Subsystems and regular quotients of C-systems[1]

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Abstract

C-systems were introduced by J. Cartmell under the name “contextual categories”. In this note we study sub-objects and quotient-objects of C-systems. In the case of the sub-objects we consider all sub-objects while in the case of the quotient-objects only *regular* quotients that in particular have the property that the corresponding projection morphism is surjective both on objects and on morphisms.

It is one of several short papers based on the material of the ”Notes on Type Systems” by the same author.

1 Introduction

1-2

C-systems where introduced by John Cartmell ([1], [2, p.237]) and studied further by Thomas Streicher (see [3, Def. 1.2, p.47]). Both authors used the name contextual categories for these structures. We feel it to be important to use the word “category” only for constructions which are invariant under equivalences of categories. For the essentially algebraic structure with two sorts “morphisms” and “objects” and operations “source”, “target”, “identity” and “composition” we suggest to use the word pre-category. Since the additional structures introduced by Cartmell are not invariant under equivalences we can not say that they are structures on categories but only that they are structures on pre-categories. Correspondingly, Cartmell objects should be called “contextual pre-categories”. We suggest to use the name C-systems instead.

1-3

Our first result, Proposition 2.3, shows that C-systems can be defined in two equivalent ways: one, as was originally done by Cartmell, using the condition that certain squares are pull-back and another using a new operation $f \mapsto s_f$ which is almost everywhere defined and satisfies simple algebraic conditions.

This description is useful for the study of quotients and homomorphisms of C-systems.

To any C-system CC we associate a set $\widetilde{Ob}(CC)$ and eight partially defined operations on the pair of sets $(Ob(CC), \widetilde{Ob}(CC))$.

In Proposition 4.3 we construct a bijection between C-subsystems of a given C-system CC and pairs of subsets $(C, \widetilde{C})$ in $(Ob(CC), \widetilde{Ob}(CC))$ which are closed under the eight operations.

In Proposition 5.4 we construct a bijection between *regular congruence relations* on CC and pairs of equivalence relations on $(Ob(CC), \widetilde{Ob}(CC))$ which are compatible with the eight operations and satisfy some additional properties.

These two results strongly suggest that the theory of C-systems is equivalent to the theory with the sorts $(Ob, \widetilde{Ob})$ and the eight operations which we consider together with some relations among these operations.

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The essentially algebraic version of this other theory is called the theory of B-systems and will be consider in the sequel [5].

This is one of the short papers based on the material of [4] by the same author. I would like to thank the Institute Henri Poincare in Paris and the organizers of the "Proofs" trimester for their hospitality during the preparation of this paper. The work on this paper was facilitated by discussions with Richard Garner and Egbert Rijke.

2 C-systems

By a pre-category C we mean a pair of sets $Mor(C)$ and $Ob(C)$ with four maps

$$\partial_0, \partial_1 : Mor(C) \to Ob(C)$$

$$Id : Ob(C) \to Mor(C)$$

and

$$\circ : Mor(C)_{\partial_1} \times_{\partial_0} Mor(C) \to Mor(C)$$

which satisfy the well known conditions of unity and associativity (note that we write composition of morphisms in the form $f \circ g$ or fg where $f : X \to Y$ and $g : Y \to Z$). These objects would be usually called categories but we reserve the name "category" for those uses of these objects that are invariant under the equivalences.

Definition 2.1 *A C0-system is a pre-category CC with additional structure of the form*

1. *a function $l : Ob(CC) \to \mathbf{N}$,*
2. *an object pt,*
3. *a map $ft : Ob(CC) \to Ob(CC)$,*
4. *for each $X \in Ob(CC)$ a morphism $p_X : X \to ft(X)$,*
5. *for each $X \in Ob(CC)$ such that $l(X) > 0$ and each morphism $f : Y \to ft(X)$ an object f^*X and a morphism $q(f, X) : f^*X \to X$,*

which satisfies the following conditions:

