On the zero slice of the sphere spectrum

Vladimir Voevodsky

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1 Introduction

In [3] we introduced the slice filtration on the motivic stable homotopy category which is a motivic analog of the filtration by the subcategories of n-connected spectra in topology. Since the inclusion functors between different terms of the filtration have right ajoints it makes sense to speak of the projection $s_n(E)$ of a spectrum E to the n-th quotient of this filtration. This projection which is again an object of the motivic stable homotopy category is called the n-th slice of E. Its topological analog is the spectrum $\Sigma^n H_{\pi_n(E)}$ where $\pi_n(E)$ is the n-th stable homotopy group of E and H_A is the Eilenberg-MacLane spectrum corresponding to the abelian group A. In this sense slices provide a motivic replacement of the stable homotopy groups.

The goal of this paper is to prove over fields of characteristic zero the main conjecture of [3] which asserts that the zero slice of the sphere spectrum $\mathbf{1}$ is the motivic Eilenberg-MacLane spectrum $H_{\mathbf{Z}}$. Using the analogy between slices and stable homotopy groups one may interpret this result as a motivic version of the statement that $\pi^s_0(S^0) = \mathbf{Z}$. As an immediate corollary one gets that the slices of any spectrum are modules over $H_{\mathbf{Z}}$.

We obtain our main result from an unstable statement about the motivic Eilenber-MacLane spaces K_n. We introduce the notion of a (homotopically) n-thick space and show that on the one hand the suspension spectrum of any n-thick space belongs to the n-th stage of the slice filtration and on the other hand that for $n > 0$ the cone of the natural map $T^n \to K_n$ is $(n+1)$-thick.

The restriction on the characteristic of the base field k appears in our approach twice. Firstly, we use the model of K_n based on the symmetric powers of $T^n = \mathbf{A}^n/(\mathbf{A}^n - \{0\})$ which is only valid over fields of characteristic zero. Secondly, in order to analyze the structure of the d-th symmetric power we need invertibility of $d!$ in k. We expect that the main result of the paper remains valid in any characteristic even though the intermediate steps we use do not.

The first draft of this paper (with some mistakes) was announced on the Great Lakes K-theory meeting in Evanston in 2001 and I would like to thank all the listeners for pointing out gaps in the original argument.

2 Thom spaces of linear representations and their quotients

We fix a base field k and let Sch/k denote the category of quasi-projective schemes over k. For a finite group scheme G over k we denote by $G - Sch/k$ the category of G-objects in Sch/k. In [2] we defined an analog of the Nisnevich topology on $G - Sch/k$. We let $(G - Sch/k)_{Nis}$ denote the corresponding site and $Spc_\bullet(G)$ denote the category of pointed simplicial sheaves on $(G - Sch/k)_{Nis}$. Following [2] one introduces the class of $\mathbf{A}^1$-weak equivalences on $Spc_\bullet(G)$ and defines the corresponding $\mathbf{A}^1$-homotopy category as the localization with respect to this class.

We let Σ^1_s and Σ^1_T denote the suspensions by the simplicial circle and the sphere $T = \mathbf{A}^1/(\mathbf{A}^1 - \{0\})$ respectively, and $\tilde{\Sigma}_s$ denote the unreduced s-suspension:

$$\tilde{\Sigma}_s F = cone(F_+ \to S^0)$$

For a G-equivariant vector bundle E over X we let $Th(E)$ denote the Thom space $E/E - s_0(X)$ where s_0 is the zero section. For $X = Spec(k)$, equivariant vector bundles are linear representations V of G and $Th(V)$ is the object $V/V - \{0\}$. Note that $Th(V \oplus W) \cong Th(V) \wedge Th(W)$ and $Th(0) = S^0$. Note also that since V is $\mathbf{A}^1$-contractible as a G-scheme there is a natural $\mathbf{A}^1$-equivalence of the form

$$Th(V) \cong \tilde{\Sigma}_s(V - \{0\})$$

We let $Quot_G : Spc_\bullet(G) \to Spc_\bullet$ denote the functor which commutes with colimits and such that $Quot(X_+) = (X/G)_+$ (see [2, Sec. 5.1]).

In this section we prove several general results about the structure of the quotients $Quot_G(Th(V))$. For simplicity, we assume in this section that G is a *finite group*. The following lemma is straightforward.

