A C-system defined by a universe category[1]

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Abstract

This is the third paper in a series started in [7]. In it we construct a C-system $CC(\mathcal{C}, p)$ starting from a category $\mathcal{C}$ together with a morphism $p : \widetilde{U} \to U$, a choice of pull-back squares based on p for all morphisms to U and a choice of a final object of $\mathcal{C}$. Such a quadruple is called a universe category. We then define universe category functors and construct homomorphisms of C-systems $CC(\mathcal{C}, p)$ defined by universe category functors. As a corollary of this construction and its properties we show that the C-systems corresponding to different choices of pull-backs and final objects are constructively isomorphic.

1 Introduction

The concept of a C-system in its present form was introduced in [7]. The type of the C-systems is constructively equivalent to the type of contextual categories defined by Cartmell in [3] and [2] but the definition of a C-system is slightly different from the Cartmell's foundational definition.

In [6] we constructed for any pair (R, LM) where R is a monad on $Sets$ and LM a left R-module with values in $Sets$ a C-system $CC(R, LM)$. In the particular case of pairs (R, LM) corresponding to signatures as in [4, p.228] or to nominal signatures the regular sub-quotients of $CC(R, LM)$ are the C-systems corresponding to dependent type theories of the Martin-Lof genus.

In this paper we describe another construction that generates C-systems. This time the input data is a quadruple that consists of a category $\mathcal{C}$, a morphism $p : \widetilde{U} \to U$ in this category, a choice of pull-back squares based on p for all morphisms to U and a choice of a final object in $\mathcal{C}$. Such a quadruple is called a universe category. For any universe category we construct a C-system that we denote by $CC(\mathcal{C}, p)$.

We then define the notion of a universe category functor and construct homomorphisms of C-systems of the form $CC(\mathcal{C}, p)$ corresponding to universe category functors. For universe category functors satisfying certain conditions these homomorphisms are isomorphisms. In particular, any equivalence $F : \mathcal{C} \to \mathcal{C}'$ together with an isomorphism $F(p) \cong p'$ (in the category of morphsims) defines a universe category functor whose associated homomorphism of C-systems is an isomorphism.

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To the best of our knowledge it is the only known construction of a C-system from a category level data that transforms equivalences into isomorphisms. Because of this fact we find it important to present both the construction of the C-system and the construction of the homomorphisms defined by universe functors in detail.

To avoid the abuse of language inherent in the use of the Theorem-Proof style of presenting mathematics when dealing with constructions we use the pair of names Problem-Construction for the specification of the goal of a construction and the description of the particular solution.

In the case of a Theorem-Proof pair one usually refers (by name or number) to the statement when using both the statement and the proof. This is acceptable in the case of theorems because the future use of their proofs is such that only the fact that there is a proof but not the particulars of the proof matter.

In the case of a Problem-Construction pair the content of the construction often matters in the future use. Because of this we often have to refer to the construction and not to the problem and we assign in this paper numbers both to Problems and to the Constructions.

Following the approach used in [7] we write the composition of morphisms in categories in the diagrammatic order, i.e., for $f : X \to Y$ and $g : Y \to Z$ their composition is written as $f \circ g$. This makes it much easier to translate between diagrams and equations involving morphisms.

The methods of this paper are fully constructive.

We use the word "category" to refer to that which in the univalent formalization may be replaced by the concept of a precategory (see [1]). However, due to the invariance of our constructions under equivalences all of them should factor through the Rezk completion. This invariance also makes the use of the word "category" consistent with the practice suggested in the introduction to [7].

This paper is based almost entirely on the material of [5]. I am grateful to The Centre for Quantum Mathematics and Computation (QMAC) and the Mathematical Institute of the University of Oxford for their hospitality during my work on the previous version of the paper and to the Department of Computer Science and Engineering of the University of Gothenburg and Chalmers University of Technology for its the hospitality during my work on the present version.

