VOEVODSKY'S NORDFJORDEID LECTURES ON MOTIVIC HOMOTOPY THEORY

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Preliminary version

1. Introduction

This note is a report on Voevodsky's lectures at the summer school on motivic homotopy theory in Nordfjordeid.

2. Motivic stable homotopy theory

In this section we introduce the motivic stable homotopy category of a field k.

2.1. **Spaces.** Let Sm/k be the category of separated smooth schemes of finite type over a field k. The category of *spaces* is the category $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ of Nisnevich sheaves on Sm/k with values in simplicial sets. Any scheme X in Sm/k determines a representable space via the Yoneda embedding. This follows since the Nisnevich topology is sub-canonical. On the other hand, any simplicial set can be viewed as a constant Nisnevich sheaf on Sm/k.

A *pointed space* is a space X together with a map $x : \mathrm{Spec}(k) \to X$, where $\mathrm{Spec}(k)$ is the representable sheaf with constant simplicial structure. We denote the category of pointed spaces by $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}/k)_*$. If X is a space, let X_+ denote the pointed space $X \amalg \mathrm{Spec}(k)$. This defines a left adjoint to the forgetful functor from pointed spaces to spaces. The category of pointed spaces has a symmetric monoidal structure: Suppose X and Y are pointed spaces. Their *smash product* $X \wedge Y$ is the space associated to the presheaf $U \mapsto X(U) \wedge Y(U)$. Note that $\mathrm{Spec}(k)_+$ is the unit for the smash product.

To define the stable motivic homotopy category we employ the notion of bispectra for pointed spaces. In classical stable homotopy theory one 'inverts' the simplicial circle to construct spectra of pointed simplicial sets. This works slightly differently in the motivic context. Most notably; for reasons which will become apparent later, we need two 'circles' to obtain an adequate stable homotopy theory. Let $\Delta[n]$ denote the standard simplicial n-simplex. The *simplicial circle* is the coequalizer of the diagram $\Delta[0] \rightrightarrows \Delta[1]$. By analogy, we denote the corresponding pointed space S^1_s. Let $\mathbb{A}^1 \in \mathrm{Sm}/k$ denote the affine line. The *Tate circle* S^1_t is the space $\mathbb{A}^1 - 0$ pointed by the global section given by the identity. Note that S^1_t is the underlying scheme

of the multiplicative group. We let S_t^n denote the n-fold smash product of the Tate circle, and likewise for the n-simplicial sphere S_s^n.

2.2. **The motivic s-stable homotopy category** $\mathrm{SH}_s^{\mathbb{A}^1}(k)$**.** To invert S_s^1 we form spectra of pointed spaces. This is analogous to classical spectra where the simplicial circle is inverted.

Definition 2.3. An *s-spectrum* E is a sequence of pointed spaces $\{E_n\}_{n\geq 0}$ together with structure maps $S_s^1 \wedge E_n \to E_{n+1}$. A map $E \to E'$ of s-spectra consists of degreewise maps $E_n \to E'_n$ of spaces which are compatible with the structure maps. Let $\mathbf{Spt}_s(k)$ denote the category of s-spectra.

Example 2.4. The *s-suspension spectrum* of a pointed space X is the s-spectrum $\Sigma_s^\infty X$ with n-th term $S_s^n \wedge X$ and identity structure maps.

The next step is to define weak equivalences of s-spectra. Let (X, x) be a pointed space. If $n \geq 1$, then $\pi_n(X, x)$ is the sheaf of homotopy groups associated to the presheaf $U \mapsto \pi_n(X(U), x_U)$ where x_U is the image of x in $X(U)$. This is a sheaf of Abelian groups if $n \geq 2$. The classical suspension homomorphism for a pointed simplicial set induces a suspension homomorphism $\pi_n(X) \to \pi_{n+1}(S_s^1 \wedge X)$ of sheaves of homotopy groups. If E is an s-spectrum and $n \in \mathbb{Z}$, we obtain a sequence

$$\pi_{n+m}(E_m) \longrightarrow \pi_{n+m+1}(S_s^1 \wedge E_m) \longrightarrow \pi_{n+m+1}(E_{m+1}) \longrightarrow \cdots$$

of sheaves of Abelian groups where $m > n$. The *sheaves of stable homotopy groups of* E are the sheaves of Abelian groups

$$\pi_n(E) := \operatorname*{colim}_m \pi_{n+m}(E_m)$$

for $n \in \mathbb{Z}$. A map $E \to E'$ of s-spectra is an *s-stable weak equivalence* if the induced map $\pi_n(E) \to \pi_n(E')$ is an isomorphism of sheaves for all $n \in \mathbb{Z}$.

Definition 2.5. Let $\mathrm{SH}_s(k)$ be the category obtained from $\mathbf{Spt}_s(k)$ by inverting the s-stable weak equivalences.

Remark 2.6. One can show that $\mathbf{Spt}_s(k)$ has the structure of a proper simplicial stable model category, with s-stable weak equivalences as weak equivalences. The homotopy category of this model category is $\mathrm{SH}_s(k)$.

There is an obvious way to smash an s-spectrum with a pointed space. When smashing with S_s^1, the induced simplicial suspension functor becomes an equivalence of $\mathrm{SH}_s(k)$. The word 's-stable' refers to this fact.

Remark 2.7. Let $\mathbf{Spt}$ denote the category of ordinary spectra. There is a canonical functor $\mathbf{Spt} \to \mathbf{Spt}_s(k)$ obtained by considering pointed simplicial sets as pointed spaces. A stable weak equivalence of spectra induces an s-stable weak equivalence of s-spectra, and there is an induced functor $\mathrm{SH} \to \mathrm{SH}_s(k)$ between the corresponding homotopy categories.

The basic organizing principle in motivic homotopy theory is to make the affine line contractible. One way to achieve this is as follows: By evaluating the spaces in an s-spectrum E at a scheme $U \in \mathrm{Sm}/k$ we obtain an ordinary spectrum $E(U)$. An s-spectrum E is called $\mathbb{A}^1$-*local* if the map $E(U) \to E(U \times \mathbb{A}^1)$ induced by the projection $U \times \mathbb{A}^1 \to U$ is a stable weak equivalence for every scheme $U \in \mathrm{Sm}/k$. Next, a map $E \to E'$ of s-spectra is an $\mathbb{A}^1$-*stable weak equivalence* if for any $\mathbb{A}^1$-local s-spectrum F the canonical map $\mathrm{Hom}_{\mathrm{SH}_s(k)}(E', F) \to \mathrm{Hom}_{\mathrm{SH}_s(k)}(E, F)$ is bijective.

