Sheaves and sheaves with transfers in the cdh-topology.

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Contents

1 Introduction.

These unfinished notes is a byproduct of the paper on cohomological operations. It can be developed into a paper about general properties of sheaves and sheaves with transfers in the cdh-topology on the categories of schemes over a base. In [**?**] I have choosen to work with the Nisnevich (upper cd-) topology instead. The plan for possible further results looks as follows.

1. Prove that the full subcategory of $D^{-}(Shv_{cdh}(Cor(Sch/S)))$ generated by finite complexes of sheaves of the form $\mathbf{Z}_{tr}(X)$ is the localization of the homotopy category of bounded complexes over $Cor(Sch/S)$ with respect to short distinguished complexes. The key step is to show that the localizing subcategory in $D^{-}(PreShv(-))$ generated by these complexes coincides with the subcategory of complexes qiuasi-isomorphic to zero in the cdh-topology.

2. Prove that $D^{\leq 0}(Sch_{cdh})$ is a localization of $Compl^{\leq 0}$.

3. Prove that $D^{\leq 0}_{finite}$ i.e. the subcategory generated by finite complexes over $Cor(Sch/S)$ concentrated in non negative homological degrees is a localization of $Compl^{\leq 0}_{finite}(Cor(Sch/S))$ with respect to the smallest class of morphisms containing what comes from the short distinguished complexes and closed uder formation of total complexes (? no good formulation so far).

4. Prove the following universality theorems:

   (a) Let $F : Cor(S) \to C$ be a functor with values in a simplicial closed model category such that:

      i. $F(X)$ are cofibrant

      ii. For a cdh-covering $p : U \to X$ the morphism $hocolim(F(\check{C}(p))) \to F(X)$ is a weak equivalence

   Then there exists a unique functor $DF : D^{\leq 0}_{finite}(Shv_{cdh}()) \to H(C)$ such that for a finite complex C over $Cor(Sch/S)$ one has $DF(C) = hocolim(\Gamma(C))$ where $\Gamma(C)$ is the simplicial object corresponding to C.

   (b) Let $F : Cor(S) \to C$ be a functor with values in a simplicial closed model category such that:

      i. $F(X)$ are cofibrant

      ii. For an elementary cdh-covering $p : U \to X$ the morphism $hocolim(F(\check{C}(p))) \to F(X)$ is a weak equivalence

   Then for ay cdh-covering p the morphism like that is a weak equivalence.

   (c) Let $F : Cor(S) \to C$ be a functor with values in a simplicial closed model category such that:

      i. $F(X)$ are cofibrant

      ii. For a cdh-covering $p : U \to X$ the morphism $hocolim(F(\check{C}(p))) \to F(X)$ is a weak equivalence

   Then there exists a unique functor $DF : D^{\leq 0}(Shv_{cdh}()) \to H(C)$ such that for a complex C over $\bar{Cor}(Sch/S)$ one has $DF(C) = hocolim(\Gamma(C))$ where $\Gamma(C)$ is the simplicial object corresponding to C and $\bar{Cor}$ is the category obtained from Cor by adding infinite direct sums.

1.1 *Finite correspondences for general schemes.*

In this section we describe some general constructions based on [**?**]. For any Noetherian scheme S we define the category $Cor(Sch/S)$ of finite correspondences over S as follows. Objects of this category are schemes of finite type over S. To distinguish a scheme considered as an object of the category of schemes from a scheme considered as an object of the category of correspondences which we are going to construct we denote the later by $[X]$. Morphisms from $[X]$ to $[Y]$ are elements of the abelian group

$$c(X,Y) = c(X \times_S Y/X, 0)$$

defined in [**?**, after Lemma 3.3.9]. The composition of morphisms should be given by homomorphisms of abelian groups

$$c(X,Y) \otimes c(Y,Z) \to c(X,Z)$$

i.e.

$$c(X \times_S Y/X, 0) \to c(Y \times_S Z/Y, 0) \to c(X \times_S Z/X, 0)$$

Denote by $p_X : X \to S$, $p_Y : Y \to S$ the canonical morphisms. Then for $f \in c(X \times_S Y/X, 0)$ and $g \in c(Y \times_S Z/Y, 0)$ we define the composition $g \circ f$ as $(p_Y)_* Cor(cycl(p_X)(g), f)$ where $Cor(-,-)$ is the correspondence homomorphism constructed in [**?**, §3.7].

The lemma below follows immediately from the definition of $Cor(-,-)$ and the fact that the (proper) push-forward commutes $cycl(-)$ homomorphisms ([**?**, Prop. 3.6.2]).

Lemma 1.1.1 *[**missing**] Let $Y \to X \to S$ be a sequence of morphisms of finite type, $p : Y \to Y'$ a morphism over X, $\mathcal{Y} \in Cycl(Y/X, r) \otimes \mathbf{Q}$ and $\mathcal{X} \in Cycl(X/S, s) \otimes \mathbf{Q}$. Assume that p is proper on the support of $\mathcal{Y}$. Then*

$$p_* Cor(\mathcal{Y}, \mathcal{X}) = Cor(p_*(\mathcal{Y}, \mathcal{X})).$$

Proposition 1.1.2 *[**associativity**] For any $f \in c(X,Y)$, $g \in c(Y,Z)$, $h \in c(Z,T)$ one has*

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Proof: Let us write both sides explicitly:

$$(h \circ g) \circ f = (p_Y)_* Cor(cycl(p_X)((p_Z)_* Cor(cycl(p_Y)(h), g)), f)$$

$$h \circ (g \circ f) = (p_Z)_* Cor(cycl(p_X)(h), (p_Y)_* Cor(cycl(p_X)(g), f))$$

For the first expression we have:

$$(p_Y)_* Cor(cycl(p_X)((p_Z)_* Cor(cycl(p_Y)(h), g)), f) =$$

$$= (p_Y)_* Cor((p_Z)_* cycl(p_X) Cor(cycl(p_Y)(h), g), f) =$$

$$= (p_Y)_* (p_Z)_* Cor(cycl(p_X) Cor(cycl(p_Y)(h), g), f) =$$

$$= (p_Y)_* (p_Z)_* Cor(Cor(cycl(p_{X \times_Z Y}(h), cycl(p_X)(g)), f)$$

where the first equality holds by [**?**, Prop. 3.6.2], the second by Lemma 1.1.1 and the third by [**?**, Th. 3.7.3]. For the second expression we have

$$(p_Z)_* Cor(cycl(p_X)(h), (p_Y)_* Cor(cycl(p_X)(g), f)) =$$

$$= (p_Z)_* (p_Y)_* Cor(cycl(p_{X \times_Z Y})(h), Cor(cycl(p_X)(g), f))$$

by [**?**, Lemma 3.7.1]. This too expressions are now equal by [**?**, Prop. 3.7.7].