1. *$l^{-1}(0) = \{pt\}$*
2. *for X such that $l(X) > 0$ one has $l(ft(X)) = l(X) - 1$*
3. *$ft(pt) = pt$*
4. *pt is a final object,*
5. *for $X \in Ob(CC)$ such that $l(X) > 0$ and $f : Y \to ft(X)$ one has $ft(f^*X) = Y$ and the square*

$$\begin{array}{ccc} f^*X & \xrightarrow{q(f,X)} & X \\ {\scriptstyle p_{f^*X}}\downarrow & & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow{f} & ft(X) \end{array} \qquad (1)$$

commutes,

6. *for* $X \in Ob(CC)$ *such that* $l(X) > 0$ *one has* $id^*_{ft(X)}(X) = X$ *and* $q(id_{ft(X)}, X) = id_X$*,*

7. *for* $X \in Ob(CC)$ *such that* $l(X) > 0$*,* $g : Z \to Y$ *and* $f : Y \to ft(X)$ *one has* $(gf)^*(X) = g^*(f^*(X))$ *and* $q(gf, X) = q(g, f^*X)q(f, X)$*.*

For $f : X \to Y$ in CC we let $ft(f) : X \to ft(Y)$ denote the composition $f \circ p_Y$.

Definition 2.2 *A C-system is a C0-system together with an operation* $f \mapsto s_f$ *defined for all* $f : X \to Y$ *such that* $l(Y) > 0$ *and such that*

1. $s_f : X \to (ft(f))^*(Y)$*,*

2. $s_f \circ p_{(ft(f))^*(Y)} = Id_X$*,*

3. $f = s_f \circ q(ft(f), Y)$*,*

4. *if* $Y = g^*(Z)$ *where* $g : ft(Y) \to ft(Z)$ *then* $s_f = s_{f \circ q(g,Z)}$*.*

Proposition 2.3 *Let* CC *be a C0-system. Then the following are equivalent:*

1. *the canonical squares (1) of* CC *are pull-back squares,*

2. *there is given a structure of a C-system on* CC*.*

Proof: Let us show first that if we are given an operation $f \mapsto s_f$ satisfying the conditions of Definition 2.2 then the canonical squares of CC are pull-back squares.

Let $l(X) > 0$ and $f : Y \to ft(X)$. We want to show that for any Z the map

$$(g : Z \to f^*(X)) \mapsto (ft(g), g \circ q(f, X))$$

is injective and that for any $g_1 : Z \to Y$, $g_2 : Z \to X$ such that $g_1 \circ f = ft(g_2)$ there exists a unique $g : Z \to Y$ such that $ft(g) = g_1$ and $g \circ q(f, X) = g_2$.

Let $g, g' : Z \to f^*(X)$ be such that $ft(g) = ft(g')$ and $g \circ q(f, X) = g' \circ q(f, X)$. Then

$$g = s_g \circ q(ft(g), f^*(X)) = s_{g \circ q(f,X)} \circ q(ft(g), f^*(X)) =$$

$$s_{g' \circ q(f,X)} \circ q(ft(g'), f^*(X)) = s_{g'} \circ q(ft(g'), f^*(X)) = g'.$$

If we are given g_1, g_2 as above let $g = s_{g_2} \circ q(g_1, f^*(X))$. Then:

$$ft(g) = s_{g_2} \circ ft(q(g_1, f^*(X))) = s_{g_2} \circ p_{g_1^*(f_*(X))} \circ g_1 = g_1$$

$$g \circ q(f, X) = s_{g_2} \circ q(g_1, f^*(X)) \circ q(f, X) = s_{g_2} \circ q(g_1 \circ f, X) = s_{g_2} \circ q(ft(g_2), X) = g_2.$$

If on the other hand the canonical squares of CC are pull-back then we can define the operation s_f in the obvious way and moreover such an operation is unique because of the uniqueness part of the definition of pull-back. This implies the assertion of the proposition.

