PIECE-WISE EUCLIDEAN APPROXIMATIONS OF JACOBIANS OF ALGEBRAIC CURVES.

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0. *Introduction*

In the present paper we prove the result, the particular case of which was announced in [SV2]. Both results present the sequences of finite-term constructions that approximate the jacobians of complex algebraic curves.

The problems, to which this paper is devoted, originated from A,Grothendieck's unpublished paper [Gr]. In this paper the unexpected connection between the combinatorial topology of graphs on the surfaces and algebraic curves over number fields was revealed. Namely, Grothendieck established the equivalence between "dessins d'enfant" - the graphs on the compact oriented surfaces with the complement homeomorphic to disjoint union of open cells - and the pairs (X,β) of complex algebraic curves X and rational functions $\beta: X \longrightarrow \mathbb{P}^1(\mathbb{C})$, whose only finite critical values are 0 and 1; in this equivalence the graphs are restored as the preimages $\beta^{-1}[1,\infty]$ of the real segment of $\mathbb{P}^1(\mathbb{C})$. Previously G.Belyi [Be] proved, that such function β exists on a complex curve X if and only if X is a complexification of some curve, defined over a number field; these two results sum up in the above mentioned relation between combinatorial topology and arithmetic. See [SV2] for the details.

So the problem of describing the algebraic curves over fields of complex and algebraic numbers in combinatorial terms arises. In the present paper the only graphs on the surfaces we deal with are triangulations. There is another way to get the complex structure on the surface, starting from triangulation. Namely, the complex structure is defined by the introduction of a Euclidean structure on all the triangles (i.e., the specification of lengths of all the edges), similarly to the introduction of a Riemannian metric on a smooth surface; see 1.1 below. As it was shown in [SV1], the curves over number fields (and all of them!) arise from the equilateral structures; in the present paper we deal with the arbitrary lengths and in particular show, that we obtain all the complex curves (theorem 1.2.2 below).

The problem we solve is to define the jacobian of the curve in terms of the piecewise euclidean structure on it. To do this, we develop, following [Do], some piecewise-linear analogue of the smooth Hodge theory on the riemannian surfaces and define the approximate jacobians, whose limits under the infinite refinement of the triangulations are the classical ones.

The problem of the effective estimates will be treated in a separate paper, as well as some explicit calculations.

1. *Euclidean simplicial surfaces and associated complex structures.*

1.0 Notations

Let S be a simplicial scheme of dimention 2. We shall denote by S_0, S_1 and S_2 the sets of its 0-,1- and 2-dimensional simplexes respectively. S is called pseudo-surface if for each $\sigma \in S_1$ there exist exactly two 2-simplexes σ_+, σ_- such that $\sigma \in \sigma_+, \sigma \in \sigma_-$. The orientation of the simplex is by the definition the cyclic order on the set of its vertices. The set of oriented q-simplexes will be denoted by S_q^*. Note that $S_0^*=S_0$. The orientation of S is the function $\kappa: S_2^* \longrightarrow \{+1,-1\}$ such that for each $\sigma \in S_1^*$ one has $\kappa(\sigma_+)=-\kappa(\sigma_-)$, where σ_+, σ_- are oriented with respect to the orientation of σ. The pair (S,κ) is called the oriented pseudo-surface. The unordered sets of objects will be listed inside { }, the objects with the cyclic order - inside <>, and the ordered objects - inside ( ). We shall also use the standard notations from the combinatorial topology: st(σ) for the star of the simplex, sk_q for the q-skeleton of the simplicial scheme S and $|S|$ for its geometrical realization (see [Spa]).

1.1 Euclidean structures

A *Euclidean structure* on the simplicial surface S is defined by the length function

$$\lambda: S_1 \longrightarrow \mathbb{R}^+,$$

satisfying

$$\lambda(v_0,v_2)<\lambda(v_0,v_1)+\lambda(v_1,v_2)$$

for all $\{v_0,v_1,v_2\} \in S_2$. We shall call a simplicial surface S together with the length function λ the euclidean simplicial surface.