Lemma 1.1.3 *[***left***] For a morphism of schemes $f : X \to Y$ let Γ_f be its graph considered as an element of $c(X \times_S Y/X, 0)$. Then for any $g \in c(Y \times_S Z/Y, 0)$ we have*

$$g \circ \Gamma_f = cycl(f)(g)$$

Proof: This follows immediately from the defnition of $Cor(-,-)$.

Lemma 1.1.4 *[***right***] For a morphism of schemes $g : Y \to Z$ let Γ_g be its graph considered as an element of $c(Y \times_S Z/Y, 0)$. Then for any $f \in c(X \times_S Y/X, 0)$ we have*

$$\Gamma_g \circ f = (id_X \times g)_*(f)$$

Proof: We have

$$\Gamma_g \circ f = (p_Y)_* Cor(cycl(p_X)(\Gamma_g), f) = (p_Y)_* Cor(id_{X \times_Z Y} \times g)_*(1), f) =$$

$$= (p_Y)_* (Id_{X \times_Z Y} \times g)_* Cor(1, f) = (p_Y)_* (Id_X \times g)_*(f) =$$

$$= (id_X \times g)_*(f)$$

where 1 is the tautological cycle on $X \times_S Y$ over itself and the second equality holds by Lemma 1.1.1.

Applying Lemmas 1.1.3 and 1.1.4 in the case of the identy morphisms we see that the graph of the identity morphism gives a unit for the composition of finite correspondences. Together with Proposition 1.1.2 this proves that we indeed constructed a category. One verifies easily that this category is additive with the direct sum given by $[X] \oplus [Y] = [X \coprod Y]$. This category is denoted by $Cor(Sch/S)$ and is called the category of finite correspondences over S.

Using again Lemmas 1.1.3 and 1.1.4 we see that the mapping which sends a scheme X to $[X]$ and a morphism of schemes $f : X \to Y$ to its graph which we will denote by $[f]$ is a functor from schemes to finite correspondences.

This category has an important version which is called the category of equidimensional finite correspondences $Cor_{equi}(Sch/k)$. It is a subcategory of $Cor(Sch/S)$ which has the same objects as $Cor(Sch/S)$ but where morphisms from $[X]$ to $[Y]$ are given by $c_{equi}(X \times_S Y/Y, 0)$ (see [**?**, after Lemma 3.3.9]). We further define $Cor_{equi}(Sm/S)$ as the full subcategory in $Cor_{equi}(Sch/S)$ which consists of objects of the form $[X]$ for smooth schemes X over S. If $S = Spec(k)$ then $Cor_{equi}(Sm/S)$ is the catgeory $SmCor(k)$ defined in [**?**, §2.1].

1.2 *More on the cdh-topology.*

We start by giving a new definition of the cdh-topology. Its equivalence with the old definition (see [**?**, §4.1]) is proved in Lemma 1.2.5 below.

Definition 1.2.1 *Let $p : U \to X$ be a morphism of schemes of finite type over S. A sequence*

$$\emptyset = Z_{n+1} \subset Z_n \subset Z_{n-1} \subset \ldots \subset Z_0 = X$$

closed subschemes of X is called a a splitting sequence for p if the morphisms $p^{-1}(Z_i - Z_{i+1}) \to Z_i - Z_{i+1}$ have sections.

Definition 1.2.2 *[***upandlow***] The upper cd-topology on Sch/S is the Grothendieck topology generated by coverings of the form $\{p_i : U_i \to U\}$ such that p_i are etale and $\coprod p_i : \coprod U_i \to U$ has a splitting sequence.*

The lower cd-topology on Sch/S is the Grothendieck topology generated by coverings of the form $\{p_i : U_i \to U\}$ such that p_i are proper and $\coprod p_i : \coprod U_i \to U$ has a splitting sequence.

The cdh-topology on Sch/S is the Grothendieck topology generated by the upper and lower cd-topologies.

Remark: In this definition *cd−* is the abbreviation of "completely decomposed" introduced by Nisnevich in []. The adjectives "upper" and "lower" come from [].

The following lemma is straightforward.

Lemma 1.2.3 *[***cdhcov***] A morphism $p : U \to X$ is a cdh-covering if and only if there exists a composable sequence of morphisms $p_1, q_1, \ldots, p_n, q_n$ which goes from a scheme Y to X and a morphism $s : Y \to U$ such that*

1. *the morphisms p_i are proper and have splitting sequences*
2. *the morphisms q_i are etale and have splitting sequences*
3. *$ps = q_n p_n q_{n-1} p_{n-1} \ldots q_1 p_1$*

Lemma 1.2.4 *[***Niscom***] The upper cd-topology is the Nisnevich topology.*

Proof: The fact that any upper cd-covering is a Nisnevich covering is obvious from the definitions. The fact that Nisnevich coverings have splitting sequences is proved in [**?**, Lemma 3.1.5].

Lemma 1.2.5 *[***cdcov***] The definition of the cdh-topology given above is equivalent to the definition given in [***?***, §4.1].*

Proof: For the proof of this lemma let us call the topology defined in [**?**, §4.1] the cdh'-topology. It is obvious from definitions and Lemma 1.2.4 that any cdh'-covering is a cdh-covering. It is also obvious that any etale morphism with a splitting sequence is a cdh'-covering. Thus the only fact which needs a proof is that a proper morphism $p : U \to X$ with a splitting sequence is a cdh'-covering.

Let $Z_1 \subset \ldots \subset Z_n \subset X$ be a a splitting sequence for p. We proceed by induction on the length of this sequence. If its length is zero than p has a section and thus it is a covering in any topology. For the inductive step consider the pull-back square

$$\begin{array}{ccc} U_1 & \stackrel{\tilde{j}_1}{\to} & U \\ \downarrow & & \downarrow \\ X - Z_1 & \stackrel{j_1}{\to} & X \end{array}$$

and let $s : X - Z_1 \to U_1$ be a section which exists by definition of a splitting sequence. Set $V_1 = U - \tilde{j}_1(U_1 - s(X - Z_1))$. Since p is separated $s(X - Z_1)$ is closed and thus V_1 is a closed subset of U. In particular $V_1 \to X$ is proper and one verifies easily that $V_1 \coprod Z_1 \to X$ is an "elementary" cdh'-covering. To show that p is a cdh'-covering it is sufficient now to show that $U \times_X V_1 \to V_1$ and $U \times_X Z_1 \to Z_1$ are cdh'-coverings. The first of these morphisms has a section. The second has a splitting sequence of length less than n.