Lemma 2.1 *Let V_1, V_2 be linear representations of G_1, G_2. Then there is a natural isomorphism*

$$Quot_{G_1\times G_2}(Th(V_1\times V_2)) = Quot_{G_1}(V_1)\wedge Quot_{G_2}(V_2)$$

In particular if $\mathcal{O}^n$ is the trivial representation of G of dimension n then

$$Quot_G(Th(V\oplus\mathcal{O}^n)) = \Sigma^n_T Quot_G(Th(V))$$

For a subgroup H in G we let $V^{\geq H}$ denote the closed subset of H-invariant elements in V and by $V^{>H}$ the subset of elements whose stabilizer is strictly greater than H. Note that for $H_1\neq H_2$ one has $V^{\geq H_1}\cap V^{\geq H_2}\subset V^{>H_1}\cap V^{>H_2}$ and in particular

$$(V^{\geq H_1}-V^{>H_1})\cap(V^{\geq H_2}-V^{>H_2}) = \emptyset$$

For a closed subset Z of V we let GZ denote the orbit of Z i.e. the closed subset $Im(Z\times G\to V\times G\to V)$ where the first map is the closed embedding and the second one is the action.

Lemma 2.2 *Let H be a closed subgroup in G and $N(H)$ the normalizer of H. Then one has*

$$\begin{aligned} &Quot_G((V-GV^{>H})/(V-GV^{\geq H})) = \\ &= Quot_{N(H)}((V-V^{>H})/(V-V^{\geq H})) \end{aligned} \tag{2.1}$$

Proof: Let $A = G/N(H)$ be the set of subgroups adjoint to H. The scheme $GV^{\geq H}-GV^{>H}$ is (non canonically) isomorphic to $(V^{\geq H}-V^{>H})\times A$ and there is a G-equivariant map $GV^{\geq H}-GV^{>H}\to A$. Consider the section of the projection $(GV^{\geq H}-GV^{>H})\times A\to(GV^{\geq H}-GV^{>H})$ defined by this map. Since this projection is etale the image of this section is open and we may consider the closed complement C to this image. Consider the square:

$$\begin{array}{ccc} (V-GV^{\geq H})\times A & \longrightarrow & (V-GV^{>H})\times A - C \\ \downarrow & & \downarrow p \\ V-GV^{\geq H} & \xrightarrow{j} & V-GV^{>H} \end{array}$$

where the vertical morphisms come from the obvious projections and the horizontal morphisms from the obvious embeddings. Let us show that this is an upper distinguished square in $G - Sch/k$. It is clearly a pull-back square, p is etale and j is an open embedding. It remains to check that $p^{-1}(GV^{\geq H} - GV^{>H}) \to GV^{\geq H} - GV^{>H}$ is an isomorphism which follows from our choice of C. We conclude that

$$(V - GV^{>H})/(V - GV^{\geq H}) = (V - GV^{\geq H}) \times A/((V - GV^{>H}) \times A - C)$$

and therefore

$$Quot_G((V - GV^{>H})/(V - GV^{\geq H})) =$$
$$= Quot_G((V - GV^{\geq H}) \times A/((V - GV^{>H}) \times A - C))$$

We further have:

$$Quot_G((V - GV^{\geq H}) \times A) = Quot_{N(H)}(V - GV^{\geq H})$$

and

$$Quot_G((V - GV^{>H}) \times A - C) = Quot_{N(H)}(V - GV^{>H} - (GV^{\geq H} - V^{\geq H}))$$

Since $GV^{\geq H} - GV^{>H} = (V^{\geq H} - V^{>H}) \coprod (GV^{\geq H} - V^{\geq H} - GV^{>H})$ we have

$$(V - GV^{>H} - (GV^{\geq H} - V^{\geq H}))/(V - GV^{\geq H}) = (V - V^{>H})/(V - V^{\geq H})$$

and (2.1) follows.

Remark 2.3 An analog of Lemma 2.2 holds for any finite etale group scheme G and any X in $G - Sch/k$.

Let H be a normal subgroup of G. Then we may consider the relative analog of the $Quot$ functor

$$Quot_{G,H} : Spc_\bullet(G) \to Spc_\bullet(G/H)$$

which commutes with colimits and such that $Quot(X_+) = (X/H)_+$ where X/H is considered with the natural action of G/H. One verifies easily that one has $Quot_G = Quot_{G/H} Quot_{G,H}$.

Lemma 2.4 *If H is a normal subgroup of G of order prime to $char(k)$ then one has:*

$$Quot_G((V - V^{>H})/(V - V^{\geq H})) =$$
$$= Quot_{G/H}((V^{\geq H} - V^{>H})_+ \wedge Quot_{G,H}(Th(V/V^{\geq H})))$$

Proof: Since

$$Quot_G = Quot_{G/H} Quot_{G,H},$$

it is sufficient to show that $Quot_{G,H}((V - V^{>H})/(V - V^{\geq H}))$ is isomorphic to $(V^{\geq H} - V^{>H})_+ \wedge Quot_{G,H}(Th(V/V^{\geq H}))$ as a G/H-space. Since the order of H is prime to $char(k)$ there is an isomorphism $V = V^{\geq H} \oplus (V/V^{\geq H})$. Using this isomorphism we get an isomorphism

$$(V - V^{>H})/(V - V^{\geq H}) = (V^{\geq H} - V^{>H})_+ \wedge Th(V/V^{\geq H})$$

Since the action of H on $V^{\geq H}$ is trivial we get

$$Quot_{G,H}((V - V^{>H})/(V - V^{\geq H})) = (V^{\geq H} - V^{>H})_+ \wedge Quot_{G,H}(Th(V/V^{\geq H}))$$

Combining Lemmas 2.2 and 2.4 we get the following result.