Remark 2.4 Note that the additional structure on a pre-category which defines a C0-system is not an additional essentially algebraic structure. Indeed, the pre-category underlying the product of two C0-systems (defined as the categorical product in the category of C0-systems and their "homomorphisms") is not the product of the underlying pre-categories but a sub-pre-category in this product which consists of pairs of objects (X, Y) such that $l(X) = l(Y)$. This gives another reason for our suggestion to use the name C0-systems ands C-systems instead of the name "contextual categories".

Remark 2.5 Let

$$Ob_n(CC) = \{X \in Ob(CC) \,|\, l(X) = n\}$$

$$Mor_{n,m}(CC) = \{f : Mor(CC) | \partial_0(f) \in Ob_n \, and \, \partial_1(f) \in Ob_m\}.$$

One can reformulate the definitions of C0-systems and C-systems using $Ob_n(CC)$ and $Mor_{n,m}(CC)$ as the underlying sets together with the obvious analogs of maps and conditions of the definition given above. In this reformulation there will be no use of the function l and of the condition $l(X) > 0$.

This shows that C0-systems and C-systems can be considered as models of algebraic theories with sorts Ob_n, and $Mor_{n,m}$ and in particular all the results of [3] are applicable to C-systems.

Remark 2.6 Note also that as defined C0-systems and C-systems can not be described, in general, by generators and relations. For example, for is a C0-system generated by $X \in Ob$? There is no such universal object because we do not know what is $l(X)$.

This problem is, of course, eliminated by using the definition with two infinite families of sorts Ob_n and $Mor_{n,m}$.

3 The set $\widetilde{Ob}$ of a C-system.

For a C-system CC denote by $\widetilde{Ob}(CC)$ the subset of $Mor(CC)$ which consists of elements s of the form $s : ft(X) \to X$ where $l(X) > 0$ and such that $s \circ p_X = Id_{ft(X)}$. In other words, $\widetilde{Ob}$ is the set of sections of the canonical projections p_X for X such that $l(X) > 0$.

For $X \in Ob(CC)$ and $i \geq 0$ such that $l(X) \geq i$ denote by $p_{X,i}$ the composition of the canonical projections $X \to ft(X) \to \ldots \to ft^i(X)$ such that $p_{X,0} = Id_X$ and for $l(X) > 0$, $p_{X,1} = p_X$. If $l(X) < i$ we will consider $p_{X,i}$ to be undefined. *All of the considerations involving $p_{X,i}$'s below are modulo the qualification that $p_{X,i}$ is defined, i.e., that $l(X) \geq i$.*

For X such that $l(X) \geq i$ and $f : Y \to ft^i(X)$ denote by $f^*(X, i)$ the objects and by $q(f, X, i) : f^*(X, i) \to X$ the morphisms defined inductively by the rule

$$f^*(X, 0) = Y \qquad q(f, X, 0) = f,$$

$$f^*(X, i+1) = q(f, ft(X), i)^*(X) \qquad q(f, X, i+1) = q(q(f, ft(X), i), X).$$

If $l(X) < i$, then $q(f, X, i)$ is undefined since $q(-, X)$ is undefined for $X = pt$ and again, as in the case of $p_{X,i}$, *all of the considerations involving $q(f, X, i)$ are modulo the qualification that $l(X) \geq i$.*

For $i \geq 1$, $(s : ft(X) \to X) \in \widetilde{Ob}$ such that $l(X) \geq i$, and $f : Y \to ft^i(X)$ let

$$f^*(s, i) : f^*(ft(X), i-1) \to f^{(}ft(X), i)$$

be the pull-back of the section $ft(X) \to X$ along the morphism $q(f, ft(X), i-1)$. We again use the agreement that always when $f^*(s,i)$ is used the condition $l(X) \geq i$ is part of the assumptions.