Definition 1.2.6 *[**eldis**] A Cartesian square in Sch/S of the form*

$$\begin{array}{ccc} Y' & \to & Y \\ \downarrow & & \downarrow p \\ X' & \stackrel{e}{\to} & X \end{array}$$

is called a lower distinguished square if p is proper, e is a closed embedding and $p^{-1}(X - e(X')) \to X - e(X')$ is an isomorphism.
A Cartesian square in Sch/S of the form

$$\begin{array}{ccc} Y' & \to & Y \\ \downarrow & & \downarrow p \\ X' & \stackrel{e}{\to} & X \end{array}$$

is called an upper distinguished square if p is etale, e is an open embedding and $p^{-1}(X - e(X')) \to X - e(X')$ is an isomorphism.

Remark: In [**?**, Def. 3.1.3] the upper distinguished squares where called elementary distinguished squares.

The formulation must be changed for non additive functors!.

Theorem 1.2.7 *[**general**] Let A be an abelian category and $F : Sch/S \to A$ a functor such that for any distinguished square*

$$\begin{array}{ccc} Y' & \to & Y \\ \downarrow & & \downarrow p \\ X' & \stackrel{e}{\to} & X \end{array}$$

the chain complex

$$\ldots \to F((X' \coprod Y)^3_X) \to F((X' \coprod Y)^2_X) \to F(X' \coprod Y) \to F(X)$$

where the differentials are given by alternating sums of partial projections is exact. Then for any cdh-covering $p : U \to X$ the complex

$$\ldots \to F(U \times_X U \times_X U) \to F(U \times_X U) \to F(U) \to F(X)$$

where the differential is given by the alternating sum of partial projections is exact.

Proof: Denote the complex whose exactness we want to prove by $\tilde{C}_F(p)$. In addition for a pair of morphisms $p : U \to X$, $q : V \to X$ denote by $\tilde{C}_F(p,q)$ the bicomplex whose terms are $F(U^m_X \times_X V^n_X)$.

Lemma 1.2.8 *[**splitcase**] Assume that p has a section $s : X \to U$. Then $\tilde{C}_F(p)$ is exact.*

Proof: The morphisms $F(U^n_X) \to F(U^{n+1}_X)$ given by $F(Id \times \ldots Id \times s)$ define a contracting homotopy for this complex. (? signs)

Lemma 1.2.9 *[**uppersplit**] Let $p : U \to X$ be an etale morphism with a splitting sequence*

$$Z_n \subset \ldots \subset Z_1 \subset X$$

Then the complex $\tilde{C}_F(p)$ is exact.

Proof: If p has a splitting sequence of length 0 then it has a section and thus $\tilde{C}_F(p)$ is exact by Lemma 1.2.8. Assume that we have proven that $\tilde{C}_F(q)$ is exact for any etale q which has a splitting sequence of length $< n$.

Consider the pull-back square

$$\begin{array}{ccc} U_n & \stackrel{\tilde{i}_n}{\to} & U \\ \downarrow & & \downarrow p \\ Z_n & \stackrel{i_n}{\to} & X \end{array}$$

and let $s_n : Z_n \to U_n$ be a section which exists by definition of a splitting sequence. Since p is unramified $s_n(Z_n)$ is an open subset in U_n and $V_n = U - \tilde{i}_n(U_n - s_n(Z_n))$ is an open subset of U. One verifies easily that the pull-back square

$$\begin{array}{ccc} W_n & \to & V_n \\ \downarrow & & \downarrow \\ X - Z_n & \to & X \end{array}$$

is an upper distinguished square.

Consider $\tilde{C}_F(p : U \to X, q : (X - Z_n) \coprod V_n \to X)$. Its first row is $\tilde{C}_F(p)$. Its other rows are of the form $\tilde{C}_F(p_l)$ for $l > 0$ where

$$p_l : U \times_X ((X - Z_n) \coprod V_n)^l_X \to ((X - Z_n) \coprod V_n)^l_X.$$

These morphisms have splitting sequences of length less than n. Indeed, p_l for $l > 1$ are obtained from p_1 by base change and for

$$p_1 : (U \times_X (X - Z_n)) \coprod (U \times_X V_n) \to (X - Z_n) \coprod V_n$$

the closed subschemes

$$(Z_{n-1} - Z_n) \coprod \emptyset \subset \ldots \subset (Z_1 - Z_n) \coprod \emptyset$$

give a splitting sequence of required length. Thus by the inductive assumption the total complex of $\tilde{C}_F(p : U \to X, q : (X - Z_n) \coprod V_n \to X)$ is quasi-isomorphic to $\tilde{C}_F(p)$. It remains to note that the columns of this bicomplex are exact by our assumption on F.

Lemma 1.2.10 *[**lowersplit**] Let $p : U \to X$ be a proper morphism with a splitting sequence*

$$Z_n \subset \ldots \subset Z_1 \subset X$$

Then the complex $\tilde{C}_F(p)$ is exact.

Proof: The proof of this lemma is analogous to the proof of Lemma 1.2.9. Now we have to find a lower distinguished square

$$\begin{array}{ccc} W_1 & \to & V_1 \\ \downarrow & & \downarrow \\ Z_1 & \to & X \end{array}$$

such that $U \times_X V_1 \to V_1$ has a section. Consider the pull-back square

$$\begin{array}{ccc} U_1 & \stackrel{\tilde{j}_1}{\to} & U \\ \downarrow & & \downarrow \\ X - Z_1 & \stackrel{j_1}{\to} & X \end{array}$$

and let $s : X - Z_1 \to U_1$ be a section which exists by definition of a splitting sequence. Set $V_1 = U - \tilde{j}_1(U_1 - s(X - Z_1))$. Since p is separated $s(X - Z_1)$ is closed and thus V_1 is a closed subset of U. In particular $V_1 \to X$ is proper and one verifies easily that the pull back-square build out of $Z_1 \to X$ and this morphism has the required properties.