Proposition 2.5 *Let G be a finite group of order prime to $char(k)$. Let further V be a linear representation of G over k such that $V^{\geq G} = 0$. Then $Quot_G(Th(V))$ belongs to the smallest class which is closed under cones, finite coproducts and $\mathbf{A}^1$-equivalences and contains the following objects:*

1. *$\tilde{\Sigma}_s(Quot_G(V_0))$ where V_0 is the open subscheme in V where G acts freely*
2. *$Quot_{N(H)/H}((V^{\geq H} - V^{>H})_+ \wedge Quot_{N(H),H}(Th(V/V^{\geq H})))$ for all subgroups H in G such that $H \neq e, G$.*

Proof: Consider the sequence of open embeddings $V_0 \to V_1 \to \ldots \to V_N = V$ where V_i is the subscheme of points with the stabilizer of order no more than $i+1$ and $N = |G| - 1$. In particular V_0 is the open subscheme where G acts freely. Under our assumption $V_{N-1} = V - \{0\}$ and hence $Th(V) = \tilde{\Sigma}_s(V_{N-1})$.

Lemma 2.6 *Let $X_0 \to X_1 \to \ldots \to X_m$ be a sequence of monomorphisms in $Spc_\bullet(G)$. Then $\tilde{\Sigma}_s(X_m)$ belongs to the smallest class which is closed under simplicial weak equivalences and cones and contains $\tilde{\Sigma}_s(X_0)$ and X_i/X_{i-1} for $i = 1, \ldots, m$.*

Proof: Denote the smallest class satisfying the conditions of the lemma by B. let us show by induction on m that $\tilde{\Sigma}_s(X_m)$ is in B. For $m = 0$ the

statement is obvious. By induction we may assume that $\tilde{\Sigma}_s X_{m-1}$ is in B. The digram

$$\begin{array}{ccccc} pt & \longrightarrow & S^0 & \longrightarrow & S^0 \\ \downarrow & & \downarrow & & \downarrow \\ X_m/X_{m-1} & \longrightarrow & \Sigma_s X_{m-1,+} & \longrightarrow & \Sigma_s X_{m,+} \\ \downarrow & & \downarrow & & \downarrow \\ X_m/X_{m-1} & \longrightarrow & \tilde{\Sigma}_s X_{m-1} & \longrightarrow & \tilde{\Sigma}_s X_m \end{array}$$

whose rows and columns are cofibration sequences, shows that $\tilde{\Sigma}_s X_m = cone(X_m/X_{m-1} \to \tilde{\Sigma}_s X_{m-1})$ and therefore $\tilde{\Sigma}_s X_m$ is in B.

In view of Lemma 2.6 it is sufficient to show that the quotients V_i/V_{i-1} for $i = 1, \ldots, N-1$ belong to the class we consider. This follows directly from Lemmas 2.2 and 2.4.

3 Thick spaces

Definition 3.1 *The class of n-thick objects is the smallest class A_n such that:*

1. *A_n is closed under $\mathbf{A}^1$-equivalences*
2. *A_n is closed under filtering colimits*
3. *if (F_i) is a simplicial object in $\Delta^{op} Spc_\bullet(G)$ such that $F_i \in A_n$ for all i and Δ is the diagonal functor then $\Delta((F_i)) \in A_n$.*
4. *for any smooth X in $G - Sch/k$ and Z closed in X everywhere of codimension $\geq n$, $X/(X - Z)$ is in A_n*

We say that an object is thick if it is 1-thick.

Remark 3.2 Note that we constructed our definition in such a way that an object of $Spc_\bullet(G)$ is 0-thick if and only if it can be build out of smooth schemes by means of homotopy colimits. In particular, unless k has resolution of singularities, it is not clear that any object is 0-thick.

Lemma 3.3 *The class A_n of n-thick objects has the following properties:*

1. *A_n is closed under coproducts*

2. *for a morphism $f : X \to Y$ where X, Y are in A_n one has $cone(f) \in A_n$*

3. *for X in A_n one has $\Sigma^1_T X \in A_{n+1}$*

Proof: Since A_n is closed under filtering colimits and any coproduct is a filtering colimit of finite coproducts it is sufficient to show that if X and Y are in A_n then $X \vee Y$ is in A_n. Let us show first that for any space X in A_n, any smooth scheme U and any Z of codimension at least $n+1$ in U, $X \vee U/(U - Z)$ is in A_n. Indeed the class of X for which this holds clearly satisfies the conditions of Definition 3.1 and therefore contains A_n. Similarly, the class of all Y such that $X \vee Y$ is in A_n clearly satisfies the first three conditions of Definition 3.1 and we have just shown that it satisfies the fourth. Hence it contains A_n.