We shall also use the *area function* $S: S_2 \longrightarrow \mathbb{R}^+$, associated with the length function: $S\{v_0,v_1,v_2\}$ is, by definition, the area of the triangle with the sides $\lambda(v_0,v_1), \lambda(v_1,v_2), \lambda(v_2,v_0)$.

The euclidean simplicial surface (S,λ) is called *equilateral* iff its length function is constant. Note that in this case the area function is also constant and equal to $\sqrt{3}\lambda^2/4$.

1.2 Associated complex structures

Let (S,λ) be an euclidean simplicial surface. We are going to define a complex Riemann surface $X_{S,\lambda}$ which we shall call the *complex realisation* of (S,λ). Denote by $T_{<a,b,c>}$ the oriented euclidean triangle with the sides $<a,b,c>$. The underlying topological surface $X_{S,\lambda}$ of $X_{S,\lambda}$ is obtained by pasting along the common sides all the triangles $T_{<\lambda(v_0,v_1),\lambda(v_1,v_2),\lambda(v_2,v_0)>}$ corresponding to positevely oriented 2-simplexes (v_0,v_1,v_2) of S.

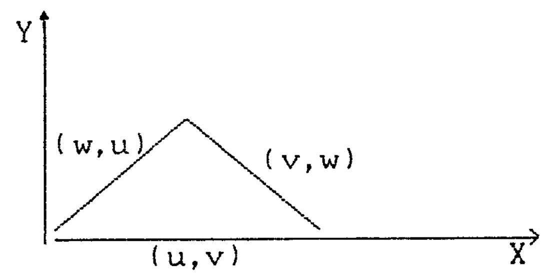
Note that it is naturally homeomorphic to the usual geometric realisation $|S|$ of the simplicial scheme S, and in particular is an oriented closed topological surface.

The $\mathbb{C}$-valued continuous function on an open subset of $X_{S,\lambda}$ is called holomorphic iff its restriction on each triangle $T_{<\lambda(v_0,v_1),\lambda(v_1,v_2),\lambda(v_2,v_0)>}$ is holomorphic with respect to the natural complex structure defined by its embeding into $\mathbb{R}^2=\mathbb{A}^1(\mathbb{C})$. This construction defines the sheaf of rings on topological space $X_{S,\lambda}$ which we denote by $\mathcal{O}_{S,\lambda}$.

Theorem 1.2.1 *The pair $(X_{S,\lambda},\mathcal{O}_{S,\lambda})$ is a compact complex Riemann surface.*

Proof: We have to prove, that each point of $X_{S,\lambda}$ has a neighbourhood isomorphic to complex disc as a ringed space. It is obvious for all the points of $X_{S,\lambda}$, except those corresponding to the vertices of S. Therefore it would be sufficient to show that for each such point there exist its neighbourhood U on $X_{S,\lambda}$

such that $O_{S,\lambda}(U)\neq\mathbb{C}$. For each (u,v,w) such that $\langle u,v,w\rangle$ is the positively oriented 2-simplex of S choose an embedding of $T_{\langle\lambda(u,v),\lambda(v,w),\lambda(w,u)\rangle}$ into $\mathbb{R}^2$ as follows:



Denote this planar triangle by $T_{(u,v,w)}$.

Let v be a vertex of S and $\langle v,v_0,v_1\rangle, \langle v,v_1,v_2\rangle, \ldots, \langle v,v_n,v_0\rangle$ be positively oriented 2-simplexes of the star of v ordered clock-wise. Denote by α_i the angle between the edges (v,v_{i-1}) and (v,v_i) on the triangle $T_{(v,v_{i-1},v_i)}$ $(1\leq i\leq n)$. For $z\in T_{(v,v_{i-1},v_i)}\subset\mathbb{C}$ set

$$F_i(z) = e^{\sqrt{-1}\,\omega(\alpha_1+\alpha_2+\cdots+\alpha_{i-1})} z^{\alpha_i \omega}, \text{ where}$$

$$\omega = \frac{2\pi}{\alpha_1+\alpha_2 \ldots\ldots+\alpha_n}.$$

All the F_i's together define a holomorphic function F on the star of v in $X_{S,\lambda}$. ■

We shall denote $(X_{S,\lambda}, \mathcal{O}_{S,\lambda})$ by $X_{S,\lambda}$. One can easily see, that for $\lambda'=c\lambda$, where c is a positive constant, there is a natural isomorphism $X_{S,\lambda} \longrightarrow X_{S,\lambda'}$. In particular there is a unique Riemann surface X_S defined by the equilateral euclidean structure on S.