Lemma 1.2.11 *[**gen0**] Consider two morphisms $p : U \to X$ and $q : Y \to X$ and suppose that*

1. *$\tilde{C}_F(Y \times_X U^n_X \stackrel{q \times Id}{\to} U^n_X)$ are exact for all $n \geq 0$*

2. *$\tilde{C}_F(Y^m_X \times_X U \stackrel{Id \times p}{\to} Y^m_X)$ are exact for all $m \geq 1$.*

Then $\tilde{C}_F(p)$ is exact.

Proof: Consider the bicomplex whose terms are $F(Y^m_X \times_X U^n_X)$ and the differentials are given by the alternating sums of partial projections. Its rows are exactly the complexes $\tilde{C}_F(Y \times_X U^n_X \stackrel{q\times Id}{\to} U^n_X)$ and by our first condition they are exact. Thus the total complex of this bicomplex is exact. On the other hand all the columns of our complex but the first one are exact by the second condition. Thus the total complex must by quasi-isomorphic to its first column i.e. to $\tilde{C}_F(p)$. Combining we get that $\tilde{C}_F(p)$ is exact.

Lemma 1.2.12 *[**trrr**] Let $X \stackrel{f}{\to} Y \stackrel{g}{\to} Z$ be a composable pair of morphisms such that $\tilde{C}_F((X \stackrel{gf}{\to} Z) \times_Z Y^n)$ are exact for all $n \geq 0$ then $\tilde{C}_F(g)$ is exact.*

Proof: Consider the morphisms $g : Y \to Z$ and $gf : X \to Z$. The morphisms $X^m_Z \times_Z Y \to X^m_Z$ have sections for $m \geq 1$ and thus by Lemma 1.2.8 $\tilde{C}_F$ of these morphisms are exact. The complexes $\tilde{C}_F(X \times_Z Y^n \to Y^n_Z)$ are exact by assumption. Thus $\tilde{C}_F(g)$ is exact by Lemma 1.2.11.

Proposition 1.2.13 *[**comp**] Let $X \stackrel{f}{\to} Y \stackrel{g}{\to} Z$ be a composable pair of morphisms such that*

1. *$\tilde{C}_F((Y \to Z) \times_Z X^n_Z)$ are exact for all $n \geq 0$*
2. *$\tilde{C}_F((X \to Y) \times_Z Y^l_Z \times_Z X^n_Z)$ are exact for all $n, l \geq 0$.*

Then $\tilde{C}_F(gf)$ is exact.

Proof: We need a lemma.

Lemma 1.2.14 *[**tech**] If the second condition of the proposition holds then $\tilde{C}_F((X \to Z) \times_Z Y^{l+1}_Z)$ are exact for $l \geq 0$.*

Proof: Observe first that the morphisms

$$\phi_m : (Y^l_Z \times_Z X)^m_{Y^{l+1}_Z} \times_{Y^{l+1}_Z} (X \times_Z Y^{l+1}_Z) \to (Y^l_Z \times_Z X)^m_{Y^{l+1}_Z}$$

have sections for $m > 0$. Indeed it is sufficient to show this for $m = 1$ in which case the corresponding morphism is isomorphic to the morphism

$$Y^l_Z \times_Z X \times_Z X \to Y^l_Z \times_Z X$$

which has a section given by the diagonal of $X \times_Z X$. By Lemma 1.2.8 we conclude that $\tilde{C}_F(\phi_m)$ are exact for $m > 0$. By the second condition of the proposition $\tilde{C}_F(Y^l \times_Z X^n_Z \to Y^{l+1}_Z \times_Z X^n_Z)$ are exact for $l \geq 0$. Thus by Lemma 1.2.11

$$\tilde{C}_F(\phi_0) = \tilde{C}_F((X \to Z) \times_Z Y^{l+1}_Z)$$

are exact for all $l \geq 0$.

To prove the proposition we apply Lemma 1.2.11 to morphisms $gf : X \to Z$ and $g : Y \to Z$. By Lemma 1.2.14 $\tilde{C}_F((X \to Z) \times_Z Y^m)$ are exact for all $m \geq 1$ and by the first condition of the proposition $\tilde{C}_F((Y \to Z) \times_Z X^n_Z)$ are exact for all $n \geq 0$. Thus $\tilde{C}_F(gf)$ is exact.

Now we can finish the proof of Theorem 1.2.7. Let $p : U \to X$ be the cdh-covering for which we want to prove the exactness of $\tilde{C}_F(p)$. Assume first that it is a composition of etale and proper morphisms which have splitting sequences. Then applying inductively Lemmas 1.2.9, 1.2.10 and Proposition 1.2.13 we conclude that $\tilde{C}_F(p)$ is exact. By Lemma 1.2.3 for general cdh-covering p there exists a morphism s such that $ps = p'$ where p' is of the form which we just considered. Applying Lemma 1.2.12 we conclude that $\tilde{C}_F(p)$ is exact in this case as well. Theorem is proven.

Recall that a functor F from Sch/S to an additive category A is called additive if $F(\emptyset) = 0$ and for any X, Y over S the canonical morphism $F(X) \oplus F(Y) \to F(X \coprod Y)$ is an isomorphism.

Proposition 1.2.15 *[***additivecase***] Let $F : Sch/S \to A$ be an additive functor to an abelian category. Then the following two conditions are equivalent:*

1. *For any upper distinguished (resp. lower distinguished, distinguished) square*

$$\begin{array}{ccc} Y' & \to & Y \\ \downarrow & & \downarrow p \\ X' & \stackrel{e}{\to} & X \end{array} \qquad (1)$$

   the complex $\tilde{C}_F((X' \coprod Y) \to X)$ is exact.

2. *For any upper distinguished (resp. lower distinguished, distinguished) square as above the complex*

$$0 \to F(Y') \to F(X') \oplus F(Y) \to F(X) \to 0 \qquad (2)$$

is exact.

Proof: It is obviously sufficient to prove the proposition for upper and lower distinguished squares. We will say that a functor which satisfies the condition of the proposition for upper (resp. lower) distinguished squares is called upper exact (resp. lower exact).

Definition 1.2.16 *[**plain**] A distinguished square is called a plain distinguished square if p is a monomorphism.*

The following two lemmas is straightforward.