Consider the following operations on the pair of sets $Ob = Ob(CC)$ and $\widetilde{Ob} = \widetilde{Ob}(CC)$:

1. $pt \in Ob$,
2. $ft : Ob \to Ob$,
3. $\partial : \widetilde{Ob} \to Ob$ of the form $(s : ft(X) \to X) \mapsto X$,
4. T which is defined on pairs $(Y, X) \in Ob \times Ob$ such that $l(Y) > 0$ and there exists (a necessarily unique) $i \geq 1$ with $ft(Y) = ft^i(X)$ and $T(Y, X) = p_Y^*(X, i)$,
5. $\widetilde{T}$ which is defined on pairs $(Y, (r : ft(X) \to X)) \in Ob \times \widetilde{Ob}$ such that $l(Y) > 0$ and there exists (a necessarily unique) $i \geq 1$ such that $ft(Y) = ft^i(X)$ and $\widetilde{T}(Y, r) = p_Y^*(r, i)$,
6. S which is defined on pairs $((s : ft(Y) \to Y), X) \in \widetilde{Ob} \times Ob$ such that there exists (a necessarily unique) $i \geq 1$ such that $Y = ft^i(X)$ and $S(s, X) = s^*(X, i)$,
7. $\widetilde{S}$ which is defined on pairs $((s : ft(Y) \to Y), (r : ft(X) \to X)) \in \widetilde{Ob} \times \widetilde{Ob}$ such that there exists (a necessarily unique) $i \geq 1$ such that $Y = ft^i(X)$ and $\widetilde{S}(s, r) = s^*(r, i)$,
8. δ which is defined on elements $X \in Ob$ such that $l(X) > 0$ and $\delta(X) \in \widetilde{Ob}$ is $s_{p_X} : X \to p_X^*(X)$.

4 C-subsystems.

A C-subsystem CC' of a C-system CC is a sub-pre-category of the underlying pre-category which is closed, in the obvious sense under the operations which define the C-system on CC.

A C-subsystem is itself a C-system with respect to the induced structure.

Lemma 4.1 *Let CC be a C-system and CC', CC'' be two C-subsystems such that $Ob(CC') = Ob(CC'')$ (as subsets of $Ob(CC)$) and $\widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$ (as subsets of $\widetilde{Ob}(CC)$). Then $CC' = CC''$.*

Proof: Let $f : Y \to X$ be a morphism in CC'. We want to show that it belongs to CC''. Proceed by induction on $m = l(X)$. For $m = 0$ the assertion is obvious. Suppose that $m > 0$. Since CC' is a C-subsystem we have a commutative diagram

$$\begin{array}{ccc} Y & & \\ {\scriptstyle s_f}\downarrow & & \\ (f \circ p_X)^* X & \xrightarrow{q(f \circ p_X, X)} & X \\ \downarrow & & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow{f \circ p_X} & ft(X) \end{array} \tag{2}$$

in CC' such that $f = s_f\, q(p_X f, X)$. By the inductive assumption the square is a canonical pull-back square in CC'' as well. Since $\widetilde{ob}(CC') = \widetilde{ob}(CC'')$ we have $s_f \in CC''$ and therefore $f \in CC''$.

Remark 4.2 In Lemma 4.1, it is sufficient to assume that $\widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$. The condition $Ob(CC') = Ob(CC'')$ is then also satisfied. Indeed, let $X \in Ob(CC')$ and $l(X) > 0$. Then $p_X^* X$ is the product $X \times X$ in CC. Consider the diagonal section $\delta_X : X \to p_X^* X$ of $p_{p_X^*(X)}$. Since CC' is assumed to be a C-subsystem we conclude that $\delta_X \in \widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$ and therefore $X \in Ob(CC'')$. It is however more convenient to think of C-subsystems in terms of subsets of both Ob and $\widetilde{Ob}$.

Proposition 4.3 *A pair* $(C, \widetilde{C})$ *where* $C \subset ob(CC)$ *and* $\widetilde{C} \subset \widetilde{Ob}(CC)$ *corresponds to a C-subsystem of* CC *if and only if the following conditions hold:*

1. $pt \in C$,
2. *if* $X \in C$ *then* $ft(X) \in C$,
3. *if* $s \in \widetilde{C}$ *then* $\partial(s) \in C$,
4. *if* $Y \in C$ *and* $r \in \widetilde{C}$ *then* $\widetilde{T}(Y, r) \in \widetilde{C}$,
5. *if* $s \in \widetilde{C}$ *and* $r \in \widetilde{C}$ *then* $\widetilde{S}(s, r) \in \widetilde{C}$,
6. *if* $X \in C$ *then* $\delta(X) \in \widetilde{C}$.