The cone $cone(f : X \to Y)$ is the diagonal of a bisimplicial object with terms $Y, Y \vee X, Y \vee X \vee X$ etc. which one obtains if one writes the definition of the cone using second dimension for the simplicial interval Δ^1_+. Hence, any class closed under finite coproducts and diagonals is also closed under the cones.

To verify that last condition observe that the class of X such that $\Sigma^1_T X$ is in A_{n+1} satisfies the first three conditions of Definition 3.1. It satisfies the fourth one since

$$\Sigma^1_T(X/(X - Z)) = X \times \mathbf{A}^1/(X \times \mathbf{A}^1 - Z \times \{0\})$$

and codim $Z \times \{0\} = codim Z + 1$.

For a finite etale G-scheme W consider the functor $F \mapsto F^{\wedge W}$ introduced in [2].

Proposition 3.4 *Let W be a finite etale G-scheme of degree d. Then the functor $F \mapsto F^{\wedge W}$ takes n-thick objects to nd-thick objects.*

Proof: It is sufficient to show that the class of F such that $F^{\wedge W}$ is nd-thick satisfies the conditions of Definition 3.1. The first condition follows from the fact that $F \mapsto F^{\wedge W}$ preserves $\mathbf{A}^1$-equivalences (see [2, p.63, Prop. 5.2.11]). The second from the fact that $F \mapsto F^{\wedge W}$ commutes with filtering colimits. The third condition is obvious.

To see that it satisfies the fourth condition consider X in $G - Sch/k$ and Z closed in X everywhere of codimension $\geq n$. An easy generalization of [2, p.63, Rem. 5.2.8] shows that

$$(X/(X-Z))^{\wedge W} = X^W/(X^W - Z^W).$$

and it remains to note that if W is etale then $codim(Z^W) = dcodim(Z)$.

Lemma 3.5 *Let X be a scheme with a free action of G and F be a pointed solid G-sheaf which is n-thick. Then $Quot_G(X_+ \wedge F)$ is n-thick.*

Proof: Let $R : Spc_\bullet(G) \to Spc_\bullet(G)$ be the resolution functor introduced in [2, Constr. 3.6.3, p.43] which takes a simplicial sheaf to a weakly equivalent one with terms being coproducts of representable sheaves. By [2, Prop. 5.1.4, p.57] functor $Quot \circ R$ respects $\mathbf{A}^1$-equivalences and for a solid F the natural morphism $Quot(R(F)) \to Quot(F)$ is a weak equivalence. Hence it is sufficient to show that for F satisfying the conditions of the lemma, $Quot(R(X_+ \wedge F))$ is n-thick.

Since F is assumed to be n-thick it is sufficient to check that the class of F such that $Quot(R(X_+ \wedge F))$ is n-thick satisfies the conditions of Definition 3.1. The first condition holds since the functor $Quot(R(X_+ \wedge -))$ preserves $\mathbf{A}^1$-equivalences. To see the second one recalls that R and $Quot$ commute with filtering colimits. Similarly the third condition follows from the fact that both $Quot$ and R commute with the diagonal functor.

To check the fourth condition we have to verify that $Quot(R(X_+ \wedge (Y/(Y - Z))))$ is n-thick for any closed pair (Y, Z) where $codim(Z) \geq n$. Using again [2, Prop. 5.1.4, p.57] we see that

$$Quot(R(X_+ \wedge (Y/(Y-Z)))) \to Quot(X_+ \wedge (Y/(Y-Z)))$$

is an $\mathbf{A}^1$-equivalence. We further have

$$Quot(X_+ \wedge (Y/(Y-Z))) = Quot(X \times Y)/(Quot(X \times Y) - Quot(X \times Z))$$

Since the action of G on X is free, the scheme $Quot(X \times Y)$ is smooth and since the projection $X \times Y \to Quot(X \times Y)$ is finite, $codim(Quot_G(X \times Z)) = codim(X \times Z) = codim(Z)$.

In the following two results G is the trivial group.

Proposition 3.6 *Let X be an n-thick space over a perfect field k. Then the suspension spectrum $\Sigma^\infty_T(X)$ is in $\Sigma^n_T SH^{eff}$.*

Lemma 3.7 *Let U be a smooth rational scheme over k. Then $\tilde{\Sigma}_s(U)$ is thick.*

Proof: Since U is rational there is a dense open subscheme V in U such that V is an open subscheme in $\mathbf{A}^n$ for some n. Then $\tilde{\Sigma}_s(V) \cong \mathbf{A}^n/V$ is thick and U/V is thick. We conclude by Lemma 2.6 and Lemma 3.3(2) that $\tilde{\Sigma}_s U$ is thick.