Definition 2.8. Let the *motivic s-stable homotopy category* $\mathrm{SH}_s^{\mathbb{A}^1}(k)$ be the category obtained from $\mathbf{Spt}_s(k)$ by inverting the $\mathbb{A}^1$-stable weak equivalences.

Lemma 2.9. *Assume $X \in \mathrm{Sm}/k$. Then the canonical map $X \times \mathbb{A}^1 \to X$ induces an $\mathbb{A}^1$-stable weak equivalence of s-suspension spectra. In particular, $\Sigma_s^\infty(\mathbb{A}^1, 0)$ is zero in the motivic s-stable homotopy category of k.*

We note that the simplicial suspension functor is an equivalence of $\mathrm{SH}_s^{\mathbb{A}^1}(k)$. The last step in the construction of the motivic stable homotopy category is to invert the Tate circle. This will be made precise in the next section.

2.10. **The motivic stable homotopy category** $\mathrm{SH}(k)$**.** The definition we will give of $\mathrm{SH}(k)$ combines the category of s-spectra and the Tate circle S_t^1. In order to make this statement precise we use bispectra as a convenient technical device. An (s,t)-*bispectrum* E consists of pointed spaces $E_{m,n}$ together with structure maps $\sigma_s : S_s^1 \wedge E_{m,n} \to E_{m+1,n}$ and $\sigma_t : S_t^1 \wedge E_{m,n} \to E_{m,n+1}$. In addition, the structure maps have to be compatible in the sense that the following diagram is required to commute.

$$\begin{array}{ccccc} S_s^1 \wedge S_t^1 \wedge E_{m,n} & & \xrightarrow[\cong]{\tau \wedge E_{m,n}} & & S_t^1 \wedge S_s^1 \wedge E_{m,n} \\ \downarrow{\scriptstyle S_s^1 \wedge \sigma_t} & & & & \downarrow{\scriptstyle S_t^1 \wedge \sigma_s} \\ S_s^1 \wedge E_{m,n+1} & \xrightarrow{\sigma_s} & E_{m+1,n+1} & \xleftarrow{\sigma_t} & S_t^1 \wedge E_{m+1,n} \end{array}$$

Here, τ flips the copies of S_s^1 and S_t^1. There is an obvious notion of maps between bispectra. An (s,t)-bispectrum can be interpreted as a spectrum object in the category of s-spectra. That is, a collection $E_n := E_{*,n}$ of s-spectra together with maps of s-spectra $S_t^1 \wedge E_n \to E_{n+1}$ induced by the structure maps in the t-direction. We let $\mathbf{Spt}_{s,t}(k)$ denote the category of (s,t)-bispectra. If X is a pointed space, we denote the corresponding suspension (s,t)-bispectrum by $\Sigma_{s,t}^\infty X$.

If E is an (s,t)-bispectrum and p, q are integers, we let $\pi_{p,q}(E)$ denote the sheaf of bigraded stable homotopy groups associated to the presheaf

$$U \longmapsto \operatorname*{colim}_m \mathrm{Hom}_{\mathrm{SH}_s^{\mathbb{A}^1}(k)}(S_s^{p-q} \wedge S_t^{q+m} \wedge \Sigma_s^\infty U_+, E_m).$$

In this expression we use the observation that smashing with S_s^1 yields an equivalence of categories $\mathrm{SH}_s^{\mathbb{A}^1}(k) \to \mathrm{SH}_s^{\mathbb{A}^1}(k)$, $E \mapsto S_s^1 \wedge E$. Moreover, we assume $q + m \geq 0$.

Definition 2.11. A map $E \to E'$ of (s,t)-bispectra is a *stable weak equivalence* if for all $p, q \in \mathbb{Z}$, the induced map $\pi_{p,q}(E) \to \pi_{p,q}(E')$ is an isomorphism of sheaves.

We are ready to define our main object of study:

Definition 2.12. The *motivic stable homotopy category* $\mathrm{SH}(k)$ of k is the category obtained from $\mathbf{Spt}_{s,t}(k)$ by inverting the stable weak equivalences.

Remark 2.13. We remark that there is an underlying model category structure on $\mathbf{Spt}_{s,t}(k)$ where the weak equivalences are the stable weak equivalences defined in 2.11. Moreover, this model structure is stable, proper and simplicial.

By construction, suspension Σ^1_t with the Tate circle induces an equivalence on the motivic stable homotopy category. The same is true for the simplicial suspension functor Σ^1_s, and the analogue of 2.9 holds in $\mathrm{SH}(k)$.

Lemma 2.14. *The canonical map $\mathbb{A}^1 \to \mathrm{Spec}(k)$ induces a stable weak equivalence of suspension (s,t)-bispectra.*

In 2.15 we show that maps between suspension spectra in $\mathrm{SH}(k)$ can be expressed in terms of maps in $\mathrm{SH}^{\mathbb{A}^1}_s(k)$. This result depends essentially on the fact that schemes have finite cohomological dimension in the Nisnevich topology. For example, it is not clear whether there is a similar result for the etale topology.

Proposition 2.15. *For $X \in \mathrm{Sm}/k$ and $E \in \mathbf{Spt}_{s,t}(k)$ there is an isomorphism*

$$\mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_{s,t}\ X_+, E) = \operatorname*{colim}_m\ \mathrm{Hom}_{\mathrm{SH}^{\mathbb{A}^1}_s(k)}(S^m_t \wedge \Sigma^\infty_s X_+, E_m).$$

Suppose $X, Y \in \mathrm{Sm}/k$. As a special case of 2.15 we obtain the isomorphism

$$\mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_{s,t}\ X_+, \Sigma^\infty_{s,t}\ Y_+) = \operatorname*{colim}_m\ \mathrm{Hom}_{\mathrm{SH}^{\mathbb{A}^1}_s(k)}(S^m_t \wedge \Sigma^\infty_s X_+, S^m_t \wedge \Sigma^\infty_s Y_+).$$

Remark 2.16. We will not discuss the notoriously difficult notion of an adequate smash product for (bi)spectra. One solution is to consider symmetric spectra instead of spectra. An alternative solution is discussed in the lectures of B. I. Dundas.
For our purposes it suffices to know that the handicrafted smash product of bispectra induces a symmetric monoidal structure on $\mathrm{SH}(k)$. The proof of this fact is tedious but straight-forward. The unit for the monoidal structure is the 'sphere spectrum' or the (s,t)-suspension spectrum $\Sigma^\infty_{s,t}\,\mathrm{Spec}(k)_+$ of k and there is an isomorphism between the smash product $\Sigma^\infty_{s,t}X \wedge \Sigma^\infty_{s,t}Y$ and $\Sigma^\infty_{s,t}(X \times Y)$ for $X, Y \in \mathrm{Sm}/k$.