Remark: Note that besides the complex structure there exists one more additional structure on $X_{S,1}$, namely almost everywhere flat metric with the singularities in the vertices of S.

Theorem 1.2.2 For any compact complex Riemann surface X there exists a euclidean simplicial surface (S,λ) such that $X_{S,\lambda} \simeq X$.

Proof: Let $f:X \longrightarrow \mathbb{P}^1(\mathbb{C})$ be any non-constant rational function unramified over infinity and $p_1,\dots p_n$ its critical values. Choose a conformal isomorphism $i:\mathbb{P}^1(\mathbb{C})\setminus\{\infty\} \longrightarrow \mathbb{R}^2$. Obviously there exists a triangulation $\tau: |S| \longrightarrow \mathbb{P}^1(\mathbb{C})$ of $\mathbb{P}^1(\mathbb{C})$ such that:

a) $\tau(|sk_1(S)|)$ does not contain an infinity

b) for each 1-simplex σ of S $i\tau(|\sigma|)$ is a linear segment on $\mathbb{R}^2$

c) for each i=1,..,n there exist 0-simplex σ_i of S such that $\tau(|\sigma_i|)=p_i$.

We can define a euclidean structure λ on S setting for $\sigma \in S_1$, $\lambda(\sigma)$ to be the usual length of the segment $i\tau(|\sigma|)$. Denote by $f^{-1}(S)$ the simpicial surface corresponding to the preimage of (S,τ) with respect to f. The euclidean structure λ on S defines the euclidean structure $f^{-1}(\lambda)$ on $f^{-1}(S)$ and ,therefor we obtain an euclidean simplicial surface $(f^{-1}(S),\ f^{-1}(\lambda))$. It is easy to show that $X_{(f^{-1}(S),f^{-1}(\lambda))}$ is isomorphic to X. ■

Remark: It can be shown, that there exists such S, that, varying λ, we cover all the moduli space. It seems to be very likelywise that this construction is closely connected with Penner's construction [Pen] of the triangulations of the Teichmuller spaces.

Theorem 1.2.3 Let X be a compact complex Riemann surface, then it can be realized as X_S for some simplicial surface S iff it is defined over $\bar{\mathbb{Q}}$.

Proof: see [SV1],[SV2].■

2. *Hodge Theory on the euclidean simplicial surfaces.*

2.0 Cohomologies and Whitney maps.

Let (S,λ) be an euclidean simplicial surface. We denote by $C^*(S,?)$ $(?=\mathbb{R},\mathbb{C},\mathbb{Z})$ the usual complex of simplicial ?-valued cochains and by $A^*(S,\lambda,?)$ $(?=\mathbb{R},\mathbb{C})$ the complex of ?-valued C^∞-forms on $X_{S,\lambda}$. We shall omit the coefficient groups where possible.

There is an obvious *De Ram map* R^* from A^* to C^* which is well known to be quasi-isomorphism. In the paper [W] the "quasi-invers" for this map was introduced; following [Do], we call it the *Whitney map*. It is defined in the following way.

For each vertex v of S define the continuous function

$b_v : X_{S,\lambda} \longrightarrow \mathbb{R}$ called the *barycentric coordinate* associated with v. This function is uniquely defined by the following conditions:

(i) b_v is identically zero outside the star st(v) of v on $X_{S,\lambda}$;

(ii) $b_v(v)=1$;

(iii) b_v is linear on the triangles $T_{<u,v,w>}$ for all $\{u,v,w\}\in S_2$.

Note that b_v is C^∞ almost everywhere (with respect to the C^∞-structure, defined by the complex structure).