4 Symmetric powers of T

Let S_n be the symmetric group. In this section we assume that $char(k) = 0$ or $n < char(k)$. Consider the symmetric power

$$Symm^n(T^m) := Quot_{S_n}((T^m)^{\wedge n})$$

Our goal is to prove the following result.

Theorem 4.1 *For any $n \geq 2$ and $m \geq 1$ the space $Symm^n(T^m)$ is $(m+1)$-thick.*

Consider the linear representations $P(m,n)$ of S^n defined by the permutation action on $(\mathbf{A}^m)^n$. Then

$$Symm^n(T^m) = Quot_{S_n}(Th(P(m,n)))$$

Let $V(m,n)$ be the reduced version of $P(m,n)$

$$V(m,n) = ker(p : (\mathbf{A}^m)^n \to \mathbf{A}^m)$$

where $p(x_1, \ldots, x_n) = x_1 + \cdots + x_m$. Under our assumptions on $char(k)$ there is an isomorphism

$$P(m,n) \cong V(m,n) \times \mathbf{A}^m$$

where the action on $\mathbf{A}^m$ is trivial. Therefore by Lemma 2.1

$$Symm^n(T^m) = Quot_{S_n}(Th(P(m,n))) = \Sigma^m_T Quot_{S_n}(Th(V(m,n)))$$

and Theorem 4.1 follows from Lemma 3.3(3) and Theorem 4.3 below.

Let (n) be the standard set of n-elements and $j_1, \ldots, j_n$ integers ≥ 0 such that

$$\sum ij_i = n. \tag{4.1}$$

Then there is a bijection

$$\phi : (n) \cong (1)^{j_1} \amalg (2)^{j_2} \amalg \cdots \amalg (n)^{j_3} \tag{4.2}$$

and any such bijection defines an embedding

$$e_\phi : S_1^{j_1} \times S_2^{j_2} \times \cdots \times S_n^{j_n} \to S_n$$

The stabilizer in S_n of a point x in $P(m,n)$ is determined by the set of diagonals $x_i = x_j$ which contain x and subgroups of the form $Im(e_\phi)$ are exactly the subgroups which occur as stabilizers of different points.

The ajunction class of a subgroup of the form $Im(e_\phi)$ is determined by the integers $j_1, \ldots, j_n$. The case $j_1 = n$ and $j_i = 0$ for $i \neq 1$ corresponds to a point with all components being different and the case $j_n = 1$ and $j_i = 0$ for $i \neq n$ to a point with all components being the same. We choose one isomorphism of the form (4.2) for each collection $j_1, \ldots, j_n$ satisfying (4.1) and let $H_{j_1,\ldots,j_n}$ denote the corresponding subgroup.

The same classification of stabilizers applies to $V(m,n)$ since $V(m,n)$ is a subspace of $P(m,n)$.

The normalizer of $H = H_{j_1,\ldots,j_n}$ is of the form $G_{j_1,1} \times \cdots \times G_{j_n,n}$ where $G_{j,i}$ is the semi-direct product of S_j and S_i^j with respect to the obvious permutational action. The quotient $N(H)/H$ is of the form $S_{j_1} \times \cdots \times S_{j_n}$.

Consider now the quotient $V(m,n)/V(m,n)^{\geq H}$. One can easily see that it is isomorphic to the sum $\oplus V(m,i)^{\oplus j_i}$, the action of $N(H)$ on it is the direct product of actions of $G_{j_i,i}$ on $V(m,i)^{\oplus j_i}$ and the action of $G_{j_i,i}$ is given by the product of the standard actions of j_i-copies of S_i on $V(m,i)$ and the permutation action of S_{j_i}. Summing this up we get the following result:

Lemma 4.2 *One has an isomorphism of $N(H)/H$ spaces of the form*

$$Quot_{N(H),H}(Th(V(m,n)/V(m,n)^{\geq H})) = \wedge_{i=1}^n Quot_{S_i}(Th(V(m,i)))^{j_i}$$

Theorem 4.3 *For $n \geq 2$ and $m > 0$, the space $Quot_{S_n}(Th(V(m,n)))$ is thick.*

Example 4.4 For $n = 2$, $V(m,n)$ is $\mathbf{A}^m$ with the sign action of $S_2 = Z/2$. Over a field of odd characteristic one has

$$Quot_{S_2}(V(m,2) - \{0\}) = \mathcal{O}(-2)_{\mathbf{P}^{m-1}} - z(\mathbf{P}^{m-1})$$

where z is the sero section. In particular it is a rational variety and therefore

$$Quot_{S_2}(Th(V(m,2))) = \tilde{\Sigma}_s(Quot_{S_2}(V(m,2) - \{0\}))$$

is thick by Lemma 3.7.