Conditions (4) and (5) are illustrated by the following diagrams:

$$
\begin{array}{ccc}
p_Y^*(ft(X), i-1) & \xrightarrow{q(p_Y, ft(X), i-1)} & ft(X) \\
\downarrow{\scriptstyle q(p_Y, ft(X), i-1)^*(r)} & & \downarrow{\scriptstyle r} \\
p_Y^*(X, i) & \xrightarrow{q(p_Y, X, i)} & X \\
\downarrow & & \downarrow{\scriptstyle p_X} \\
p_Y^*(ft(X), i-1) & \xrightarrow{q(p_Y, ft(X), i-1)} & ft(X) \\
\downarrow & & \downarrow \\
\dots & & \dots \\
\downarrow & & \downarrow \\
Y & \xrightarrow{p_Y} & ft^i(X)
\end{array}
\qquad
\begin{array}{ccc}
s^*(ft(X), i-1) & \xrightarrow{q(s, ft(X), i-1)} & ft(X) \\
\downarrow{\scriptstyle q(s, ft(X), i-1)^*(r)} & & \downarrow{\scriptstyle r} \\
s^*(X, i) & \xrightarrow{q(s, X, i)} & X \\
\downarrow & & \downarrow{\scriptstyle p_X} \\
s^*(ft(X), i-1) & \xrightarrow{q(s, ft(X), i-1)} & ft(X) \\
\downarrow & & \downarrow \\
\dots & & \dots \\
\downarrow & & \downarrow \\
ft^{i+1}(X) & \xrightarrow{s} & ft^i(X)
\end{array}
$$

Proof: The "only if" part of the proposition is straightforward. Let us prove that for any $(C, \widetilde{C})$ satisfying the conditions of the proposition there exists a C-subsystem CC' of CC such that $C = Ob(CC')$ and $\widetilde{C} = \widetilde{Ob}(CC')$.

Define a candidate subcategory CC' setting $Ob(CC') = C$ and defining the set $Mor(CC')$ of morphisms of CC' inductively by the conditions:

1. $Y \to pt$ is in $Mor(CC')$ if and only if $Y \in C$,
2. $f : Y \to X$ is in $Mor(CC')$ if and only if $X \in C$, $ft(f) \in Mor(CC')$ and $s_f \in \widetilde{C}$.

(note that the for $(f : Y \to X) \in Mor(CC')$ one has $Y \in C$ since $s_f : Y \to X_f$).

Let us show that if the conditions of the proposition are satisfied then $(Ob(CC'), Mor(CC'))$ form a C-subsystem of CC.

The subset $Ob(CC')$ contains pt and is closed under ft map by the first two conditions. The following lemma shows that $Mor(CC')$ contains identities and the compositions of the canonical projections.

Lemma 4.4 *Under the assumptions of the proposition, if $X \in C$ and $i \geq 0$ then $p_{X,i} : X \to ft^i(X)$ is in $Mor(CC')$.*

Proof: Let $l(X) = n$. Then $p_{X,n} \in Mor(CC')$ by the first constructor of $Mor(CC')$. By induction it remains to show that if $X \in C$ and $p_{X,i} \in Mor(CC')$ then $p_{X,i-1} \in Mor(CC')$. We have $ft(p_{X,i-1}) = p_{X,i}$ and $s_{p_{X,i-1}}$ is the pull-back of the diagonal $ft^{i-1}(X) \to (p_{ft^{i-1}(X)})^*(ft^{i-1}(X))$ with respect to $p_{X,i-1} : X \to ft^{i-1}(X)$. The diagonal is in $\widetilde{C}$ by condition (6) and therefore $s_{p_{X,i-1}}$ is in $\widetilde{C}$ by repeated application of condition (4).