Let us briefly summarize the above constructions:

$$\mathrm{Sm}/k \xrightarrow{\Sigma^\infty_s} \mathrm{SH}^{\mathbb{A}^1}_s(k) \xrightarrow{\Sigma^\infty_t} \mathrm{SH}(k)$$

The spaces in our context are the simplicial sheaves on the Nisnevich site of Sm/k. By using the circles S^1_s and S^1_t one defines spectra of pointed spaces as for spectra

of simplicial pointed sets. From the principle that $\mathbb{A}^1$ should be contractible we use spectra in the s-direction to define the motivic s-stable homotopy category $\mathrm{SH}_s^{\mathbb{A}^1}(k)$. Finally, the motivic stable homotopy category $\mathrm{SH}(k)$ is obtained from (s,t)-bispectra by formally inverting the class of stable weak equivalences. If $\Sigma^\infty_{s,t} : \mathrm{Sm}/k \to \mathrm{SH}(k)$ denotes the (s,t)-suspension spectrum functor, then $\Sigma^\infty_{s,t} = \Sigma^\infty_t \circ \Sigma^\infty_s$.

Remark 2.17. Work in progress by Voevodsky suggests a different construction of the motivic stable homotopy category. The idea is to use framed correspondences, an algebraic notion of tubular neighborhoods, to construct a category equivalent to $\mathrm{SH}(k)$.

Next we specify a triangulated structure on the category $\mathrm{SH}(k)$.[1] As for a wide range of other examples, this additional structure provides a convenient tool to construct long exact sequences in $\mathrm{SH}(k)$.

The category $\mathbf{Spt}_{s,t}(k)$ of (s,t)-bispectra is complete and cocomplete by forming limits and colimits degreewise in pointed spaces. In particular, arbitrary coproducts exist. Furthermore, the coproduct of (s,t)-bispectra induces a coproduct $\vee$ in the motivic stable homotopy category. It is easy to show that $\mathrm{SH}(k)$ is an additive category: The simplicial circle is a cogroup object in the classical unstable homotopy category. Since every spectrum in $\mathrm{SH}(k)$ is a twofold simplicial suspension, it follows that all objects in $\mathrm{SH}(k)$ are Abelian cogroup objects. The set of maps out of any object is thus an Abelian group.

Let us define the distinguished triangles and the shift functor: If E is an (s,t)-bispectrum, then the *shift* $E[1]$ of E is $\Sigma^1_s E$. A map $f : E \to E'$ of (s,t)-bispectra has an associated *cofibration sequence* $E \to E' \to \mathrm{Cone}(f) \to \Sigma^1_s E$. Here, the cone of f is defined by a push-out square in $\mathbf{Spt}_{s,t}(k)$ where $\Delta[1]$ is pointed by 0:

$$\begin{array}{ccc} E & \xrightarrow{E\wedge 0} & E \wedge \Delta[1] \\ {\scriptstyle f}\downarrow & & \downarrow \\ E' & \longrightarrow & \mathrm{Cone}(f) \end{array}$$

The map $\mathrm{Cone}(f) \to \Sigma^1_s E$ collapses E' to a point. A *distinguished triangle* in $\mathrm{SH}(k)$ is a sequence that is isomorphic to the image of a cofibration sequence in $\mathbf{Spt}_{s,t}(k)$. Suppose E is an (s,t)-bispectrum and let $[-,-]$ denote $\mathrm{Hom}_{\mathrm{SH}(k)}$. It follows that any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathrm{SH}(k)$ induces long exact sequences of Abelian groups:

$$\cdots \longrightarrow [E, X[n]] \longrightarrow [E, Y[n]] \longrightarrow [E, Z[n]] \longrightarrow [E, X[n+1]] \longrightarrow \cdots$$

$$\cdots \longrightarrow [Z[n], E] \longrightarrow [Y[n], E] \longrightarrow [X[n], E] \longrightarrow [Z[n-1], E] \longrightarrow \cdots$$

[1]The homotopy categories $\mathrm{SH}_s(k)$ and $\mathrm{SH}_s^{\mathbb{A}^1}(k)$ are also triangulated.

We include some examples of distinguished triangles.

Lemma 2.18. *If $X \hookrightarrow Y$ is a monomorphism of pointed spaces, then*

$$\Sigma^\infty_{s,t} X \longrightarrow \Sigma^\infty_{s,t} Y \longrightarrow \Sigma^\infty_{s,t} Y/X \longrightarrow \Sigma^1_s \Sigma^\infty_{s,t} X$$

is a distinguished triangle in $\mathrm{SH}(k)$.

Proof. It follows from the model structure in 2.13 that the canonically induced map $\mathrm{Cone}(\Sigma^\infty_{s,t}(X \hookrightarrow Y)) \to \Sigma^\infty_{s,t} Y/X$ is a stable weak equivalence. □

Remark 2.19. Note that 2.18 applies to open and closed embeddings in Sm/k.