Proof of Theorem 4.3: We proceed by induction on $n \geq 2$. Let V_0 be the open subscheme of $V(m,n)$ where S_n acts freely. By Lemma 4.5 below $\tilde{\Sigma}_s Quot_{S_n}(V_0)$ is thick. In view of Proposition 2.5 and Lemma 3.3(1,2) it remains to show that for $H = H_{j_1,\ldots,j_n}$, $H \neq e, S_n$ the space

$$Quot_{N(H)}((V^{\geq H} - V^{>H})_+ \wedge Quot_{N(H),H}(Th(V/V^{\geq H})))$$

is thick. For $n = 2$ any subgroup of S_n is e or S_n and our statement is trivial. Hence we may assume inductively that $n \geq 3$ and the theorem is proved for $n-1$. Since the action of $N(H)$ on $V^{\geq H} - V^{>H}$ is free and the space $F = Quot_{N(H),H}(Th(V/V^{\geq H}))$ is solid it is sufficient by Lemma 3.5 to see that F is thick as a $N(H)/H$-space. Since $H \neq e, S_n$ there exists $n > i \geq 2$ such that $j_i \neq 0$. By the inductive assumption $Quot_{S_i}(Th(V(m,i)))$ is thick and therefore $Quot_{S_i}(Th(V(m,i)))^{j_i}$ is thick as a S_{j_i}-space by Lemma 3.4. By Lemma 4.2, F is the smash product of the form $F' \wedge Quot_{S_i}(Th(V(m,i)))^{j_i}$ and $N(H)/H = G \times S_{j_i}$ where G acts on the first factor and S_{j_i} on the second and we conclude that F is thick.

Lemma 4.5 *The object $Quot_{S_n}(\tilde{\Sigma}_s V_0)$ is thick.*

Proof: By Lemma 3.7 it is sufficient to show that $Quot_{S_n}(V_0)$ is rational. Let $V_{0,0}$ be the open subset in V_0 which consists of $x_1, \ldots, x_n$ such that the first components of all $x_i \in \mathbf{A}^m$ are different. The projection to the first component

$$V_{0,0}(m,n) \to V_0(1,n) \tag{4.3}$$

makes $V_{0,0}(m,n)$ into an equivariant vector bundle over $V_0(1,n)$. Since the action of S_n on $V_0(1,n)$ is free, the map of quotient schemes defined by (4.3) is a vector bundle as well. Hence it is sufficient to show that $Quot(V_0(1,n))$ or, equivalently, $Quot(V(1,n))$ is rational.

The quotient $Quot(P(1,n)) = \mathbf{A}^n/S_n$ can be identified in the standard way with $\mathbf{A}^n$ where the first coordinate is given on $P(1,n)$ by the sum of components. It follows that $Quot(V(1,n)) = \mathbf{A}^{n-1}$ and in particular that it is rational.

5 Reformulation for smooth schemes

Denote for a moment the (pointed, simplicial) sheaves on smooth schemes by $Spc_\bullet(Sm)$ and sheaves on all quasi-projective schemes by $Spc_\bullet(Sch)$. Let A_n be the class of n-thick spaces in $Spc_\bullet(Sch)$ and A_n^{Sm} the similarly defined class in $Spc_\bullet(Sm)$. The functor

$$\pi_* : Spc_\bullet(Sch) \to Spc_\bullet(Sm)$$

which takes a sheaf on Sch/k to its restriction on Sm/k respects limits, colimits and $\mathbf{A}^1$-equivalences. This implies immediately that π_* takes A_n to A_n^{Sm}. Therefore, all the results about thickness proved above remain valid if we work in the context of sheaves on smooth schemes.

6 Motivic Eilenberg-MacLane spaces

In this section k is a field of characteristic zero and we work in the context of sheaves on smooth schemes. Let $K_n = \mathbf{Z}_{tr}(\mathbf{A}^n)/\mathbf{Z}_{tr}(\mathbf{A}^n - \{0\})$ be the n-th motivic Eilenberg-Maclane space over k. Consider the obvious morphism $T^n \to K_n$ where $T^n = T^{\wedge n} = \mathbf{A}^n/(\mathbf{A}^n - \{0\})$. The goal of this section is to prove the following theorem.

Theorem 6.1 *For $n > 0$, the space $\Sigma_s cone(T^n \to K_n)$ is $(n+1)$-thick.*

Example 6.2 Consider the case $n = 1$. It is easy to see using Lemma 6.3 below that K_1 is $\mathbf{A}^1$-equivalent to $(\mathbf{P}^\infty, *)$ where $*$ is a rational point and the morphism $T \to K_1$ corresponds to the standard embedding $(\mathbf{P}^1, *) \to (\mathbf{P}^\infty, *)$. The cone is given by $\mathbf{P}^\infty/\mathbf{P}^1$ and its s-suspension is 2-thick by the reduced analog of Lemma 2.6 and Lemma 3.3(2).