2 Construction of $CC(\mathcal{C}, p)$.

Definition 2.1 *Let $\mathcal{C}$ be a category. A universe structure on a morphism $p : \widetilde{U} \to U$ in $\mathcal{C}$ is a mapping that assigns to any morphism $f : X \to U$ in $\mathcal{C}$ a pull-back square*

$$\begin{array}{ccc} (X; f) & \xrightarrow{Q(f)} & \widetilde{U} \\ {\scriptstyle p_{X,f}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & U \end{array}$$

A universe in $\mathcal{C}$ is a morphism p together with a universe structure on it.

In what follows we will write $(X; f_1, \dots, f_n)$ for $(\dots((X; f_1); f_2)\dots; f_n)$.

Example 2.2 Let G be a group. Consider the category BG with one object pt whose monoid of endomorphisms is G. Recall that any commutative square where all four arrows are isomorphisms is a pull-back square. Let $p : pt \to pt$ be the unit object of G. Then a universe structure on p can be defined by specifying, for every $g : pt \to pt$, of the horizontal morphism $Q(g)$ in the corresponding canonical square. There are no restrictions on the choice of $Q(g)$ since for any such choice one can take the vertical morphism to be $Q(g)g^{-1}$ obtaining a pull-back square. Therefore, the set of universe structures on p is G^G. The automorphisms of BG are given by $Aut(G)$ (with two automorphisms being isomorphic as functors if they differ by an inner automorphisms of G). Therefore, there are $(G^G)/Aut(G)$ isomorphism classes of categories with universes with the underlying category BG and the underlying universe morphism being $Id : pt \to pt$. Note that in this case all auto-equivalences of the category are automorphisms and so simply saying that we will consider universes up to an equivalence of the underlying category does not change the answer. To have, as is suggested by category-theoretic intuition, no more than one universe structure on a morphism one needs to consider categories with universes up to equivalences of categories with universes and then one has the obligation to prove that the constructions that are supposed to produce objects such as C-systems map equivalences of categories with universes to isomorphisms. In the case of the main construction of this paper it is achieved in Lemma 3.4 and with respect to universe category functors of a somewhat wider class than the class of universe category equivalences.

Definition 2.3 *A universe category is a triple $(\mathcal{C}, p, pt)$ where $\mathcal{C}$ is a category, $p : \widetilde{U} \to U$ is a morphism in $\mathcal{C}$ with a universe structure on it and pt is a final object in $\mathcal{C}$.*

Problem 2.4 *For each universe category $(\mathcal{C}, p, pt)$ to define a C-system $CC = CC(\mathcal{C}, p)$.*

Construction 2.5 The set of objects of CC is the set of sequences of the form $(F_1, \dots, F_n)$ where $F_1 \in Hom(pt, U)$ and $F_{i+1} \in Hom((pt; F_1, \dots, F_i), U)$. Morphisms from $(G_1, \dots, G_n)$ to $(F_1, \dots, F_m)$ are given by

$$Hom_{CC}((G_1, \dots, G_n), (F_1, \dots, F_m)) = Hom_{\mathcal{C}}((pt; G_1, \dots, G_n), (pt; F_1, \dots, F_m))$$

and units and compositions are defined as units and compositions in $\mathcal{C}$ such that the mapping $(F_1, \dots, F_n) \to (pt; F_1, \dots, F_n)$ is a full embedding of the underlying category of CC to $\mathcal{C}$. The image of this embedding consists of objects X for which the canonical morphism $X \to pt$ is a composition of morphisms which are (canonical) pull-backs of p. We will denote this embedding by int.