Let $L^2(A^q)(S,\lambda)$ be the L^2-completions of the spaces $A^q(S,\lambda)$ with respect to the metric of constant negative curvative on $X_{S,\lambda}$ corresponding to the complex structure. For vertex v of S denote by db_v the "differential" of the barycentric coordinat b_v. By definition it is an element of $L^2(A^q)(S,\lambda)$ represented by the 1-form which is equal to the usual differential of b_v in the points where it is smooth and is zero in other points. The Whitney maps $W^q : C^q(S,\lambda) \longrightarrow L^2(A^q)(S,\lambda)$ are defined as follows:

$$W^q(\delta_{(v_0,\ldots v_q)}) = \sum_{i=0}^{q} (-1)^i b_{v_i} db_{v_1} \wedge \ldots \wedge \widehat{db_{v_i}} \wedge \ldots \wedge db_{v_q}$$

Lemma 2.0.1 $dW^{q-1} = W^q d$

Proof: see [Do].■

2.1. Scalar products and harmonic cochains.

It is not difficult to show that for $c\in C^q(S,\lambda)$ the cochain R(W(c)) is well defined and equal to c. In particular, W^q are embeddings. Therefore, we can define *scalar products* on $C^q(S,\lambda)$, setting

$$(x,y)_{S,\lambda}=(W^q(x),W^q(y)) \quad \text{(see Appendix for explicit formulae).}$$

The scalar products on the components of the cochain complex define a *Hodge decomposition* as follows. Denote by d^* the adjoint linear operator to d. Then one has:

$$C^q(S,\lambda)= \text{Im}\, d \oplus (\ker d \cap \ker d^*) \oplus \text{Im}\, d^*,$$

where $\oplus$ is, in fact, the orthogonal decomposition. The natural projection from $\ker d \cap \ker d^*$ to $H^1(S,\lambda)$ is an isomorphism and the inverse map defines the scalar product on the cohomology space. We shall denote this product also by $(\ ,\ )_{S,\lambda}$. Besides this one, there is another scalar product $(\ ,\ )_{L^2}$ on this space, defined by its embeding into the space of harmonic 1-forms on $X_{S,\lambda}$. Generally these products do not coinside.

The 1-cochains from the subspace $\ker d \cap \ker d^*$ will be called *piecewise-euclidean harmonic* (or PE-harmonic) cochains. For a cohomology class c we shall denote by $[c]_{S,\lambda}$ its PE-harmonic representative.

2.2. Piecewise euclidean jacobians.

The *Hodge operator* $*_{S,\lambda}:H^1(S,\lambda)\longrightarrow H^1(S,\lambda)$ is defined as the unique linear operator, satisfying

$$(a,b)_{S,\lambda}=a\wedge(*_{S,\lambda}b)$$

for all $a,b\in H^1(S,\lambda)$, where $\wedge$ is the wedge cohomology product.

*Proposition 2.2.1 All the eigenvalues of * are purely imaginary complex numbers.*

Proof: For any $c \in H^1(S,\lambda)\setminus\{0\}$ we have $(c,\bar{c})>0$. On the other hand, $c\wedge\bar{c}$ is purely imaginary, since $\overline{c\wedge\bar{c}} = \bar{c}\wedge c = -c\wedge\bar{c}$. For the eigenvector c, satisfying $*c=\mu c$,

$$0 < (c,\bar{c}) = c\wedge\overline{*c} = c\wedge\overline{\mu c} = \bar{\mu}\, c\wedge\bar{c}. \blacksquare$$

Denote $\Omega^{0,1}(S,\lambda)$ (resp. $\Omega^{1,0}(S,\lambda)$) the subspace of $H^1(S,\mathbb{C})$, generated by the eigenvectors of $*_{S,\lambda}$, corresponding to the eigenvalues with negative (resp. positive) imaginary parts.