Consider the sheaf K_n^{eff} associated to the presheaf of the form

$$K_n^{eff} : U \mapsto c^{eff}(U \times \mathbf{A}^n/U)/c^{eff}(U \times (\mathbf{A}^n - \{0\})/U) \qquad (6.1)$$

where $c^{eff}(X/U)$ is the monoid of effective finite cycles on X over U and the quotient on the right hand side means that two cycles Z_1, Z_2 on $U \times \mathbf{A}^n$ are identified if $Z_1 - Z_2$ is in $U \times (\mathbf{A}^n - \{0\})$.

Lemma 6.3 *The obvious map $K_n^{eff} \to K_n$ is an $\mathbf{A}^1$-equivalence for $n \geq 1$.*

Proof: Let F^{eff} be the presheaf (6.1) and let F be the presheaf given by

$$U \mapsto \mathbf{Z}_{tr}(\mathbf{A}^n)(U)/\mathbf{Z}_{tr}(\mathbf{A}^n - \{0\})(U)$$

such that the sheaf associated with F is K_n. Consider the singular simplicial presheaves $C_*(F^{eff})$ and $C_*(F)$. The standard argument shows that the morphisms

$$K_n^{eff} = a_{Nis}F^{eff} \to a_{Nis}C_*(F^{eff})$$
$$K_n = a_{Nis}F \to a_{Nis}C_*(F)$$

are $\mathbf{A}^1$-equivalences. For any smooth U the abelian group associated with the monoid $F^{eff}(U)$ coincides with $F(U)$. Therefore, $a_{Nis}C_*(F)$ is the sheaf of simplicial abelian groups associated with the sheaf of simplicial monoids $a_{Nis}C_*(F^{eff})$. Let us show that the natural map

$$a_{Nis}C_*(F^{eff}) \to a_{Nis}C_*(F)$$

is a local equivalence in the Nisnevich topology i.e. that for any henselian local S the map of simplicial sets

$$C_*(F^{eff})(S) \to C_*(F)(S)$$

is a weak equivalence.

For any i, $C_i(F^{eff})(S) = F^{eff}(\mathbf{A}^i_S)$ is the free commutative monoid generated by irreducible closed subsets in $\Delta^i_S \times \mathbf{A}^n$ which are finite and equidimensional over Δ^i_S and have nontrivial intersection with $\Delta^i_S \times \{0\}$.

Therefore, by Lemma 6.4 below it is sufficient to show that

$$\pi_0(C_*(F^{eff})(S)) = 0.$$

An element in $C_0(F^{eff})(S)$ may be represented by a cycle on $\mathbf{A}^n_S$ of the form $Z = \sum n_i Z_i$ where Z_i are closed irreducible subsets of $\mathbf{A}^n_S$ and $n_i > 0$. Since Z_i are finite over S they are local (since S is henselian) and therefore we may assumed that the closed points of all Z_i lie in $\{0\}_S$. Consider now the cycle H on $\mathbf{A}^1_S \times \mathbf{A}^n$ obtained from Z by the pull-back with respect to the map $(t, x) \mapsto x - t$. This cycle is finite over $\mathbf{A}^1_S$. Its restriction to $t = 0$ is Z and the restriction to $t = 1$ has all the closed points in $\{1\}_S$ and therefore lies in $(\mathbf{A}^n - \{0\})_S$. Hence the image h of H in $C_1(F^{eff})(S)$ has the property $\partial_0 h = Z$ and $\partial_1 h = 0$ and we conclude that $\pi_0 = 0$.

Lemma 6.4 *Let M be a commutative simplicial monoid and $f : M \to M^+$ the canonical map from M to the associated simplicial group. If for any n the monoid M_n is a free commutative monoid and $\pi_0(M)$ is a group then f is a weak equivalence.*

Proof: If $\pi_0(K)$ is a group the topological monoid $|M|$ is a group-like associative H-space. For such H-spaces the canonical map $|M| \to \Omega B(|M|)$ is a weak equivalence (follows from [1]). The same argument shows that $|M^+| \to \Omega B(|M^+|)$ is a weak equivalence. It remains to show that $B(|M|) \to B(|M^+|)$ is a weak equivalence. By construction $B(|M|)$ is the same as $|B(M)|$ where $B(M)$ is the diagonal of the bisimplicial object obtained by applying the classifying space construction to each term of M. Since the class of weak equivalences of simplicial sets is closed under taking diagonals it remains to show that for a free commutative monoid M the map $B(M) \to B(M^+)$ is a weak equivalence. For this case see [,].