The following objects are of fundamental interest in motivic homotopy theory: An *upper distinguished square* in Sm/k is a pullback square

(1)
$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

where i is an open embedding, p is an etale map, and $p^{-1}(X-U) \to (X-U)$ induces an isomorphism of reduced schemes. For example, a Zariski open cover $U \cup V$ of X with $W = U \cap V$ gives an upper distinguished square. The Nisnevich topology is generated by coverings $\{i : U \to X, p : V \to X\}$ where i and p give an upper distinguished square with $W = p^{-1}(U)$. Since spaces are sheaves in the Nisnevich topology, we have

Lemma 2.20. *The square of representable spaces corresponding to an upper distinguished square is a pushout square.*

In the following we will use upper distinguished squares to generate examples of distinguished triangles in $\mathrm{SH}(k)$. Part (a) of the next result can be thought of as a generalized Mayer-Vietoris property:

Corollary 2.21. *Consider the upper distinguished square (1).*

(a) *There is a distinguished triangle in* $\mathrm{SH}(k)$*:*

$$\Sigma^\infty_{s,t} W_+ \longrightarrow \Sigma^\infty_{s,t} U_+ \vee \Sigma^\infty_{s,t} V_+ \longrightarrow \Sigma^\infty_{s,t} X_+ \longrightarrow \Sigma^1_s \Sigma^\infty_{s,t} W_+.$$

(b) *There is a naturally induced stable weak equivalence:*

$$\Sigma^\infty_{s,t} V/W_+ \longrightarrow \Sigma^\infty_{s,t} X/U_+.$$

Proof. By 2.19, the pushout square of representable spaces associated to an upper distinguished square is also a homotopy pushout square. This implies (a) since the suspension spectrum functor from spaces to bispectra preserves homotopy pushout squares. To prove (b) we use 2.20 to conclude that there is even an isomorphism between the underlying pointed spaces. □

In the following, we consider the projective line $\mathbb{P}^1$ pointed by the rational point at infinity. Corollary 2.21 allows us to give different models for the mixed spheres.

Lemma 2.22. *In* $\mathrm{SH}(k)$ *there are canonical isomorphisms:*

(a) $\Sigma^\infty_{s,t}\ (\mathbb{A}^1/\mathbb{A}^1-0) = \Sigma^\infty_{s,t}\ (S^1_s \wedge S^1_t)$.

(b) $\Sigma^\infty_{s,t}\ (\mathbb{P}^1,\infty) = \Sigma^\infty_{s,t}\ (S^1_s \wedge S^1_t)$.

Proof. Part (a) follows from 2.14 and the following distinguished triangle obtained from 2.18

$$\Sigma^\infty_{s,t}\ (\mathbb{A}^1-0,1) \longrightarrow \Sigma^\infty_{s,t}\ (\mathbb{A}^1,1) \longrightarrow \Sigma^\infty_{s,t}\ (\mathbb{A}^1/\mathbb{A}^1-0) \longrightarrow \Sigma^1_s\Sigma^\infty_{s,t}\ (\mathbb{A}^1-0,1).$$

To prove (b), one covers the projective line by two affine lines. The choice of a point on $\mathbb{P}^1$ is not important since all points are $\mathbb{A}^1$-homotopic to each other. At any rate, by 2.21(b), we have the distinguished triangle

$$\Sigma^\infty_{s,t}\ (\mathbb{A}^1-0)_+ \longrightarrow \Sigma^\infty_{s,t}\ \mathbb{A}^1_+ \vee \Sigma^\infty_{s,t}\ \mathbb{A}^1_+ \longrightarrow \Sigma^\infty_{s,t}\ \mathbb{P}^1_+ \longrightarrow \Sigma^1_s\Sigma^\infty_{s,t}\ (\mathbb{A}^1-0)_+.$$

To get a distinguished triangle of spaces without disjoint basepoints, we point all spaces by $1 : \mathrm{Spec}(k) \to \mathbb{A}^1 - 0$ and form quotients, e.g.

$$\Sigma^\infty_{s,t}(\mathbb{A}^1-0)_+/\Sigma^\infty_{s,t}\mathrm{Spec}(k)_+ = \Sigma^\infty_{s,t}(\mathbb{A}^1-0,1).$$

In this way we obtain a distinguished triangle, and the claim follows from homotopy invariance 2.14. □

Remark 2.23. The pointed space $T := \mathbb{A}^1/\mathbb{A}^1-0$ is called the *Tate sphere*. There is an 'unstable motivic homotopy category' for spaces and 2.22 holds unstably.

We place emphasize on a typical instance of 2.21(b):

Corollary 2.24. *Suppose* $p : V \to X$ *is an etale morphism and* Z *is a closed subscheme of* X *such that* $p^{-1}(Z) \to Z$ *is an isomorphism. Then there is a canonical isomorphism in* $\mathrm{SH}(k)$*:*

$$\Sigma^\infty_{s,t}\ V/V - p^{-1}(Z) = \Sigma^\infty_{s,t}\ X/X - Z.$$

As in classical homotopy theory, there is a notion of Thom spaces in motivic homotopy theory.

Definition 2.25. Suppose $\mathcal{E} \to X$ is a vector bundle over a smooth scheme X. Let $i : X \to \mathcal{E}$ denote the zero-section. Then the *Thom space* $\mathrm{Th}(\mathcal{E}/X)$ is the space $\mathcal{E}/\mathcal{E} - i(X)$ pointed by the image of $\mathcal{E} - i(X)$.

Example 2.26. We list some examples of Thom spaces.

(a) The Thom space of the trivial 1-bundle $\mathbb{A}^1 \times X \to X$ is the smash product $T \wedge X_+$.

(b) If $\mathcal{E}_1 \to X_1$ and $\mathcal{E}_2 \to X_2$ are vector bundles over smooth k-schemes, then $\mathrm{Th}(\mathcal{E}_1 \times \mathcal{E}_2/X_1 \times X_2) \cong \mathrm{Th}(\mathcal{E}_1/X_1) \wedge \mathrm{Th}(\mathcal{E}_2/X_2)$ as pointed spaces.

For Thom spaces we have the homotopy purity theorem.

Theorem 2.27. *Let $i : Z \hookrightarrow X$ be a closed embedding of smooth schemes over the field k. Denote the corresponding normal vector bundle of Z in X by $\mathbb{N}_{X,Z}$. Then there is a canonical isomorphism in* $\mathrm{SH}(k)$*:*

$$\Sigma^\infty_{s,t}\ \mathrm{Th}(\mathbb{N}_{X,Z}) \cong \Sigma^\infty_{s,t}\ X/X - i(Z).$$

Let us prove the homotopy purity theorem in the following special case: Assume k' is a finite separable extension of k. By the Primitive Element Theorem there exists an element α such that $k' = k(\alpha)$. We obtain a surjective map $\phi : k[x] \to k'$, $x \mapsto \alpha$. Let f be the minimal polynomial of α. Then the induced closed embedding $\mathrm{Spec}(k') \hookrightarrow \mathbb{A}^1_k$ sends the closed point of $\mathrm{Spec}(k')$ to the closed point f of $\mathbb{A}^1_k$.

