns to any morphism $f: X \rightarrow U$ in $\mathcal{C}$ a pull-back square

$A$ universe in $\mathcal{C}$ is a morphism $p$ together with a universe structure on it.

In what follows we will write $\left(X ; f_{1}, \ldots, f_{n}\right)$ for $\left(\ldots\left(\left(X ; f_{1}\right) ; f_{2}\right) \ldots ; f_{n}\right)$.

Definition 2.2 $A$ (pre-)category with a universe is a triple $(\mathcal{C}, p, f t)$ where $\mathcal{C}$ is a (pre)category, $p: \widetilde{U} \rightarrow U$ is a morphism in $\mathcal{C}$ with a universe structure on it and pt is a final object in $\mathcal{C}$.

Problem 2.3 For each (pre-) category with a universe ( $\mathcal{C}, p, p t$ ) to define a $C$-system $C C=$ $C C(\mathcal{C}, p)$.

Construction 2.4 We define the set of objects of $C C$ as the set of sequences of the form $\left(F_{1}, \ldots, F_{n}\right)$ where $F_{1} \in \operatorname{Hom}_{\mathcal{C}}(p t, U)$ and $F_{i+1} \in \operatorname{Hom}_{\mathcal{C}}\left(\left(p t ; F_{1}, \ldots, F_{i}\right), U\right)$. Morphisms from $\left(G_{1}, \ldots, G_{n}\right)$ to $\left(F_{1}, \ldots, F_{m}\right)$ are given by

$$
\operatorname{Hom}_{C C}\left(\left(G_{1}, \ldots, G_{n}\right),\left(F_{1}, \ldots, F_{m}\right)\right)=\operatorname{Hom}_{\mathcal{C}}\left(\left(p t ; G_{1}, \ldots, G_{n}\right),\left(p t ; F_{1}, \ldots, F_{m}\right)\right)
$$

and units and compositions are defined as units and compositions in $\mathcal{C}$ such that the mapping $\left(F_{1}, \ldots, F_{n}\right) \rightarrow\left(p t ; F_{1}, \ldots, F_{n}\right)$ is a full embedding of the underlying category of $C C$ to $\mathcal{C}$. The image of this embedding consists of objects $X$ for which the canonical morphism $X \rightarrow p t$

[^1]is a composition of morphisms which are (canonical) pull-backs of $p$. We will denote this embedding by int.
The final object of $C C$ is the empty sequence (). The map $f t$ sends $\left(F_{1}, \ldots, F_{n}\right)$ to $\left(F_{1}, \ldots, F_{n-1}\right)$. The canonical morphism $p_{\left(F_{1}, \ldots, F_{n}\right)}$ is the projection
$$
p_{\left(\left(p t ; F_{1}, \ldots, F_{n-1}\right) ; F_{n}\right)}:\left(\left(p t ; F_{1}, \ldots, F_{n-1}\right) ; F_{n}\right) \rightarrow\left(p t ; F_{1}, \ldots, F_{n-1}\right)
$$

For an object $\left(F_{1}, \ldots, F_{m+1}\right)$ and a morphism $f:\left(G_{1}, \ldots, G_{n}\right) \rightarrow\left(F_{1}, \ldots, F_{m}\right)$ the canonical pull-back square is of the form

where $p_{F}=p_{\left(F_{1}, \ldots, F_{m+1}\right)}, p_{G}=p_{\left(G_{1}, \ldots, G_{n}, f \circ F_{m+1}\right)}$ and $q(f)$ is the unique morphism such that $q(f) \circ p_{F}=p_{G} \circ f$ and $\operatorname{int}(q(f)) \circ Q\left(F_{m+1}\right)=f \circ Q\left(F_{m+1}\right)$. The unity and composition axioms for the canonical squares follow immediately from the unity and associativity axioms for compositions of morphisms in $C$.