Lemma 1.2.17 *[**lplain**] Let $F : Sch/S \to A$ be an additive functor to an additive catgeory which has kernels of projectors. Then for a plain distinguished square the normalized chain complex of the simplicial object $\check{C}((X' \coprod Y) \to X)$ with terms*

$$\check{C}((X' \coprod Y) \to X)_n = (X' \coprod Y)^n_X$$

is isomorphic to $F(Y') \to F(X') \oplus F(Y)$.

Lemma 1.2.18 *[**sections**] Let*

$$\begin{array}{ccc} Y' & \to & Y \\ \downarrow & & \downarrow p \\ X' & \stackrel{e}{\to} & X \end{array} \tag{3}$$

be an upper (resp. lower) distinguished square and assume that p has a section s. Then the square

$$\begin{array}{ccc} X' & \stackrel{e}{\to} & X \\ s' \downarrow & & \downarrow s \\ Y' & \to & Y \end{array} \tag{4}$$

where s' is the induced section of $Y' \to X'$ is a plain upper (resp. lower) distinguished square.

Lemma 1.2.19 *[**sections2**] Let F be an upper (resp. lower) exact functor. Then for an upper (resp. lower) distinguished square of the form (3) satisfying the condition of Lemma 1.2.18 the sequence*

$$0 \to F(Y') \to F(X') \oplus F(Y) \to F(X) \to 0$$

is exact.

Proof: The sequence in question is the cone of the morphism of complexes which are the rows of the diagram

$$\begin{array}{ccc} F(Y') & \to & F(Y) \\ \downarrow & & \downarrow p \\ F(X') & \stackrel{e}{\to} & F(X) \end{array}$$

The sections s and s' give a section of this morphism. The cone of this section is the complex associated with the square (4). By Lemma 1.2.17 and our assumption on F it is exact. Thus the section is a quasi-isomorphism and therefore the morphism itself is a quasi-isomorphism and its cone is exact.

We can now finish the proof of Proposition 1.2.15. Assume first that F is such that for any upper (resp. lower) distinguished square of the form (1) the complex (2) is exact. Consider the complex of complexes

$$\tilde{C}_F(Y' \times_X (X' \coprod Y) \to Y') \to$$

$$\to \tilde{C}_F(X' \times_X (X' \coprod Y) \to X') \oplus \tilde{C}_F(Y \times_X (X' \coprod Y) \to Y) \to$$

$$\to \tilde{C}_F((X' \coprod Y) \to X)$$

as a bicomplex. Then its rows are complexes of the form (2) for upper (resp. lower) distinguished squares and thus are exact. On the other hand the morphisms

$$Y' \times_X (X' \coprod Y) \to Y'$$

$$X' \times_X (X' \coprod Y) \to X'$$

$$Y \times_X (X' \coprod Y) \to Y$$

have sections and thus the first two columns are exact by Lemma 1.2.8. We conclude that the last column is also exact.

Let now F be an upper (resp. lower) exact functor. Consider the same bicomplex as before. Now we know that its columns are exact and we want to show that the first row is. One can easily see that all the rows but the fisrt one are direct sums of complexes of the form (2) for upper (resp. lower) distinguished squares such that either e is an isomorphism or p has a section. In the first case the complex (2) is obviously exact. In the second it is exact by Lemma 1.2.19. Proposition is proved.

1.3 *Presheaves and sheaves with transfers.*

Definition 1.3.1 *A presheaf with transfers on Sch/S is an additive contravariant functor from $Cor(Sch/S)$ to the category of abelian groups.*

The category of presheaves with transfers on $Cor(Sch/S)$ is denoted by $PreShv(Cor(Sch/S))$. It is an abelian category. The presheaf with transfers represented on $Cor(Sch/S)$ by $[X]$ is denoted by $\mathbf{Z}_{tr}(X)$. **Remark**: Note that this notation is not compatible with the notation of [**?**, §1]. For a smooth scheme X over a field k the restriction of $\mathbf{Z}_{tr}(X)$ to the category of smooth schemes does not coincide with the functor denoted by $\mathbf{Z}_{tr}(X)$ in [**?**]. The later is denoted here by $\mathbf{Z}_{tr}^{equi}(X)$. It is the functor represented by $[X]$ on the subcategory $Cor^{equi}(Sch/S)$ of $Cor(Sch/S)$ (see Definition 1.3.8).

Lemma 1.3.2 *[**representare**] For any X the presheaf with transfers $\mathbf{Z}_{tr}(X)$ considered as a presheaf on Sch/S is a cdh-sheaf.*

Proof: See [**?**, Th. 4.2.9].

Proposition 1.3.3 *[**step1**] Let $p : U \to X$ be a cdh-covering. Then the complex*

$$\ldots \to \mathbf{Z}_{tr}(U \times_X U \times_X U) \to \mathbf{Z}_{tr}(U \times_X U) \to \mathbf{Z}_{tr}(U) \to \mathbf{Z}_{tr}(X)$$

considered as a complex of sheaves in the cdh-topology on Sch/S is exact.

Proof: It is sufficient to show that the conditions of Theorem 1.2.7 hold for the functor $\mathbf{Z}_{tr}(-)$ from the category Sch/S to the category of sheaves of abelain groups in the cdh-topology on Sch/S. The part of the conditions of Theorem 1.2.7 related to the upper distinguished squares follows now from [**?**, Prop. 4.3.9] and the exactness of the associated sheaf functor. The part related to lower distinguished squares follows from [**?**, Prop. 4.3.3].

Definition 1.3.4 *A presheaf with transfers F is called a cdh-sheaf with transfers if considered as a functor on Sch/S via the functor $X \mapsto [X]$ it is a cdh-sheaf.*

Applying Proposition 1.3.3 and using the same reasoning as in [**?**, ...] we get the following sequence of corollaries.

Corollary 1.3.5 *[**cor1**] For any presheaf with transfers F on Sch/S the associated cdh-sheaf F_{cdh} has a unique structure of a presheaf with transfers such that the morphism $F \to F_{cdh}$ is a morphism of presheaves with transfers.*

Denote by $Shv_{cdh}(Cor(Sch/S))$ the full subcategory of $PreShv(Cor(Sch/S))$ which consists of cdh-sheaves with transfers.

Corollary 1.3.6 *[**cor2**] The category $Shv_{cdh}(Cor(Sch/S))$ is abelian and the forgetful functor*

$$Shv_{cdh}(Cor(Sch/S)) \to Shv_{cdh}(Sch/S)$$

is exact.