This construction can be described more formally as follows. One defines, by induction on n, pairs $(Ob_n, int_n : Ob_n \to \mathcal{C})$ where Ob_n is a set and int_n is a function from Ob_n to objects of $\mathcal{C}$. One starts with $Ob_0 = Hom(pt, pt)$ and int_0 mapping Ob_0 to pt. Then

$$Ob_{n+1} = \amalg_{\Gamma \in Ob_n} Hom(int_n(\Gamma), U)$$

and

$$int_{n+1}(\Gamma, F) = (int_n(\Gamma); F)$$

The morphisms in $CC(\mathcal{C}, p)$ are defined by

$$Hom_{CC(\mathcal{C},p)}(\Gamma, \Gamma') := Hom_{\mathcal{C}}(int(\Gamma), int(\Gamma'))$$

The final object of CC is the empty sequence (). The map ft sends $(F_1, \ldots, F_n)$ to $(F_1, \ldots, F_{n-1})$. The canonical morphism $p_{(F_1,\ldots,F_n)}$ is the projection

$$p_{((pt;F_1,\ldots,F_{n-1}),F_n)} : ((pt; F_1, \ldots, F_{n-1}); F_n) \to (pt; F_1, \ldots, F_{n-1})$$

For an object $(F_1, \ldots, F_{m+1})$ and a morphism $f : (G_1, \ldots, G_n) \to (F_1, \ldots, F_m)$ the canonical pull-back square in CC is of the form

$$\begin{array}{ccc} (G_1, \ldots, G_n, f \circ F_{m+1}) & \xrightarrow{q(f)} & (F_1, \ldots, F_{m+1}) \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_F} \\ (G_1, \ldots, G_n) & \xrightarrow{f} & (F_1, \ldots, F_m) \end{array} \tag{1}$$

where $p_F = p_{(F_1,\ldots,F_{m+1})}$, $p_G = p_{(G_1,\ldots,G_n,f\circ F_{m+1})}$ and $q(f)$ is the unique morphism such that $q(f) \circ p_F = p_G \circ f$ and $int(q(f)) \circ Q(F_{m+1}) = f \circ Q(F_{m+1})$. The unity and composition axioms for the canonical squares follow immediately from the unity and associativity axioms for compositions of morphisms in C.

3 Functoriality of $CC(\mathcal{C}, p)$.

Definition 3.1 *Let $(\mathcal{C}, p, pt)$ and $(\mathcal{C}', p', pt')$ be universe categories. A functor of universe categories from $(\mathcal{C}, p, pt)$ to $(\mathcal{C}', p', pt')$ is a triple $(\Phi, \phi, \widetilde{\phi})$ where $\Phi : \mathcal{C} \to \mathcal{C}'$ is a functor and $\phi : \Phi(U) \to U'$, $\widetilde{\phi} : \Phi(\widetilde{U}) \to \widetilde{U}'$ are morphisms such that:*

1. *Φ takes the canonical pull-back squares based on p to pull-back squares,*
2. *Φ takes pt to a final object of $\mathcal{C}'$,*
3. *the square*

$$\begin{array}{ccc} \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

   is a pull-back square.

Problem 3.2 *Let*

$$(\Phi, \phi, \widetilde{\phi}) : (\mathcal{C}, p, pt) \to (\mathcal{C}', p', pt')$$

be a functor of universes categories. To define a homomorphism $H = H(\Phi, \phi, \widetilde{\phi})$ from $CC(\mathcal{C}, p)$ to $CC(\mathcal{C}', p')$.

Construction 3.3 Denote by ψ the isomorphism $\psi : pt' \to \Phi(pt)$. We define by induction on n objects $H(F_1, \ldots, F_n) \in CC(\mathcal{C}', p')$ and isomorphisms

$$\psi_{(F_1,\ldots,F_n)} : int'(H(F_1, \ldots, F_n)) \to \Phi(int(F_1, \ldots F_n))$$

where int and int' are the canonical functors $CC(\mathcal{C}, p) \to \mathcal{C}$ and $CC(\mathcal{C}', p') \to \mathcal{C}'$ respectively. For $n = 0$ we set $H(()) = ()$ and $\psi_{()} = \psi$. For $n > 0$ let

$$(F_1', \ldots, F_{n-1}') = H(F_1, \ldots, F_{n-1})$$

and let $F_n : int(F_1, \ldots, F_{n-1}) \to U$. Define F_n' as the composition

$$int'(F_1', \ldots, F_{n-1}') \stackrel{\psi_{(F_1,\ldots,F_{n-1})}}{\longrightarrow} \Phi(int(F_1, \ldots, F_{n-1})) \stackrel{\Phi(F_n)}{\to} \Phi(U) \stackrel{\phi}{\to} U' \tag{2}$$