Lemma 2.28. *Suppose k'/k is a finite separable extension, and let f denote the minimal polynomial of a primitive element α for k' over k. Then there is a canonical isomorphism in* $\mathrm{SH}(k)$

$$\Sigma^\infty_{s,t}\ \mathrm{Th}(\mathbb{N}_{\mathbb{A}^1_k,k'}) \cong \Sigma^\infty_{s,t}\ \mathbb{A}^1_k/\mathbb{A}^1_k - f.$$

Proof. We can identify the normal bundle of the closed embedding $i : \mathrm{Spec}(k') \hookrightarrow \mathbb{A}^1_k$ with the affine line $\mathbb{A}^1_{k'}$. By definition, the pointed space $\mathrm{Th}(\mathbb{N}_{\mathbb{A}^1_k,k'})$ is $\mathbb{A}^1_{k'}/\mathbb{A}^1_{k'} - \{0\}$. Recall from above that $\mathbb{A}^1_k/\mathbb{A}^1_k - i(\mathrm{Spec}(k')) = \mathbb{A}^1_k/\mathbb{A}^1_k - \{f\}$. In $k'[x]$, f factors into irreducible polynomials, say $f_1, \ldots, f_q$. Since k'/k is a separable extension we may assume $f_1 = x - \alpha$, and $f_i(\alpha) \neq 0$ if $i \neq 1$. Define $\rho_\alpha : k'[x] \to k'[x]$ by $x \mapsto x - \alpha$. There is an induced upper distinguished square:

$$\begin{array}{ccc} \mathbb{A}^1_{k'} - \{f_1\} & \longrightarrow & \mathbb{A}^1_{k'} \\ {\scriptstyle \rho^*_\alpha}\downarrow & & \downarrow{\scriptstyle \rho^*_\alpha} \\ \mathbb{A}^1_{k'} - \{0\} & \longrightarrow & \mathbb{A}^1_{k'} \end{array}$$

By 2.21(b) we conclude that there is a canonical isomorphism

$$\Sigma^\infty_{s,t}\ \mathrm{Th}(\mathbb{N}_{\mathbb{A}^1_k,k'}) \cong \Sigma^\infty_{s,t}\ \mathbb{A}^1_{k'}/\mathbb{A}^1_{k'} - \{f_1\}$$

in $\mathrm{SH}(k)$. The closed embedding $\mathbb{A}^1_{k'} - \{f_1\} \hookrightarrow \mathbb{A}^1_{k'}$ fits into the upper distinguished square:

$$\begin{array}{ccc} \mathbb{A}^1_{k'} - \{f_1, f_2, \ldots, f_q\} & \longrightarrow & \mathbb{A}^1_{k'} - \{f_2, \ldots, f_q\} \\ \downarrow & & \downarrow \\ \mathbb{A}^1_{k'} - \{f_1\} & \longrightarrow & \mathbb{A}^1_{k'} \end{array}$$

Similarly to the above, we obtain a canonical isomorphism

$$\Sigma^\infty_{s,t}\ \mathrm{Th}(\mathbb{N}_{\mathbb{A}^1_k,k'}) \cong \Sigma^\infty_{s,t}\ \mathbb{A}^1_{k'} - \{f_2,\dots,f_q\}/\mathbb{A}^1_{k'} - \{f_1,f_2,\dots,f_q\}$$

in $\mathrm{SH}(k)$. The right hand side of the isomorphism is related to $\mathbb{A}^1_k/\mathbb{A}^1_k - \{f\}$ via the following upper distinguished square.

$$\begin{array}{ccc} \mathbb{A}^1_{k'} - \{f_1,f_2,\dots,f_q\} & \longrightarrow & \mathbb{A}^1_{k'} - \{f_2,\dots,f_q\} \\ \downarrow & & \downarrow \\ \mathbb{A}^1_k - \{f\} & \longrightarrow & \mathbb{A}^1_k \end{array}$$

There results a canonical isomorphism

$$\Sigma^\infty_{s,t}\ \mathbb{A}^1_{k'} - \{f_2,\dots,f_q\}/\mathbb{A}^1_{k'} - \{f_1,f_2,\dots,f_q\} \cong \Sigma^\infty_{s,t}\ \mathbb{A}^1_k/\mathbb{A}^1_k - \{f\}$$

in $\mathrm{SH}(k)$. This allows us to conclude the proof. □

3. Cohomology theories

The introduction of the motivic stable homotopy category via spectra provides a framework for defining cohomology theories of algebraic varieties. We will consider the following examples: Motivic cohomology, algebraic K-theory and algebraic cobordism.

In 2.22(b) we noted that the (s,t)-suspension spectra of $(\mathbb{P}^1,\infty)$ and $S^1_s \wedge S^1_t$ are canonically isomorphic in $\mathrm{SH}(k)$. In fact something better is a foot; one can replace the two suspension coordinates S^1_s and S^1_t by the single suspension coordinate $\mathbb{P}^1$ without changing the stable motivic homotopy category. This approach is often convenient in order to construct spectra which represent cohomology theories on Sm/k.

3.1. **The motivic Eilenberg-MacLane spectrum $\mathbf{H}\mathbb{Z}$.** Let us first motivate the construction of the motivic Eilenberg-MacLane spectrum $\mathbf{H}\mathbb{Z}$. In stable homotopy theory, the key to understand cohomology theories is to study their representing spectra. An important example is singular cohomology. We let $K(\mathbb{Z},n)$ denote the Eilenberg-MacLane simplicial set with homotopy groups

$$\pi_i K(\mathbb{Z},n) = \begin{cases} \mathbb{Z} & i=n, \\ 0 & i\neq n. \end{cases} \tag{2}$$

To describe a specific model, let $\mathbb{Z}(-)$ be the functor which associates to a pointed simplicial set (X,x) the simplicial Abelian group $\mathbb{Z}[X]/\mathbb{Z}[x]$. That is, the simplicial free Abelian group generated by X modulo the copy of the integers generated by the basepoint. As $K(\mathbb{Z},n)$ one can choose the underlying simplicial set of the simplicial Abelian group $\mathbb{Z}(\Delta^n/\partial\Delta^n)$. The $K(\mathbb{Z},n)$'s define a spectrum $H\mathbb{Z}$ which represents singular cohomology. If X is a simplicial set, then the integral singular cohomology groups $H^n(X,\mathbb{Z})$ coincide with $\mathrm{Hom}_{\mathrm{SH}}(\Sigma^\infty X_+, H\mathbb{Z})$.