We are going now to analyze the structure of K_n^{eff}. Let $K_n^{\leq d}$ be the subsheaf in K_n^{eff} associated with the subpresheaf which takes U to the image in $K_n^{eff}(U)$ of the set of finite cycles on $\mathbf{A}^n_U/U$ of degree everywhere $\leq d$ over U. Clearly, $K_n^{eff} = colim_d K_n^{\leq d}$. The sheaf $K_n^{\leq 1}$ coincides with $T^n = \mathbf{A}^n/(\mathbf{A}^n - \{0\})$ naturally embedded into K_n^{eff}. Consider the quotients $K_n^{\leq d}/K_n^{\leq d-1}$.

Lemma 6.5 *Let k be a field of characteristic zero. Then there is an isomorphism*

$$K_n^{\leq d}/K_n^{\leq d-1} \cong Symm^d(T^n) \tag{6.2}$$

where $Symm^d(T^n) := Quot_{S_d}((T^n)^{\wedge d})$.

Proof: Let $c^{\leq d}(\mathbf{A}^n)$ be the sheaf of finite cycles of degree $\leq d$ on $\mathbf{A}^n$. By [,], for k of characteristic zero this sheaf is represented by

$$\coprod_{i \leq d} Symm^i(\mathbf{A}^n) = \coprod_{i \leq d} Quot_{S_i}(\mathbf{A}^{ni})$$

Since the right hand side of (6.2) is given by $Quot_{S_d}(\mathbf{A}^{nd}/(\mathbf{A}^{nd} - \{0\}))$ it remains to show that the restriction of the map

$$c^{\leq d}(\mathbf{A}^n) \to K_n^{\leq d}/K_n^{\leq d-1}$$

to $Quot_{S_d}(\mathbf{A}^{nd})$ is surjective and that two sections of $Quot_{S_d}(\mathbf{A}^{nd})$ over a henselian local scheme S project to the same section of $K_n^{\leq d}/K_n^{\leq d-1}$ if and only if they project to the same section of $Quot_{S_d}(\mathbf{A}^{nd}/(\mathbf{A}^{nd}-\{0\}))$.

The surjectivity is clear. Let $Z=\sum n_iZ_i$, $W=\sum m_jW_j$ be two sections of $c^{\leq d}(\mathbf{A}^n)$ over S of degree strictly d. By definition they project to the same section of $K_n^{\leq d}/K_n^{\leq d-1}$ if both Z and W are equivalent modulo the components lying in $\mathbf{A}^n-\{0\}$ to cycle of degree $\leq d-1$. Since Z_i and W_j are local this means that not all of the closed points of the Z_i's (resp. W_j's) lie in $\{0\}$. This is equivalent to the condition that Z and W lie in $Quot_{S_d}(\mathbf{A}^{nd}-\{0\})$.

Proof of Theorem 6.1: By Lemma 6.3, it is sufficient to show that the space $\Sigma_s(K_n^{eff}/T^n)$ is $(n+1)$-thick. Since $T^n=K_n^{\leq 1}$ the space K_n^{eff}/T^n has a filtration which starts with $K_n^{\leq 2}/K_n^{\leq 1}$ and has quotients of the form $K_n^{\leq d}/K_n^{\leq d-1}$ for $d\geq 3$. Since the class of thick objects is closed under filtering colimits it is sufficient by the obvious reduced analog of Lemma 2.6 and Lemma 3.3(2) to show that $K_n^{\leq d}/K_n^{\leq d-1}$ are $(n+1)$-thick for $d\geq 2$. This follows immediately from Lemma 6.5 and Theorem 4.1.

As a corollary of Theorem 6.1 we can prove the main conjecture of [3] over fildes of characteristic zero.

Theorem 6.6 *Let k be a field of characteristic zero. Then $s_0(\mathbf{1})=H_{\mathbf{Z}}$.*

Proof: Consider the unit map $e:\mathbf{1}\to H_{\mathbf{Z}}$. Since $s_0(H_{\mathbf{Z}})=H_{\mathbf{Z}}$ it is sufficient to show that $s_0(e)$ is an isomorphism i.e. that $s_0(cone(e))=0$. By definition of s_0 this would follow if we can show that $cone(e)$ belongs to $\Sigma_T^1SH^{eff}$. The map e is given on the level of spectra by the maps $T^n\to K_n$ considered above. In particular we have

$$cone(e)=hocolim_n\Sigma_T^{-n}\Sigma_T^{\infty}(cone(T^n\to K_n))$$

Since $\Sigma_T^1SH^{eff}$ is closed under homotopy colimits it remains to check that

$$\Sigma_T^{-n}\Sigma_T^{\infty}(cone(T^n\to K_n))\in\Sigma_T^1SH^{eff}$$

i.e. that

$$\Sigma_T^{\infty}(cone(T^n\to K_n))\in\Sigma_T^{n+1}SH^{eff}$$

Since $\Sigma_T^{n+1}SH^{eff}$ is stable under Σ_s it follows from Theorem 6.1 and Proposition 3.6.

References

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