and let $H(F_1, \ldots, F_n) = (F_1', \ldots, F_{n-1}', F_n')$. Then

$$int'(H(F_1, \ldots, F_n)) = (int'(H(F_1, \ldots, F_n)), F_n')$$

To define

$$\psi_{(F_1,\ldots,F_n)} : int'(H(F_1, \ldots, F_n)) \to \Phi(int(F_1, \ldots, F_n))$$

observe that by our conditions on $\phi, \widetilde{\phi}$ and Φ the squares of the diagram

$$\begin{array}{ccccc}
\Phi(int(F_1, \ldots, F_n)) & \xrightarrow{\Phi(Q(F_n))} & \Phi(\widetilde{U}) & \longrightarrow & \widetilde{U}' \\
\downarrow & & \downarrow & & \downarrow \\
\Phi(int(F_1, \ldots, F_{n-1})) & \xrightarrow{\Phi(F_n)} & \Phi(U) & \xrightarrow{\phi} & U'
\end{array}$$

are pull-back. Therefore there is a unique morphism $\psi_{(F_1,\ldots,F_n)}$ such that the diagram

$$\begin{array}{ccccc}
int'(H(F_1, \ldots, F_n)) & \xrightarrow{\psi_{(F_1,\ldots,F_n)}} & \Phi(int(F_1, \ldots, F_n)) & \xrightarrow{\Phi(Q(F_n))\circ\widetilde{\phi}} & \widetilde{U}' \\
\downarrow & & \downarrow & & \downarrow \\
int'(H(F_1, \ldots, F_{n-1})) & \xrightarrow{\psi_{(F_1,\ldots,F_{n-1})}} & \Phi(int(F_1, \ldots, F_{n-1})) & \xrightarrow{\Phi(F_n)\circ\phi} & U'
\end{array} \tag{3}$$

commutes and

$$\psi_{(F_1,\ldots,F_n)} \circ \Phi(Q(F_n)) \circ \widetilde{\phi} = Q(\psi_{(F_1,\ldots,F_{n-1})} \circ \Phi(F_n) \circ \phi) \tag{4}$$

and this morphism is an isomorphism.

To define H on morphisms we use the fact that morphisms $\psi_{(F_1,\ldots,F_n)}$ are isomorphisms and for $f : (F_1, \ldots, F_n) \to (G_1, \ldots, G_m)$ we set

$$H(f) = \psi_{(F_1,\ldots,F_n)} \circ \Phi(f) \circ \psi^{-1}_{(G_1,\ldots,G_m)} \tag{5}$$

The fact that this construction gives a functor i.e. satisfies the unity and composition axioms is straightforward.

It remains to verify that it respects the rest of the operations of the C-system. It is clear that it respects the length function and the ft maps. The fact that it takes the canonical projections to canonical projections is equivalent to the commutativity of the left hand side square in (3).

Consider a canonical square of the form (1). Its image is a square of the form

$$\begin{array}{ccc} (G'_1,\ldots,G'_n,G'_{n+1}) & \xrightarrow{H(q(f))} & (F'_1,\ldots,F'_{m+1}) \\ {\scriptstyle H(p_G)}\downarrow & & \downarrow{\scriptstyle H(p_F)} \\ (G'_1,\ldots,G'_n) & \xrightarrow{H(f)} & (F'_1,\ldots,F'_m) \end{array} \tag{6}$$