Motivic cohomology is supposed to be an algebro-geometric analogue of singular cohomology. In motivic homotopy theory it is clear that (2) fails; however, one can construct a spectrum $\mathbf{H}\mathbb{Z}$ that represents motivic cohomology. This uses the theory of algebraic cycles. Next, we will indicate how one defines the integral motivic Eilenberg-MacLane spectrum.

For every scheme $X \in \mathrm{Sm}/k$ we will describe a pointed space $L(X)$. One should think of L as a version of the functor $\mathbb{Z}(-)$. The value of $L(X)$ at $U \in \mathrm{Sm}/k$ is the free Abelian group generated by *cycles*, that is, closed irreducible subsets of $U \times X$ which are finite over U and surjective on a connected component. The graph $\Gamma(f)$ of a morphism $f : U \to X$ is an example of a cycle in $U \times X$. One can verify that $L(X)$ is a Nisnevich sheaf, and that the graph yields a map of Nisnevich sheaves $X \to L(X)$. One can extend[2] L to take values on the mixed spheres $S^p_s \wedge S^q_t$. For example, $L(\mathbb{P}^1, \infty)$ is the quotient $L(\mathbb{P}^1)/L(\mathrm{Spec}(k))$ of Nisnevich sheaves of Abelian groups considered as a pointed space. To construct the structure maps of the motivic Eilenberg-MacLane spectrum, one uses the exterior product map of cycles to define a natural map:

$$L(S^p_s \wedge S^q_t) \wedge L(S^m_s \wedge S^n_t) \longrightarrow L(S^{p+m}_s \wedge S^{q+n}_t).$$

In particular, the compositions

$$\sigma_s \colon S^1_s \wedge L(S^m_s \wedge S^n_t) \xrightarrow{\mathrm{graph}\wedge\mathrm{id}} L(S^1_s) \wedge L(S^m_s \wedge S^n_t) \longrightarrow L(S^{m+1}_s, S^n_t), \quad \text{and}$$

$$\sigma_t \colon S^1_t \wedge L(S^m_s \wedge S^n_t) \xrightarrow{\mathrm{graph}\wedge\mathrm{id}} L(S^1_t) \wedge L(S^m_s \wedge S^n_t) \longrightarrow L(S^m_s, S^{n+1}_t)$$

yield the structure maps of the motivic Eilenberg-MacLane spectrum.

Definition 3.2. The *Eilenberg-MacLane spectrum* $\mathbf{H}\mathbb{Z}$ is the (s,t)-bispectrum with pointed spaces $\mathbf{H}\mathbb{Z}_{m,n} := L(S^m_s \wedge S^n_t)$ and structure maps as above.

Remark 3.3. To define the Eilenberg-MacLane spectrum with finite coefficients $\mathbb{Z}/\ell$ one considers the reduction of $L(X)$ modulo ℓ in the category of Abelian sheaves, and proceeds as above.

We can now define the motivic cohomology groups of (s,t)-bispectra.

Definition 3.4. Let E be an (s,t)-bispectrum and let p, q be integers. The *integral motivic cohomology groups* of E are defined by

$$\mathbf{H}\mathbb{Z}^{p,q}(E) := \mathrm{Hom}_{\mathrm{SH}(k)}(E, S^{p-q}_s \wedge S^q_t \wedge \mathbf{H}\mathbb{Z}).$$

Suppose $X \in \mathrm{Sm}/k$. The integral motivic cohomology group $\mathbf{H}^{p,q}(X, \mathbb{Z})$ of X in degree p and weight q is by definition $\mathbf{H}\mathbb{Z}^{p,q}(\Sigma^\infty_{s,t} X_+)$. One can show that $\mathbf{H}^{p,q}(X, \mathbb{Z})$

[2]To be more precise, L is required to preserve colimits as a functor from pointed spaces to Nisnevich sheaves with values in simplicial Abelian groups. Since every pointed space is a colimit of representable functors, it suffices to describe the values of L on Sm/k.

is isomorphic to Bloch's higher Chow group $\mathrm{CH}^q(X, 2q-p)$. There is a spectral sequence whose E_2-terms are the integral motivic cohomology groups of X and whose target groups are the algebraic K-groups of X. In section 4 we will outline an alternative approach to construct such a spectral sequence depending on some conjectures about $\mathbf{H}\mathbb{Z}$.

3.5. **The algebraic K-theory spectrum** KGL**.** If m is a non-negative integer, we denote the Grassmannian of n-planes in the $n+m$-dimensional affine space over k by $\mathrm{Gr}_n(\mathbb{A}^{n+m})$. By letting m tend to infinity one obtains a directed system of spaces; its colimit is denoted by BGL_n. There are canonical monomorphisms $\mathrm{BGL}_n \hookrightarrow \mathrm{BGL}_{n+1}$ for all n. We write BGL for the colimit. Instead of letting $\mathbb{Z} \times \mathrm{BGL}$ be the n-th term of the algebraic K-theory spectrum, one defines KGL as a fibrant replacement of $\mathbb{Z} \times \mathrm{BGL}$ in the unstable motivic homotopy theory of k. The reason is that the classifying map

$$\mathbb{P}^1 \wedge \mathrm{KGL} \xrightarrow{\beta} \mathrm{KGL} \tag{3}$$

for the canonical Bott element of $K_0(\mathbb{P}^1 \wedge \mathbb{Z} \times \mathrm{BGL})$ is the lift of an isomorphism $\mathbb{P}^1 \wedge \mathbb{Z} \times \mathrm{BGL} \xrightarrow{\cong} \mathbb{Z} \times \mathrm{BGL}$ in the unstable motivic homotopy category. We will not describe this isomorphism in detail.