## 3 Functoriality of $C C(\mathcal{C}, p)$.

Definition 3.1 Let $(\mathcal{C}, p, p t)$ and $\left(\mathcal{C}^{\prime}, p^{\prime}, p t^{\prime}\right)$ be (pre-) categories with universes. A functor of categories with universes from $(\mathcal{C}, p, p t)$ to $\left(\mathcal{C}^{\prime}, p^{\prime}, p t^{\prime}\right)$ is a triple $(\Phi, \phi, \widetilde{p h i})$ where $\Phi: \mathcal{C} \rightarrow \mathcal{C}^{\prime}$ is a functor and $\phi: \Phi(U) \rightarrow U^{\prime}, \widetilde{\phi}: \Phi(\widetilde{U}) \rightarrow \widetilde{U}^{\prime}$ are morphisms such that:

1. $\Phi$ takes pull-back squares based on $p$ to pull-back squares,
2. $\Phi$ takes $p t$ to a final object of $\mathcal{C}^{\prime}$,
3. the square

is a pull-back square.

Problem 3.2 Let

$$
(\Phi, \phi, \widetilde{\phi}):(\mathcal{C}, p, p t) \rightarrow\left(\mathcal{C}^{\prime}, p^{\prime}, p t^{\prime}\right)
$$

be a functor of universes with categories. To define a homomorphism $H=H(\Phi, \phi, \widetilde{\phi})$ from $C C(\mathcal{C}, p)$ to $C C\left(\mathcal{C}^{\prime}, p^{\prime}\right)$.

Construction 3.3 Denote by $\psi$ the isomorphism $\psi: p t^{\prime} \rightarrow \Phi(p t)$. We define by induction on $n$ objects $H\left(F_{1}, \ldots, F_{n}\right) \in C C\left(\mathcal{C}^{\prime}, p^{\prime}\right)$ and isomorphisms

$$
\psi_{\left(F_{1}, \ldots, F_{n}\right)}: \operatorname{int}^{\prime}\left(H\left(F_{1}, \ldots, F_{n}\right)\right) \rightarrow \Phi\left(\operatorname{int}\left(F_{1}, \ldots F_{n}\right)\right)
$$

where int and int ${ }^{\prime}$ are the canonical functors $C C(\mathcal{C}, p) \rightarrow \mathcal{C}$ and $C C\left(\mathcal{C}^{\prime}, p^{\prime}\right) \rightarrow \mathcal{C}^{\prime}$ respectively. For $n=0$ we set $H(())=()$ and $\psi_{()}=\psi$. For $n>0$ let

$$
\left(F_{1}^{\prime}, \ldots, F_{n-1}^{\prime}\right)=H\left(F_{1}, \ldots, F_{n-1}\right)
$$

and let $F_{n}: \operatorname{int}\left(F_{1}, \ldots, F_{n-1}\right) \rightarrow U$. Define $F_{n}^{\prime}$ as the composition

$$
\begin{equation*}
\operatorname{int}^{\prime}\left(F_{1}^{\prime}, \ldots, F_{n-1}^{\prime}\right) \xrightarrow{\psi_{\left(F_{1}, \ldots, F_{n-1}\right)}} \Phi\left(\operatorname{int}\left(F_{1}, \ldots, F_{n-1}\right)\right) \xrightarrow{\Phi\left(F_{n}\right)} \Phi(U) \xrightarrow{\phi} U^{\prime} \tag{2}
\end{equation*}
$$

and let $H\left(F_{1}, \ldots, F_{n}\right)=\left(F_{1}^{\prime}, \ldots, F_{n-1}^{\prime}, F_{n}^{\prime}\right)$. Then

$$
\operatorname{int}^{\prime}\left(H\left(F_{1}, \ldots, F_{n}\right)\right)=\left(i n t^{\prime}\left(H\left(F_{1}, \ldots, F_{n}\right)\right), F_{n}^{\prime}\right)
$$