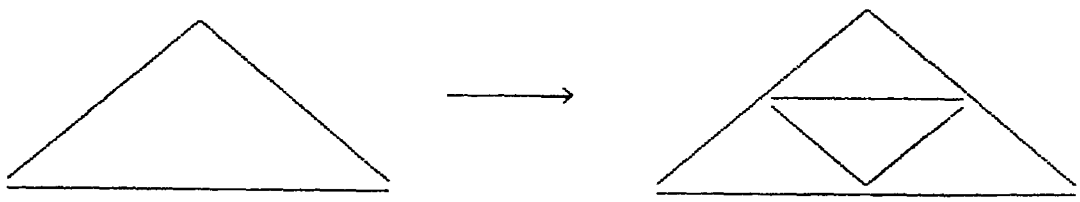
2.2.1 Definition. The piecewise-euclidean jacobian of the pair (S,λ) is the complex torus

$$J(S,\lambda)=(\Omega^{0,1}(S,\lambda))^*/H_1(S,\mathbb{Z}).$$

3. The approximation of the complex jacobians by the piecewise-euclidean ones.

3.0. Regular subdivisions and cochain refinement.

Let S be a simplicial surface. The *regular subdivision* of S is the subdivision obtained by the following refinement of the 2-simplexes of S:



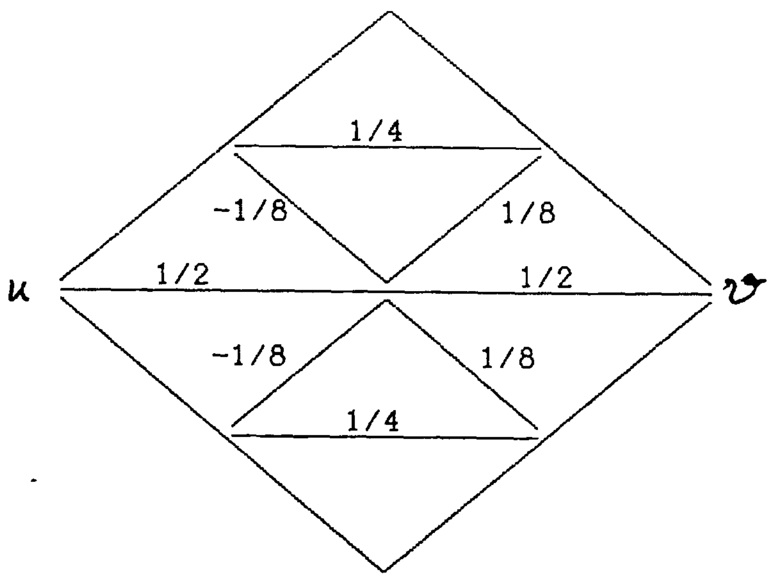
(the middlepoints on each sides are defined as new vertices).

We can also define in the obvious way a regular subdivision of a euclidean simplicial surface. Denote it by $\mathrm{sd}(S,\lambda)$ or $(\mathrm{sd}(S),\mathrm{sd}(\lambda))$.

Proposition 3.0.1. There is a natural isomorphism $\eta: X_{\mathrm{sd}(S,\lambda)} \longrightarrow X_{S,\lambda}$.

Proof: The tautological maps are obviously holomorphic.■

Define the map $\gamma: C^1(S,\lambda) \longrightarrow C^1(\mathrm{sd}(S),\mathrm{sd}(\lambda))$ by the condition that it assigns to an element of the form $\delta_{(u,v)}$ the cochain shown in the picture below:



Proposition 3.0.2. The following diagram is commutative:

$$\begin{array}{ccc} C^1(S,\lambda) & \xrightarrow{\ \gamma\ } & C^1(sd(S),sd(\lambda)) \\ \downarrow & & \downarrow \\ L^2(A^1)(S,\lambda) & \xrightarrow{\ \eta^*\ } & L^2(A^1)(sd(S),sd(\lambda)) \end{array}$$

Proof: Direct computation.■

In the sequel we shall denote by $(S[n],\lambda[n])$ the n-th iteration of sd,applied to (S,λ). We shall also identify $C^1(S,\lambda)$ with its image in $C^1(sd(S),sd(\lambda))$ and the space $L^2(A^1)(S,\lambda)$ with $L^2(A^1(sd(S),sd(\lambda))$.