Corollary 1.3.7 *[**cor3**] For any cdh-sheaf with transfers F on Sch/S and any X of finite type over S there are canonical isomorphisms*

$$H^i_{cdh}(X, F) = Ext^i_{Shv_{cdh}(Cor(Sch/S))}(\mathbf{Z}_{tr}(X), F)$$

All our constructions have immediate analogs for the Nisnevich topology (i.e. the upper cd-topology).

Definition 1.3.8 *[**equidimtransf**] A presheaf with equidimensional transfers on Sch/S is an additive contravariant functor from $Cor_{equi}(Sch/S)$ to the category of abelian groups. A presheaf with equidimensional transfers on Sm/S is an additive contravariant functor from $Cor_{equi}(Sm/S)$ to the category of abelian groups.*

The category of presheaves with equidimensional transfers on Sch/S (resp. on Sm/S) is denoted by $PreShv(Cor_{equi}(Sch/S))$ (resp. $PreShv(Cor_{equi}(Sm/S))$). It is an abelian category. We denote the functor represented by $[X]$ on $Cor_{equi}(Sch/S)$ by $\mathbf{Z}^{equi}_{tr}(X)$. The following lemma is straightforward.

Lemma 1.3.9 *[**Nisrepresentare**] For any X the presheaf $\mathbf{Z}^{equi}_{tr}(X)$ on Sch/S is a Nisnevich sheaf.*

Proposition 1.3.10 *[**step1Nis**] Let $p : U \to X$ be a Nisnevich covering. Then the complex*

$$\ldots \to \mathbf{Z}^{equi}_{tr}(U \times_X U \times_X U) \to \mathbf{Z}^{equi}_{tr}(U \times_X U) \to \mathbf{Z}^{equi}_{tr}(U) \to \mathbf{Z}^{equi}_{tr}(X)$$

considered as a complex of sheaves in the Nisnevich topology on Sch/S is exact.

Proof: An immediate analog of Theorem 1.2.7 where one has to consider only upper distinguished squares holds for the Nisnevich topology. The fact that the condition of this analog holds for the functor $X \mapsto \mathbf{Z}_{tr}^{equi}(X)$ follows from [**?**, Prop. 4.3.9].

Remark: The sequence of the functors $\mathbf{Z}_{tr}(-)$ associated with an upper distinguished square is, probably, not exact in the Nisnevich topology.

Definition 1.3.11 *A presheaf with equidimensional transfers F is called a Nisnevich sheaf with transfers if considered as a functor on Sch/S (or Sm/S) via the functor $X \mapsto [X]$ it is a Nisnevich sheaf.*

As in the cdh-case we have the following sequence of corollaries.

Corollary 1.3.12 *[**cor1Nis**] For any presheaf with equidimensional transfers F on Sch/S (or Sm/S) the associated Nisnevich sheaf F_{Nis} has a unique structure of a presheaf with equidimensional transfers such that the morphism $F \to F_{Nis}$ is a morphism of presheaves with equidimensional transfers.*

Denote by $Shv_{Nis}(Cor_{equi}(\mathcal{C}))$ the full subcategory of $PreShv(Cor_{equi}(\mathcal{C}))$ which consists of Nisnevich sheaves with transfers on $\mathcal{C}$ where $\mathcal{C}$ is Sch/S or Sm/S.

Corollary 1.3.13 *[**cor2Nis**] The category $Shv_{Nis}(Cor_{equi}(Sch/S))$ is abelian and the forgetful functor*

$$Shv_{Nis}(Cor_{equi}(Sch/S)) \to Shv_{Nis}(Sch/S)$$

is exact. The same holds for smooth schemes.

Corollary 1.3.14 *[**cor3Nis**] For any Nisnevich sheaf with transfers F on Sch/S and any X of finite type over S there are canonical isomorphisms*

$$H^i_{Nis}(X, F) = Ext^i_{Shv_{Nis}(Cor_{equi}(Sch/S))}(\mathbf{Z}_{tr}^{equi}(X), F)$$

In addition we have the following convenient fact.

Lemma 1.3.15 *[**rightexact**] The obvious restriction functor*

$$Shv_{Nis}(Cor_{equi}(Sch/S)) \to Shv_{Nis}(Cor_{equi}(Sm/S))$$

is exact.

To establish the relation between the categories of sheaves with transfesr in the Nisnevich and cdh-topologies we need the following fact which is proved in [**?**, Th. 4.2.9(2)]

Lemma 1.3.16 [**associate**] *For any scheme of finite type X over S the cdh-sheaf associated to $\mathbf{Z}_{tr}^{equi}(X)$ is $\mathbf{Z}_{tr}(X)$.*

For a small additive category A let $PreShv(A)$ be the category of contravariant additive functors from A to abelian groups. For any additive functor $\pi : A \to A'$ we denote by

$$\pi_* : PreShv(A') \to PreShv(A)$$

the functor of the form $\pi_*(F) = F \circ \pi$. One can easily see ([**?**,]) that π_* has a left adjoint π^* and if L_X is the functor represented by X one has $\pi^*(L_X) = L_{\pi(X)}$.

This construction applied to the obvious functor

$$\pi : Cor_{equi}(Sm/S) \to Cor(Sch/S)$$

gives a pair of adjoint functors which we denote by π^*_{preshv} and π_*^{preshv} respectively.

The lemma below follows immediately from the definition of sheaves with transfers.

Lemma 1.3.17 [**shtosh**] *For a cdh-sheaf with transfers F on Sch/S the presheaf $\pi_*^{preshv}(F)$ is a Nisnevich sheaf (with equidimensional transfers).*

Consider the diagram of categories and functors

$$\begin{array}{ccc}
PreShv(Cor(Sch/S)) & \underset{\pi^*_{preshv}}{\overset{\pi_*^{preshv}}{\rightleftarrows}} & PreShv(Cor_{equi}(Sm/S)) \\
i_{cdh} \uparrow \ \downarrow a_{cdh} & & i_{Nis} \uparrow \ \downarrow a_{Nis} \\
Shv_{cdh}(Cor(Sch/S)) & \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} & Shv_{Nis}(Cor_{equi}(Sm/S))
\end{array}$$

where i are the inclusions, a are the associated sheaf functors and

$$\pi^* = a_{cdh} \pi^*_{preshv} i_{cdh}$$

$$\pi_* = a_{Nis} \pi_*^{preshv} i_{Nis}.$$

Lemma 1.3.17 implies immediately that pi_* is a right adjoint to pi^*.