To define

$$
\psi_{\left(F_{1}, \ldots, F_{n}\right)}: \operatorname{int}^{\prime}\left(H\left(F_{1}, \ldots, F_{n}\right)\right) \rightarrow \Phi\left(\operatorname{int}\left(F_{1}, \ldots, F_{n}\right)\right)
$$

observe that by our conditions on $\phi, \widetilde{\phi}$ and $\Phi$ the squares of the diagram

are pull-back. Therefore there is a unique morphism $\psi_{\left(F_{1}, \ldots, F_{n}\right)}$ such that the diagram

commutes and

$$
\begin{equation*}
\psi_{\left(F_{1}, \ldots, F_{n}\right)} \circ \Phi\left(Q\left(F_{n}\right)\right) \circ \widetilde{\phi}=Q\left(\psi_{\left(F_{1}, \ldots, F_{n-1}\right)} \circ \Phi\left(F_{n}\right) \circ \phi\right) \tag{4}
\end{equation*}
$$

and this morphism is an isomorphism.
To define $H$ on morphisms we use the fact that morphisms $\psi_{\left(F_{1}, \ldots, F_{n}\right)}$ are isomorphisms and for $f:\left(F_{1}, \ldots, F_{n}\right) \rightarrow\left(G_{1}, \ldots, G_{m}\right)$ we set

$$
\begin{equation*}
H(f)=\psi_{\left(F_{1}, \ldots, F_{n}\right)} \circ \Phi(f) \circ \psi_{\left(G_{1}, \ldots, G_{m}\right)}^{-1} \tag{5}
\end{equation*}
$$

The fact that this construction gives a functor i.e. satisfies the unity and composition axioms is straightforward.

It remains to verify that it respects the rest of the operations of the C-system. It is clear that it respects the length function and the $f t$ maps. The fact that it takes the canonical projections to canonical projections is equivalent to the commutativity of the left hand side square in (3).

Consider a canonical square of the form (ㅍ). Its image is a square of the form

$$
\begin{array}{rlr}
\left(G_{1}^{\prime}, \ldots, G_{n}^{\prime}, G_{n+1}^{\prime}\right) & \xrightarrow{H(q(f))}\left(F_{1}^{\prime}, \ldots, F_{m+1}^{\prime}\right) \\
\quad H\left(p_{G}\right) \downarrow & & \downarrow H\left(p_{F}\right)  \tag{6}\\
\left(G_{1}^{\prime}, \ldots, G_{n}^{\prime}\right) & \xrightarrow{H(f)} & \left(F_{1}^{\prime}, \ldots, F_{m}^{\prime}\right)
\end{array}
$$

We already know that the vertical arrows are canonical projections. Therefore, in order to prove that (G) is a canonical square in $C C\left(\mathcal{C}^{\prime}, p^{\prime}\right)$ we have to show that $G_{n+1}^{\prime}=\operatorname{int}(H(f)) \circ$ $F_{m+1}^{\prime}$ and

$$
\begin{equation*}
\operatorname{int}(H(q(f))) \circ Q\left(F_{m+1}^{\prime}\right)=Q\left(\operatorname{int}(H(f)) \circ F_{m+1}^{\prime}\right) \tag{7}
\end{equation*}
$$

By (Z) we have

$$
\begin{gathered}
G_{n+1}^{\prime}=\psi_{\left(G_{1}, \ldots, G_{n}\right)} \circ \Phi\left(F_{m+1} f\right) \circ \phi \\
F_{m+1}^{\prime}=\psi_{\left(F_{1}, \ldots, F_{m}\right)} \circ \Phi\left(F_{m+1}\right) \circ \phi
\end{gathered}
$$

and by (回)

$$
\begin{gathered}
\operatorname{int}(H(f))=\psi_{\left(G_{1}, \ldots, G_{n}\right)} \circ \Phi(f) \circ \psi_{\left(F_{1}, \ldots, F_{m}\right)}^{-1} \\
\operatorname{int}(H(q(f)))=\psi_{\left(G_{1}, \ldots, G_{n}, F_{m+1} f\right)} \circ \Phi(q(f)) \circ \psi_{\left(F_{1}, \ldots, F_{m+1}\right)}^{-1}
\end{gathered}
$$

Therefore the relation $G_{n+1}^{\prime}=\operatorname{int}(H(f)) \circ F_{m+1}^{\prime}$ follows immediately and the relation (ШI) follows by application of ( $\mathbb{Z}$ ).

Lemma 3.4 Let $(\Phi, \phi, \widetilde{\phi})$ be as in Problem 3.7 and let $H$ be the corresponding solution of Construction [3.3. Then if $\Phi$ is a full embedding and $\phi$ and $\widetilde{\phi}$ are isomorphisms then $H$ is an isomorphism of $C$-systems.