3.1 Main theorem.

Approximation lemma 3.1.0. The union of the subspaces $W^1(dC^0(S[n],\lambda[n])$ for $n \geq 0$ is L^2-dense in the closure of the subspace of exact 1-forms on $X_{S,\lambda}$.

Proof: This assertion is just the reformulation of the fact that any smooth function can be L^2-approximated by piecewise-linear ones.■

Now we are ready to prove our main theorem:

Theorem 3.1.1. Let (S,λ) *be a euclidean simplicial surface. Then*

$$\lim_{n\longrightarrow\infty} (J(S[n],\lambda[n]) \cong J(X_{S,\lambda}),$$

where $J(X_{S,\lambda})$ *is the usual jacobian of the complex riemannian surface* $X_{S,\lambda}$.

Proof: We realize the jacobian $J(X_{S,\lambda})$ as $(\Omega^{0,1}(X_{S,\lambda}))^*/H_1(X_{S,\lambda},\mathbb{Z})$, where the cohomology group acts on the holomorphic differentials by integration. The jacobians $J(S[n],\lambda[n])$ are defined in the similar way as factors $(\Omega^{0,1}(S[n],\lambda[n]))^*/H_1((S[n],\lambda[n]),\mathbb{Z})$. All the $\Omega^{0,1}(S[n],\lambda[n])$'s can be thought of as lying in the same space $H^1(X_{S,\lambda})$; all the $H^1((S[n],\lambda[n]),\mathbb{Z})$'s as well as $H_1(X_{S,\lambda},\mathbb{Z})$ act on this space, thus being canonically identified. This means, that to prove our theorem it is sufficient to show, that the limit of the subspaces $\Omega^{0,1}(S[n],\lambda[n])$ in $H^1(X_{S,\lambda},\mathbb{C})$ is the subspace $\Omega^{0,1}(X_{S,\lambda})$.

Since the considered spaces are defined in terms of the scalar products $(\ ,\ )_{S[n],\lambda[n]}$ and $(\ ,\ )_{L^2}$, the theorem will be proved, if we show, that

$$\lim_{n\longrightarrow\infty} (\ ,\ )_{S[n],\lambda[n]} = (\ ,\ )_{L^2}.$$

By definition, the scalar products $(\ ,\ )_{S[n],\lambda[n]}$ are induced from the standard scalar product on $L^2(A^1(X_{S,\lambda}))$ via the embedding $c\longrightarrow W^1([c]_n)$, where we write $[c]_n$ instead of $[c]_{S[n],\lambda[n]}$. Therefore, it suffices to show, that $\lim_{n\longrightarrow\infty} W^1([c]_n)$ exists and is harmonic in the usual sence.

The proof proceeds in the following steps.

1. For any $c\in H^1(X,\mathbb{C})$

$$W^1([c]_n)\in Z^1(X,\mathbb{C})\cap(W^1(dC^0(S[n])))^{\perp},$$

where Z^1 is the L^2-closure of the space of closed forms and $\perp$ is the orthogonal complement in the $A^1(X)$ metric. It follows from 2.0.1.

2. $W^1(dC^0(S[n])) \subset W^1(dC^0(S[n+1]))$.

Follows from 3.0.2.

3. $W^1([c]_{n+1})$ is the orthogonal projection of $W^1([c]_n)$ onto $W^1(dC^0(S[n+1]))^{\perp}$.

To verify it it is sufficient to note that $W^1([c]_{n+1})-W^1([c]_n)=W^1([c]_{n+1}-\gamma[c]_n)\in W^1(dC^0(S[n+1]))$.

4. Denote $Z^1(X,\mathbb{C})\cap W^1(dC^0(S[n]))$ by L_n. We have $L_1\subset L_2\subset\ldots L_n\subset\ldots$, and, by the approximation lemma, $\bigcap_{n\geq 0} L_n = \mathcal{H}^1(X,\mathbb{C})$ is the space of the harmonic forms on X. The theorem follows now from 3. and the comleteness of $A^1(X)$.■

APPENDIX.