Going back to the case of an additive functor $\pi : A \to A'$ between additive catgeories define

$$L\pi^* : Compl^-(PreShv(A)) \to Compl^-(PreShv(A'))$$

as follows. For a presheaf F let $Lres(F)$ be the canonical resolution of F by representable presheaves. For a complex C bounded from the (cohomological) above we define $Lres(C)$ as the total complex of the bicomplex obtained by applying $Lres(-)$ to the terms of C. The functor $Lres : Compl^- \to Compl^-$ has the following obvious properties.

Proposition 1.3.18 *[lres1]*

1. *$Lres(C[1]) = Lres(C)[1]$*

2. *for a morphism $f : C_1 \to C_2$ there is a canonical isomorphism*

$$Lres(cone(f)) = cone(Lres(f)).$$

3. *For any set of complexes C_α the canonical morphism*

$$\oplus_\alpha Lres(C_\alpha) \to Lres(\oplus_\alpha C_\alpha)$$

   is an isomorphism.

4. *For any C the canonical morphism $Lres(C) \to C$ is a quasi-isomorphism.*

5. *If terms of C are isomorphic to direct sums of presheaves of the form $\mathbf{Z}_{tr}^{equi}(X)$ for smooth X then the canonical morphism $Lres(C) \to C$ is a homotopy equivalence.*

6. *The functor $Lres$ takes quasi-isomorphisms of complexes of presheaves to homotopy equivalences.*

Define now $L\pi^*(F)$ as $\pi^*(Lres(F))$. The following properties of $L\pi^*$ are straightforward ([,]).

Proposition 1.3.19 *[lres2]*

1. $L\pi^*(C[1]) = L\pi^*(C)[1]$
2. *for a morphism* $f : C_1 \to C_2$ *there is a canonical isomorphism*
$$L\pi^*(cone(f)) = cone(L\pi^*(f)).$$
3. *For any set of complexes* C_α *the canonical morphism*
$$\oplus_\alpha L\pi^*(C_\alpha) \to L\pi^*(\oplus_\alpha C_\alpha)$$
is an isomorphism.
4. *If terms of* C *are isomorphic to direct sums of presheaves* L_X *represented by objects* X *of* A *then the canonical morphism* $L\pi^*(C) \to \pi^*C$ *is a homotopy equivalence.*
5. *The functor* $L\pi^*$ *takes quasi-isomorphisms of complexes of presheaves to homotopy equivalences.*

In the case of presheaves with transfers this construction gives a functor

$$L\pi^* : PreShv(Cor_{equi}(Sm/S)) \to PreShv(Cor(Sch/S))$$

which in view of Lemma 1.3.2 factors through the inclusion

$$Shv_{cdh}(Cor(Sch/S)) \to PreShv(Cor(Sch/S)).$$

Proposition 1.3.20 *[lres3] Let* $f : C_1 \to C_2$ *be a quasi-isomorphism of complexes of Nisnevich sheaves with equidimensional transfers. Then* $L\pi^*(f)$ *is a quais-isomorphism of complexes of cdh-sheaves with transfers.*

Proof: By Propostion 1.3.19(1,2) it is sufficient to show that if C is a complex of Nisnevich sheaves with equidimensional transfers which is quasi-isomorphic to zero as a complex of sheaves then $L\pi^*(C)$ is quasi-isomorphic to zero. Consider $Lres(C)$. By Proposition 1.3.18 this complex is also quasi-isomorphic to zero as a complex of Nisnevich sheaves. By Lemma 1.3.16 the complex $\pi^*(Lres(C))$ is just the complex of cdh-sheaves associated with the complex of Nisnevich sheaves $Lres(C)$. Since the cdh-topology is stronger than the Nisnevich topology this finishes the proof.

Lemma 1.3.21 *[**directfirst**] Let Let k be a field which admits resolution of singularities (see [**?**, Def. 3.4]). Then for any cdh-sheaf with transfers G on Sch/k the adjunction $\pi^*\pi_*(G) \to G$ is an isomorphism.*

Proof: We have

$$\pi^*\pi_*G = a_{cdh}\underline{H}_0\pi^*_{preshv}Lres\pi_*G$$

Let $\pi^*_{pre}Lres\pi_*G$ be the complex of presheaves on Sch/S defined by the rule that

$$\pi^*_{pre}(\oplus\mathbf{Z}^{equi}_{tr}(X_\alpha)) = \oplus\mathbf{Z}^{equi}_{tr}(X_\alpha)$$

with $\mathbf{Z}^{equi}_{tr}(X_\alpha)$ considered as presheaves on Sm/k on the left hand side and presheaves on Sch/k on the right hand side. There is a canonical morphism

$$\pi^*_{pre}Lres\pi_*G \to \pi^*_{preshv}Lres\pi_*G$$

and by Lemma 1.3.16 the induced morphism of complexes of associated cdh-sheaves is an isomorphism. Thus it is sufficinet to show that the morphism

$$\pi^*_{pre}Lres\pi_*G \to G$$

is a quasi-isomorphism in the cdh-topology. For any smooth scheme U over k we have

$$(\pi^*_{pre}Lres\pi_*G)(U) = (Lres\pi_*G)(U) \cong G(U)$$

and thus it is a quasi-isomorphism of presheaves over smooth schemes. On the other hand it follows from the resolution of singularities assumption that any scheme of finite type over k has a cdh-covering which consists of smooth schemes. Therefore $\pi^*_{pre}Lres\pi_*G \to G$ induces a quasi-isomorphism of the complexes of associated cdh-sheaves.