Proof: Straightforward.

Lemma $\sqrt{3.4}$ implies in particular that considered up to a canonical isomorphism $C C(\mathcal{C}, p)$ depends only on the equivalence class of the pair ( $\mathcal{C}, p$ ) i.e. that our construction maps the type of pairs $(\mathcal{C}, p)$ to the type of C -systems.

Remark 3.5 As far as I know this is the only known functorial construction that generates a model of one of the essentially-algebraic theories that are connected to the syntax of dependent type theories from a category-level data in a functorial way. The use of representable morphisms of presheaves in [Z] does not provide set level objects defined up to an isomorphism. Even when a particular representability structure is chosen as is done in the original definition of categories with families one still does not obtain an object defined up to an isomorphism when one considers the underlying category up to an equivalence.

Let us describe now a construction which shows that any C-system is isomorphic to a Csystem of the form $C C(\mathcal{C}, p)$.

Problem 3.6 Let $C C$ be a C-system. Construct a pair $(\mathcal{C}, p)$ as above and an isomorphism $C C \cong C C(\mathcal{C}, p)$.

Construction 3.7 Denote by PreShv $(C C)$ the category of contravariant functors from the category underlying $C C$ to Sets.

Let $T y$ be the functor which takes an object $\Gamma \in C C$ to the set

$$
T y(\Gamma)=\left\{\Gamma^{\prime} \in C C \mid f t\left(\Gamma^{\prime}\right)=\Gamma\right\}
$$

and a morphism $f: \Delta \rightarrow \Gamma$ to the map $\Gamma^{\prime} \mapsto f^{*} \Gamma^{\prime}$. It is a functor due to the composition and unity axioms for $f^{*}$. Let $T m$ be the functor which takes an object $\Gamma$ to the set

$$
\operatorname{Tm}(\Gamma)=\{s \in \widetilde{C C} \mid f t \partial(s)=\Gamma\}
$$

and a morphism $f: \Delta \rightarrow \Gamma$ to the map $s \mapsto f^{*}(s)$ where $f^{*}(s)$ (or $f^{*}(s, 1)$ in the notation of [ 8$]$ ) is the pull-back of the section $s$ along $f$. Let further $p: T m \rightarrow T y$ be the morphism which takes $s$ to $\partial(s)$. It is well defined as a morphisms of families of sets and forms a morphism of presheaves since $\partial\left(f^{*}(s)\right)=f^{*}(\partial(s))$.
Let us construct an isomorphism $C C \cong C C(\operatorname{PreShv}(C C), p)$.
In what follows we identify objects of $C C$ with the corresponding representable presheaves and, for a presheaf $F$ and an object $\Gamma$, we identify morphisms $\Gamma \rightarrow F$ in $\operatorname{PreShv}(C C)$ with $F(\Gamma)$. Recall that for $X \in C C$ such that $l(X)>0$ we let $\delta(X): X \rightarrow p_{X}^{*}(X)$ denote the section of $p_{p_{X}^{*}(X)}$ given by the diagonal.

Lemma 3.8 Let $\Gamma^{\prime} \in O b(C C)$ and let $\Gamma=f t\left(\Gamma^{\prime}\right)$. Then the square

is a pull-back square.

Proof: We have to show that for any $\Delta \in C C$ the obvious map

$$
\operatorname{Hom}\left(\Delta, \Gamma^{\prime}\right) \rightarrow \operatorname{Hom}(\Delta, \Gamma) \times_{T y(\Delta)} \operatorname{Tm}(\Delta)
$$