We already know that the vertical arrows are canonical projections. Therefore, in order to prove that (6) is a canonical square in $CC(\mathcal{C}',p')$ we have to show that $G'_{n+1} = int(H(f)) \circ F'_{m+1}$ and

$$int(H(q(f))) \circ Q(F'_{m+1}) = Q(int(H(f)) \circ F'_{m+1}) \tag{7}$$

By (2) we have

$$G'_{n+1} = \psi_{(G_1,\ldots,G_n)} \circ \Phi(F_{m+1}f) \circ \phi$$
$$F'_{m+1} = \psi_{(F_1,\ldots,F_m)} \circ \Phi(F_{m+1}) \circ \phi$$

and by (5)

$$int(H(f)) = \psi_{(G_1,\ldots,G_n)} \circ \Phi(f) \circ \psi^{-1}_{(F_1,\ldots,F_m)}$$
$$int(H(q(f))) = \psi_{(G_1,\ldots,G_n,F_{m+1}f)} \circ \Phi(q(f)) \circ \psi^{-1}_{(F_1,\ldots,F_{m+1})}$$

Therefore the relation $G'_{n+1} = int(H(f)) \circ F'_{m+1}$ follows immediately and the relation (7) follows by application of (4).

Lemma 3.4 *Let $(\Phi,\phi,\widetilde{\phi})$ be as in Problem 3.2 and let H be the corresponding solution of Construction 3.3. Then if Φ is a full embedding and ϕ and $\widetilde{\phi}$ are isomorphisms then H is an isomorphism of C-systems.*

Proof: Straightforward.

Lemma 3.4 implies in particular that considered up to a canonical isomorphism $CC(\mathcal{C},p)$ depends only on the equivalence class of the pair $(\mathcal{C},p)$ i.e. that our construction maps the type of pairs $(\mathcal{C},p)$ to the type of C-systems.

Let us describe now a construction which shows that any C-system is isomorphic to a C-system of the form $CC(\mathcal{C},p)$.

Problem 3.5 *Let CC be a C-system. Construct a universe category $(\mathcal{C},p)$ and an isomorphism $CC \cong CC(\mathcal{C},p)$.*

Construction 3.6 Denote by $PreShv(CC)$ the category of contravariant functors from the category underlying CC to $Sets$.

Let Ty be the functor which takes an object $\Gamma \in CC$ to the set

$$Ty(\Gamma) = \{\Gamma' \in CC \,|\, ft(\Gamma') = \Gamma\}$$

and a morphism $f : \Delta \to \Gamma$ to the map $\Gamma' \mapsto f^*\Gamma'$. It is a functor due to the composition and unity axioms for f^*. Let Tm be the functor which takes an object Γ to the set

$$Tm(\Gamma) = \{s \in \widetilde{CC} \,|\, ft\,\partial(s) = \Gamma\}$$

and a morphism $f : \Delta \to \Gamma$ to the map $s \mapsto f^*(s)$ where $f^*(s)$ (or $f^*(s, 1)$ in the notation of [7]) is the pull-back of the section s along f. Let further $p : Tm \to Ty$ be the morphism which takes s to $\partial(s)$. It is well defined as a morphisms of families of sets and forms a morphism of presheaves since $\partial(f^*(s)) = f^*(\partial(s))$.

Let us construct an isomorphism $CC \cong CC(PreShv(CC), p)$.

In what follows we identify objects of CC with the corresponding representable presheaves and, for a presheaf F and an object Γ, we identify morphisms $\Gamma \to F$ in $PreShv(CC)$ with $F(\Gamma)$. Recall that for $X \in CC$ such that $l(X) > 0$ we let $\delta(X) : X \to p_X^*(X)$ denote the section of $p_{p_X^*(X)}$ given by the diagonal.

Lemma 3.7 *Let $\Gamma' \in Ob(CC)$ and let $\Gamma = ft(\Gamma')$. Then the square*

$$\begin{tikzcd} \Gamma' \arrow[r, "\delta(\Gamma')"] \arrow[d, "p_{\Gamma'}"'] & Tm \arrow[d, "p"] \\ \Gamma \arrow[r, "\Gamma'"] & Ty \end{tikzcd} \tag{8}$$

is a pull-back square.