Definition 3.6. Let **KGL** be the $\mathbb{P}^1$-spectrum $(\mathrm{KGL}, \mathrm{KGL}, \cdots)$ with structure maps given by (3).

As for topological K-theory, there is also Bott periodicity in algebraic K-theory: Since the structure maps of **KGL** are defined by lifting an isomorphism in the unstable motivic homotopy category, there is clearly an isomorphism $\mathbb{P}^1 \wedge \mathbf{KGL} = \mathbf{KGL}$ in $\mathrm{SH}(k)$.

3.7. **The algebraic cobordism spectrum MGL.** Recall the notation in 3.5, and let $\gamma_{n,m} \to \mathrm{Gr}_n(\mathbb{A}^{n+m})$ be the tautological vector bundle. The canonical morphism $\mathrm{Gr}_n(\mathbb{A}^{n+m}) \to \mathrm{Gr}_n(\mathbb{A}^{n+m+1})$ is covered by a bundle map $\gamma_{n,m} \to \gamma_{n,m+1}$; by taking the colimit over m we get the universal n-dimensional vector bundle $\gamma_n \to \mathrm{BGL}_n$. For example, the product $\mathbb{A}^1 \times \gamma_n \to \mathrm{BGL}_n$ with the trivial one-dimensional bundle $\mathbb{A}^1 \to \mathrm{Spec}(k)$ is classified by the canonical map $\mathrm{BGL}_n \to \mathrm{BGL}_{n+1}$. In particular, there is a bundle map $\mathbb{A}^1 \times \gamma_n \to \gamma_{n+1}$. By taking Thom spaces, we obtain the map

$$\mathrm{Th}(\mathbb{A}^1) \wedge \mathrm{Th}(\gamma_n) = (\mathbb{A}^1/\mathbb{A}^1 - 0) \wedge \mathrm{Th}(\gamma_n) \to \mathrm{Th}(\gamma_{n+1}). \tag{4}$$

Note that we may apply part (b) of 2.26, since $\gamma_n \to \mathrm{BGL}_n$ is a colimit of vector bundles of smooth schemes. By the identification of $\mathbb{P}^1$ with $\mathbb{A}^1/\mathbb{A}^1 - 0$ in 2.22 and by (4), there results maps for $n \geq 0$

$$c_n : \mathbb{P}^1 \wedge \mathrm{Th}(\gamma_n) \to \mathrm{Th}(\gamma_{n+1}). \tag{5}$$

Definition 3.8. The *algebraic cobordism spectrum* **MGL** consists of the sequence of pointed spaces $(\mathrm{Th}(\gamma_0), \mathrm{Th}(\gamma_1), \ldots, \mathrm{Th}(\gamma_n), \ldots)$, together with structure maps given by (5).

The analogy between **MGL** and the ordinary complex cobordism spectrum should be obvious. We remark that results about orientations, maps of rings and formal group laws for classical spectra carry over to $\mathbb{P}^1$-spectra. Properties of the algebraic cobordism spectrum are used in the proof of the Milnor conjecture.
(((Bloch-Kato-conjecture)))

4. The slice filtration

In classical homotopy theory, the Eilenberg-MacLane space $K(\mathbb{Z}, n)$ has a unique non-trivial homotopy group and up to homotopy equivalence there is a unique such space for each n. Thus one has an internal description of the classical Eilenberg-MacLane spectrum in the stable homotopy category. But the situation in motivic homotopy theory is quite different. For example, the homotopy groups $\pi_{p,q}\mathbf{H}\mathbb{Z}$ are often non-zero. One can hope that the so-called *slice filtration* provides an internal description of the spectrum $\mathbf{H}\mathbb{Z}$. We will discuss some conjectures for the slice filtrations of the sphere spectrum $\mathbf{1} = \Sigma^\infty_{s,t}\mathrm{Spec}(k)_+$ and of the Eilenberg-MacLane and algebraic K-theory spectra of k.

Let $\mathrm{SH}^{\mathrm{eff}}(k)$ be the smallest triangulated sub-category of $\mathrm{SH}(k)$ which is closed under direct sums and contains all the (s,t)-bispectra $\Sigma^\infty_{s,t}X_+$ where $X \in \mathrm{Sm}/k$. We point out that this category does not contain the desuspension spectra $\Sigma_t^{-n}\Sigma^\infty_{s,t}X_+$ for any $n \geq 1$. The 'effective' s-stable homotopy category $\mathrm{SH}^{\mathrm{eff}}_s(k)$ is constructed similarly by replacing $\Sigma^\infty_{s,t}$ with Σ^∞_s in the above definition.

In this section we will study the sequence of full embeddings of categories:

$$\cdots \hookrightarrow \Sigma^1_{s,t}\mathrm{SH}^{\mathrm{eff}}(k) \hookrightarrow \mathrm{SH}^{\mathrm{eff}}(k) \hookrightarrow \Sigma^{-1}_{s,t}\mathrm{SH}^{\mathrm{eff}}(k) \hookrightarrow \cdots \tag{6}$$

Each $\Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)$ is a triangulated category with arbitrary direct sums and a set of compact generators. The sequence (6) is called the *slice filtration of* k. It is a filtration of $\mathrm{SH}(k)$ in the sense that the smallest triangulated category which contains every $\Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)$ and is closed under arbitrary direct sums is the motivic stable homotopy category of k. By the theory of triangulated categories, the full inclusion $i_n : \Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k) \hookrightarrow \mathrm{SH}(k)$ allows a right adjoint $r_n : \mathrm{SH}(k) \to \Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)$ which satisfies $r_n \circ i_n \cong \mathrm{Id}$. We denote the other composition by

$$f_n : \Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k) \xrightarrow{r_n} \mathrm{SH}(k) \xrightarrow{i_n} \Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k).$$

The counit $f_n \to \mathrm{Id}$ applied to the functor f_{n-1} determines a natural transformation $f_n = f_n \circ f_{n-1} \to f_{n-1}$. If E is an (s,t)-bispectrum, the *slice tower* of E consists of

the distinguished triangles

$$f_nE \longrightarrow f_{n-1}E \longrightarrow s_nE \longrightarrow \Sigma_s f_nE \tag{7}$$

in $\mathrm{SH}(k)$ for all $n \in \mathbb{Z}$.