is a bijection. Let $f_{1}, f_{2}: \Delta \rightarrow \Gamma^{\prime}$ be two morphisms such that their images under ( ( ) coincide i.e. such that $f_{1} \circ p_{\Gamma^{\prime}}=f_{2} \circ p_{\Gamma^{\prime}}$ and $f_{1}^{*}\left(\delta\left(\Gamma^{\prime}\right)\right)=f_{2}^{*}\left(\delta(\Gamma)^{\prime}\right)$. These two conditions are equivalent to saying, in the notation of [ 8$]$, that $f t\left(f_{1}\right)=f t\left(f_{2}\right)$ and $s_{f_{1}}=s_{f_{2}}$. This implies that $f_{1}=f_{2}$. Let $f: \Delta \rightarrow \Gamma$ be a morphism and $s \in \operatorname{Tm}(\Delta)$ a section such that
$f t(\partial(s))=f^{*}\left(\Gamma^{\prime}\right)$. Then the composition $s \circ q\left(f, \Gamma^{\prime}\right)$ is a morphism $f^{\prime}: \Delta \rightarrow \Gamma^{\prime}$ such that $f^{\prime} \circ p_{\Gamma^{\prime}}=f$. We also have

$$
\left(f^{\prime}\right)^{*}\left(\delta\left(\Gamma^{\prime}\right)\right)=s^{*}\left(q\left(f, \Gamma^{\prime}\right)^{*}\left(\delta\left(\Gamma^{\prime}\right)\right)\right)=s
$$

which proves that (8) is surjective.

To construct the required isomorphism we now choose a universe structure on $p$ such that the pull-back squares associated with morphisms from representable objects are squares ( $(\mathbb{})$ ). The isomorphism is now obvious.

Another example of how this construction can be used produces a C-system from a precategory $C$ with a final object $p t$ and fiber products. This example was inspired by reading [2] and by a question from an anonymous referee of [8].
Given a pre-category $C$ as above consider the category $\operatorname{PreShv}(C)$ of presheaves of sets on $C$. Let $U$ be the presheaf that takes $X$ to the set of all pairs of morphisms $(f, g)$ such that $f: X \rightarrow Y$ and $g: Z \rightarrow Y$. The functoriality is defined by compositing $f$. Similarly let $\widetilde{U}$ be the presheaf that takes $X$ to the set of all pairs of morphisms $\left(f^{\prime}, g\right)$ such that $f^{\prime}: X \rightarrow Z, \underset{\widetilde{U}}{g}: Z \rightarrow Y$ and functoriality is again through composition of $f^{\prime}$. There is a morphism $p: U \rightarrow U$ that takes $\left(f^{\prime}, g\right)$ to $\left(f^{\prime} \circ g, g\right)$. A square

commutes if $g^{\prime}=g$ and $u \circ f=f^{\prime} \circ g^{\prime}$. It is a pull-back square if the square

is a pull-back square. In particular, if $C$ has pull-backs then the C-system $C C(\operatorname{PreShv}(C), p)$ is well defined. Note that while this construction does not require a choice of pull-backs to be well defined up to a canonical isomorphism it is not invariant under equivalences in $C$. If $C$ is replaced by an equivalent but not an isomorphic category the morphism $p$ will be replaced by a morphism that is not isomorphic to it.

Definition 3.9 Let CC be a C-system. A universe model of $C C$ is a collection of data of the following form:

1. A category $\mathcal{C}$,
2. a universe $p: \widetilde{U} \rightarrow U$ in $\mathcal{C}$ and a final object pt of $\mathcal{C}$,
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3. a C-system morphism CC }->CC(\mathcal{C},p)\mathrm{ .
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Conjecture Let $\mathcal{C}$ be a category, $C C$ be a C-system and $M: C C \rightarrow \mathcal{C}$ a functor such that $M\left(p t_{C C}\right)$ is a final object of $\mathcal{C}$ and $M$ maps distinguished squares of $C C$ to pull-back squares of $\mathcal{C}$. Then there exists a universe $p_{M}: \widetilde{U}_{M} \rightarrow U_{M}$ in $\operatorname{PreShv}(\mathcal{C})$ and a C-system morphism $M^{\prime}: C C \rightarrow C C\left(\operatorname{PreShv}(\mathcal{C}), p_{M}\right)$ such that the square

where the right hand side vertical arrow is the Yoneda embedding, commutes up to a natural isomorphism.

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[^0]:    ${ }^{1} 2000$ Mathematical Subject Classification: 03B15, 03B22, 03F50, 03G25
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[^1]:    ${ }^{4}$ For the difference between a category and a precategory see the introduction to $[\mathbb{B}]$ and $[\mathbb{T}]$.