Lemma 1.3.22 *[**withres1**] Let Let k be a field which admits resolution of singularities (see [**?**, Def. 3.4]). Then for any Nisnevich sheaf with equidimensional transfers F the canonical morphism $L\pi^*(F) \to \pi^*F$ is a quasi-isomorphism of complexes of sheaves with transfers in the cdh-topology.*

Proof: It follows immediately from the constructions that

$$\pi^*F = a_{cdh}\underline{H}^0(L\pi^*(F))$$

and all we have to show is that $a_{cdh}\underline{H}^i(L\pi^*(F)) = 0$ for $i \neq 0$. Consider $L\pi^*(F)$ as a complex of cdh-sheaves of abelian groups on Sch/k. By

Lemma 1.3.16 it is the complex of cdh-sheaves associated with the complex of presheaves $\pi^*_{pre}Lres(F)$ (see proof of Lemma 1.3.21) Since the associated sheaf functor is exact we have

$$a_{cdh}\underline{H}^i(L\pi^*(F)) = a_{cdh}\underline{H}^i(\pi^*_{pre}Lres(F)).$$

By Lemma 1.3.18(4) for any smooth scheme U over k we have

$$\underline{H}^i(\pi^*_{pre}Lres(F))(U) = 0$$

for $i \neq 0$. The resolution of singularities assumption implies that any scheme of finite type over k has a cdh-covering which consists of smooth scheme and therefore $a_{cdh}\underline{H}^i(\pi^*_{pre}Lres(F)) = 0$ for $i \neq 0$.

1.4 *Triangulated categories of motives over S.*

Definition 1.4.1 *[**dmcdh**] $DM^{-,eff}_{cdh}(Sch/S)$ is the localization of the derived category $D^-(Shv_{cdh}(Cor(Sch/S))$ of complexes bounded from the above over $Shv_{cdh}(Cor(Sch/S))$ with respect to the localizing subcategory generated by objects of the form $\mathbf{Z}_{tr}(X\times\mathbf{A}^1)\to\mathbf{Z}_{tr}(X)$ where $\mathbf{A}^1$ is the affine line over S.*

The category $DM^{-,eff}_{cdh}(Sch/S)$ is called the triangulated category of of effective motives over S in the cdh-topology. Denote by

$$M_{cdh} : Sch/S \to DM^{-,eff}_{cdh}(Sch/S)$$

the functor which takes X to $\mathbf{Z}_{tr}(X)$.

Definition 1.4.2 *[**dmequi**] $DM^{-,eff}_{Nis}(Sch/S)$ is the localization of the derived category $D^-(Shv_{Nis}(Cor_{equi}(Sch/S))$ of complexes bounded from the above over $Shv_{Nis}(Cor_{equi}(Sch/S))$ with respect to the localizing subcategory generated by objects of the form $\mathbf{Z}^{equi}_{tr}(X\times\mathbf{A}^1)\to\mathbf{Z}^{equi}_{tr}(X)$ where $\mathbf{A}^1$ is the affine line over S and X is a scheme of finite type over S.*

$DM^{-,eff}_{Nis}(Sm/S)$ is the localization of the derived category of complexes bounded from the above over $Shv_{Nis}(Cor_{equi}(Sm/S))$ with respect to the localizing subcategory generated by objects of the form

$$\mathbf{Z}^{equi}_{tr}(X\times\mathbf{A}^1)\to\mathbf{Z}^{equi}_{tr}(X)$$

where $\mathbf{A}^1$ is the affine line over S and X is a smooth scheme over S.

We denote by

$$M_{Nis} : Sch/S \to DM^{-,eff}_{cdh}(Sch/S)$$

the functor which takes X to $\mathbf{Z}^{equi}_{tr}(X)$ and use the same notation in the smooth case.

Lemma 1.4.3 *[**piupper**] There exists a unique functor*

$$L\pi^* : DM^{-,eff}_{Nis}(Sm/S) \to DM^{-,eff}_{cdh}(Sch/S)$$

such that the diagram

$$\begin{array}{ccc} Compl^-(Shv_{Nis}(Cor_{equi}(Sm/S))) & \stackrel{L\pi^*}{\to} & Compl^-(Shv_{cdh}(Cor(Sch/S))) \\ \downarrow & & \downarrow \\ DM^{-,eff}_{Nis}(Sm/S) & \stackrel{L\pi^*}{\to} & DM^{-,eff}_{cdh}(Sch/S) \end{array}$$

commutes. For a smooth scheme X over S we have a canonical isomorphism $L\pi^* M_{Nis}(X) = M_{cdh}(X)$.

Proof: This follows immediately from definitions Propositions 1.3.19 and 1.3.20.

Theorem 1.4.4 *[**trc**] Let k be a field which admits resolution of singularities (see [**?**, Def. 3.4]). Then the functor*

$$L\pi^* : DM^{-,eff}_{Nis}(Sm/k) \to DM^{-,eff}_{cdh}(Sch/k)$$

is an equivalence.

Proof: We need a lemma.

Lemma 1.4.5 *[**pilower**] Under the assumption of the theorem there exists a unique functor*

$$R\pi_* : DM^{-,eff}_{cdh}(Sch/k) \to DM^{-,eff}_{Nis}(Sm/k)$$

such that the diagram

$$\begin{array}{ccc} Compl^-(Shv_{cdh}(Cor(Sch/k))) & \stackrel{\pi_*}{\to} & Compl^-(Shv_{Nis}(Cor_{equi}(Sm/k))) \\ \downarrow & & \downarrow \\ DM^{-,eff}_{cdh}(Sch/k) & \stackrel{\pi_*}{\to} & DM^{-,eff}_{Nis}(Sm/k) \end{array}$$

commutes.

Proof: ????

Lemma 1.3.22 implies that the analog of Lemma 1.4.3 holds for the functor π^*. Since the functors π_* and π^* between the catgeories of complexes are adjoint the same holds for the corresponding functors between the categories DM. It remains to show that for any F in $Shv_{Nis}(Cor_{equi}(Sm/k))$ the adjunction $F \to \pi_*\pi^*F$ is an isomorphism in $DM_{Nis}^{-,eff}(Sm/k)$ and for any G in $Shv_{cdh}(Cor(Sch/k))$ the adjunction $\pi^*\pi_*G \to G$ is an isomorphism in $DM_{cdh}^{-,eff}(Sch/k)$. The second statement follows from Lemma 1.3.21.

To prove the first one it is clearly sufficient to show that if π^*F is zero then F is zero in $DM_{Nis}^{-,eff}$. For this it is sufficient (by [**?**,]) to prove the following lemma.

Lemma 1.4.6 *[**othogonal**] Let F be a Nisnevich sheaf with equidimensional transfers on Sm/k where k admits resolution of singularities. Then for any homotopy invariant Nisnevich sheaf with transfers Φ and any $i \geq 0$ one has*

$$Ext^i_{Shv_{Nis}(Cor_{equi}(Sm/k))}(F, \Phi) = 0$$

Proof: The proof is directly analogous to the proof of Lemma 5.4 in [**?**] with the only change being that one works with sheaves with transfers instead of the usual sheaves.