Explicit formulae

1. Scalar products in C_0.

$$(u,v) = \begin{cases} 0, \text{ if } \{u,v\} \notin S_1 \text{ and } u \neq v \\ \frac{1}{6} \sum S_i \quad \text{if } u=v \text{ (see fig.1)} \\ \frac{1}{12} (S_1+S_2) \text{ if } \{u,v\} \in S_1 \text{ (see fig.3)} \end{cases}$$

2. Scalar products in C^1.

2.1

$$(\delta_{<u,v>}, \delta_{<u,w>}) = \begin{cases} 0 \text{ if } \{u,v,w\} \notin S_2 \text{ and } \{u,v\} \neq \{u,w\} \\ \frac{1}{48S_1}(3b^2+3c^2-a^2)+\frac{1}{48S_2}(3d^2+3e^2-a^2) \text{ if } v=w \text{ (see fig.3)} \\ \frac{1}{48S}(3c^2-a^2-b^2) \text{ if } \{u,v,w\} \in S_2 \text{ (see fig.2)} \end{cases}$$

2.2 If ω,η are 1-cocycles then:

$$(\omega,\eta)=\frac{1}{8}\sum_{\sigma \in S_2}\frac{1}{S}\left[\omega(c)\eta(c)(a^2+b^2-c^2)+\omega(b)\eta(b)(a^2-b^2+c^2)+\omega(a)\eta(a)(b^2-a^2+c^2)\right]$$

(see fig.2)

2.3 If $\omega\in C^1(S,\lambda)$ and $v\in S_0$ then:

$$(\omega,d\delta_v)=\frac{1}{24}\sum_{i\in\mathbb{Z}_n}\frac{1}{S_i}\left[(a_{i+1}^2-a_i^2)(\omega(a_i)-\omega(a_{i+1})-2\omega(b_i))+3b_i^2(\omega(a_i)+\omega(a_{i+1}))\right]$$

where n is the valency of the vertex v (see fig.1).

2.4 If $f\in C^0(S,\lambda)$ and $v\in S_0$ then:

$$(df,d\delta_v)=\frac{1}{8}\sum_{i\in\mathbb{Z}_n}\frac{1}{S_i}\left[(f(v_i)-f(v_{i+1}))(a_i^2-a_{i+1}^2)+b_i^2(2f(v)-f(v_i)-f(v_{i+1}))\right]$$

where n is the valency of the vertex v (see fig.1).

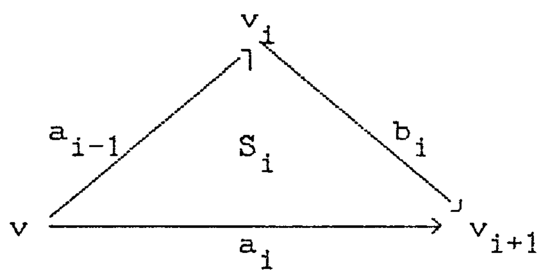


fig.1

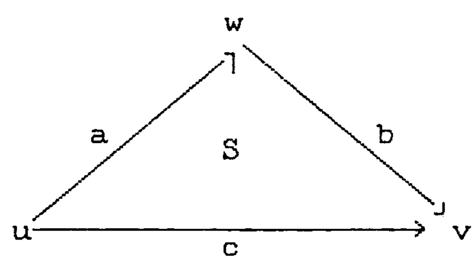


fig.2

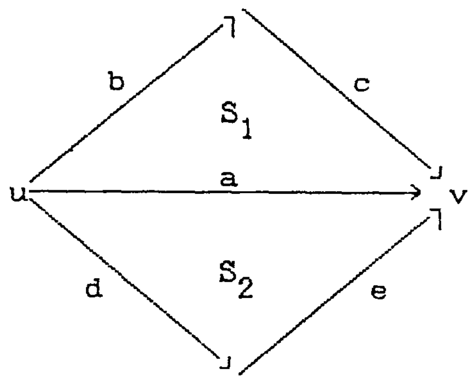


fig.3

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