Definition 4.1. The n-th *slice* of E is s_nE.

The slice tower (7) is analogous to the Postnikov tower and the slices are the cofibers or quotients. Recall that the Postnikov tower of an ordinary spectrum E is a diagram of cofibrations with colimit E:

$$\cdots \longrightarrow P_{-1}E \xrightarrow{p_0} P_0E \xrightarrow{p_1} P_1E \longrightarrow \cdots \xrightarrow{p_n} P_nE \xrightarrow{p_{n+1}} \cdots .$$

Here, the canonical map $P_nE \to E$ induces isomorphisms on all stable homotopy groups of index less or equal to n, and $\pi_i(P_nE) = 0$ for $i > n$. Thus the cofiber of the map p_n is an Eilenberg-MacLane spectrum $\Sigma^n H\pi_n(E)$. In particular, it is a module over $H\mathbb{Z}$. The zeroth stage of the Postnikov tower of the ordinary sphere spectrum is the Eilenberg-MacLane spectrum $H\mathbb{Z}$. This motivates the following fundamental conjecture:

Conjecture 4.2. $s_0\mathbf{1} = \mathbf{H}\mathbb{Z}$.

The collection of the functors s_n is compatible with the smash product: If E and F are objects of $\mathrm{SH}(k)$, there is a natural map

$$s_n(E) \wedge s_m(F) \longrightarrow s_{n+m}(E \wedge F).$$

In particular, we get a map $s_0(\mathbf{1}) \wedge s_n(E) \to s_n(E)$. If Conjecture 4.2 holds, this shows that each slice of an (s,t)-bispectrum has a natural and unique structure of a module over the motivic cohomology spectrum $\mathbf{H}\mathbb{Z}$.

At present, the following is known about Conjecture 4.2.

Theorem 4.3. *Conjecture 4.2 holds for fields of characteristic zero.*

To sketch a proof of Theorem 4.3, we will first describe the zeroth slice of $\mathbf{H}\mathbb{Z}$.

Theorem 4.4. *If k is a field of characteristic zero, then $s_0\mathbf{H}\mathbb{Z} = \mathbf{H}\mathbb{Z}$.*

The proof of 4.4 uses that $r_0\mathbf{H}\mathbb{Z} = \mathbf{H}\mathbb{Z}$, and $r_1\mathbf{H}\mathbb{Z} = 0$. By the first statement, $\mathbf{H}\mathbb{Z}$ can be build out of (s,t)-suspension spectra. The result for r_1 is a consequence of the cancellation theorem.

To finish the sketch of the proof of 4.3, it remains by 4.4 to show that the fiber of the unit map $\mathbf{1} \to \mathbf{H}\mathbb{Z}$ belongs to $\Sigma^1_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)$. This is known if the base field k has characteristic zero: The proof uses properties of the infinite symmetric products of $\mathbb{A}^n$ and $\mathbb{A}^n - 0$.

There is a slice filtration on $\mathrm{SH}^{\mathbb{A}^1}_s(k)$ constructed analogous to (6). One can show that the Tate suspension functor Σ^∞_t preserves the slice filtrations in the sense

that $\Sigma_t^\infty(\Sigma_t^n \mathrm{SH}_s^{\mathbb{A}^1}(k)) \subseteq \Sigma_{s,t}^n \mathrm{SH}(k))$. To get back into $\mathrm{SH}_s^{\mathbb{A}^1}$ we use that the Tate suspension functor has a right adjoint $\Omega_t^\infty : \mathrm{SH}(k) \to \mathrm{SH}_s^{\mathbb{A}^1}(k)$. For Ω_t^∞ there is the following conjecture.

Conjecture 4.5. *There is an inclusion* $\Omega_t^\infty(\Sigma_{s,t}^n \mathrm{SH}^{\mathrm{eff}}(k)) \subseteq \Sigma_s^n \mathrm{SH}_s(k)$.

Recall from topology that taking the suspension and then the loop space preserves connectivity. One can use an inductive argument to show that the following conjecture would imply 4.5.

Conjecture 4.6. *If* $X \in \mathrm{Sm}/k$ *and* $n \geq 0$, *then* $\Omega_t^1 \Sigma_t^1 \Sigma_s^\infty(S_t^n \wedge X_+) \in \Sigma_t^n \mathrm{SH}_s^{\mathbb{A}^1}(k)$.

This conjecture is not known at present, even if k is a field of characteristic zero. A possible approach to prove 4.6 is to develop a theory of operads, in order to have a detection principle for the Tate loop spaces $\Omega_t X_+$. Framed correspondences might be a useful tool in working out such a theory.

Let us finally relate the above machinery to a possibly new approach to some recent results in algebraic K-theory. For twenty years ago, Beilinson conjectured the existence of an Atiyah-Hirzebruch type spectral sequence for the algebraic K-groups of nice schemes. For fields, there is a construction of such a spectral sequence due to Bloch and Lichtenbaum. More general versions were developed later by Friedlander and Suslin, and also by Levine.

Assume $X \in \mathrm{Sm}/k$. From the slice tower (7) of an (s,t)-spectrum E one can construct a spectral sequence with input terms the cohomology with coefficients in the stable homotopy groups of E and target groups the cohomology groups represented by E. In the example of algebraic K-theory we get the spectral sequence

$$\mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma_{s,t}^\infty X_+, S_s^{p-q} \wedge S_t^q \wedge s_0\mathbf{KGL}) \Rightarrow K_{-p-q}(X). \tag{8}$$

The problem one encounters is to identify the input terms with motivic cohomology. This is essentially the content of the following conjecture.

Conjecture 4.7. $s_0\mathbf{KGL} = \mathbf{H}\mathbb{Z}$.

Because of the Bott periodicity, this conjecture describes indeed all slices of KGL. If Conjecture 4.7 holds, we would immediately obtain an Atiyah-Hirzebruch type spectral sequence for algebraic K-theory as conjectured by Beilinson:

$$\mathbf{H}^{p-q,q}(X, \mathbb{Z}) \Rightarrow K_{-p-q}(X). \tag{9}$$

To give some further evidence that this approach leads to the spectral sequence in (9), we remark that Conjectures 4.2 and 4.5 imply Conjecture 4.7.

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