Proof: We have to show that for any $\Delta \in CC$ the obvious map

$$Hom(\Delta, \Gamma') \to Hom(\Delta, \Gamma) \times_{Ty(\Delta)} Tm(\Delta)$$

is a bijection. Let $f_1, f_2 : \Delta \to \Gamma'$ be two morphisms such that their images under (8) coincide i.e. such that $f_1 \circ p_{\Gamma'} = f_2 \circ p_{\Gamma'}$ and $f_1^*(\delta(\Gamma')) = f_2^*(\delta(\Gamma)')$. These two conditions are equivalent to saying, in the notation of [7], that $ft(f_1) = ft(f_2)$ and $s_{f_1} = s_{f_2}$. This implies that $f_1 = f_2$. Let $f : \Delta \to \Gamma$ be a morphism and $s \in Tm(\Delta)$ a section such that $ft(\partial(s)) = f^*(\Gamma')$. Then the composition $s \circ q(f, \Gamma')$ is a morphism $f' : \Delta \to \Gamma'$ such that $f' \circ p_{\Gamma'} = f$. We also have

$$(f')^*(\delta(\Gamma')) = s^*(q(f, \Gamma')^*(\delta(\Gamma'))) = s$$

which proves that (8) is surjective.

To construct the required isomorphism we now choose a universe structure on p such that the pull-back squares associated with morphisms from representable objects are squares (8). The isomorphism is now obvious.

Example 3.8 We can use Construction 2.5 to produce a C-system from a pre-category C with a final object pt and fiber products. This example was inspired by a question from an anonymous referee of [7]. Here we have to use the word "pre-category" since this construction, unlike all other constructions of this paper, is not invariant under equivalences.

Given a pre-category C with a final object and fiber products consider the category of presheaves of sets on C. Let U be the presheaf that takes X to the set of all pairs of morphisms (f, g) such that $f : X \to Y$ and $g : Z \to Y$. The functoriality is defined by compositing f. Similarly let $\widetilde{U}$ be the presheaf that takes X to the set of all pairs of morphisms (f', g) such that $f' : X \to Z$, $g : Z \to Y$ and functoriality is again through composition of f'. There is a morphism $p : \widetilde{U} \to U$ that takes (f', g) to $(f' \circ g, g)$. A square

$$\begin{tikzcd} X' \arrow[r, "{(f',g')}"] \arrow[d, "u"'] & \widetilde{U} \arrow[d, "p"] \\ X \arrow[r, "{(f,g)}"] & U \end{tikzcd}$$

commutes if $g' = g$ and $u \circ f = f' \circ g'$. It is a pull-back square if the square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "u"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a pull-back square. In particular, if C has pull-backs then the C-system $CC(PreShv(C), p)$ is well defined.

Note that this construction is not invariant under equivalences in C. If C is replaced by an equivalent but not an isomorphic category the morphism p will be replaced by a morphism that is not isomorphic to it.

On the other hand the change in the choice of pull-backs without a change in C will lead to the change of the C-system by a constructively isomorphic one,

Definition 3.9 *Let CC be a C-system. A universe model of CC is a pair of a universe category $(\mathcal{C}, p)$ and a C-system homomorphism $CC \to CC(\mathcal{C}, p)$.*

Conjecture Let $\mathcal{C}$ be a category, CC be a C-system and $M : CC \to \mathcal{C}$ a functor such that $M(pt_{CC})$ is a final object of $\mathcal{C}$ and M maps distinguished squares of CC to pull-back squares of $\mathcal{C}$. Then there exists a universe $p_M : \widetilde{U}_M \to U_M$ in $PreShv(\mathcal{C})$ and a C-system homomorphism $M' : CC \to CC(PreShv(\mathcal{C}), p_M)$ such that the square

$$\begin{tikzcd} CC \arrow[r, "M"] \arrow[d, "M'"] & \mathcal{C} \arrow[d] \\ CC(PreShv(\mathcal{C}), p_M) \arrow[r, "int"] & PreShv(C) \end{tikzcd}$$

where the right hand side vertical arrow is the Yoneda embedding, commutes up to a functor isomorphism.

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