Lemma 4.5 *Under the assumptions of the proposition, let $(r : ft(X) \to X) \in \widetilde{C}$, $i \geq 1$, and $(f : Y \to ft^i(X)) \in Mor(CC')$. Then $f^*(s, i) : ft(f^*(X, i)) \to f^*(X, i)$ is in $Mor(CC')$.*

Proof: Suppose first that $ft^i(X) = pt$. Then $f = p_{Y,n}$ for some n and the statement of the lemma follows from repeated application of condition (4). Suppose that the lemma is proved for all morphisms to objects of length $j - 1$ and let the length of $ft^i(X)$ be j. Consider the canonical decomposition $f = s_f q_f$. The morphism q_f is the canonical pull-back of $ft(f)$ and therefore the pull-back of s relative to q_f coincides with its pull-back relative to $ft(f)$ which is $\widetilde{C}$ by the inductive assumption. The pull-back of an element of $\widetilde{C}$ with respect to s_f is in $\widetilde{C}$ by condition (5).

Lemma 4.6 *Under the assumptions of the proposition, let $g : Z \to Y$ and $f : Y \to X$ be in $Mor(CC')$. Then $gf \in Mor(CC')$.*

Proof: If $X = pt$ the the statement is obvious. Assume that it is proved for all f whose codomain is of length $< j$ and let X be of length j. We have $ft(gf) = g\,ft(f)$ and therefore $ft(gf) \in Mor(CC')$ by the inductive assumption. It remains to show that $s_{gf} \in \widetilde{C}$. We have the following diagram whose squares are canonical pull-back squares

$$\begin{array}{ccccc} X_{gf} & \longrightarrow & X_f & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow p_X \\ Z & \xrightarrow{g} & Y & \xrightarrow{ft(f)} & ft(X) \end{array}$$

which shows that $s_{gf} = g^*(s_f)$. Therefore, $s_{gf} \in Mor(CC')$ by Lemma 4.5.

Lemma 4.7 *Under the assumptions of the proposition, let $X \in C$ and let $f : Y \to ft(X)$ be in $Mor(CC')$, then $f^*(X) \in C$ and $q(f, X) \in Mor(CC')$.*

Proof: Consider the diagram

$$\begin{array}{ccccc}
f^*(X) & \xrightarrow{q(f,X)} & X & & \\
\downarrow{\scriptstyle s_{q(f,X)}} & & \downarrow{\scriptstyle s_{Id_X}} & & \\
q(f,X)^*(X) & \longrightarrow & p_X^*(X) & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
f^*(X) & \xrightarrow{q(f,X)} & X & \longrightarrow & ft(X) \\
\downarrow{\scriptstyle p_{f^*(X)}} & & \downarrow{\scriptstyle p_X} & & \\
Y & \xrightarrow{f} & ft(X) & &
\end{array}$$

where the squares are canonical. By condition (6) we have $s_{Id} \in \widetilde{C}$. Therefore, by Lemma 4.5, we have $s_{q(f,X)} \in \widetilde{C}$. In particular, $q(f,X)^*(X) \in C$ and therefore $f^*(X) = ft(q(f,X)^*(X)) \in C$. The fact that $q(f,X) \in Mor(CC')$ follows from the fact that $s_{q(f,X)} \in \widetilde{C}$ and $ft(q(f,X)) = p_{f^*(X)}f$ is in $Mor(CC')$ by previous lemmas.

Lemma 4.8 *Under the assumptions of Lemma 4.7, the square*

$$\begin{array}{ccc}
f^*(X) & \xrightarrow{q(f,X)} & X \\
\downarrow{\scriptstyle p_{f^*(X)}} & & \downarrow{\scriptstyle p_X} \\
Y & \xrightarrow{f} & ft(X)
\end{array}$$

is a pull-back square in CC'.

Proof: We need to show that for a morphism $g : Z \to f^*(X)$ such that $gp_{f^*(X)}$ and $gq(f,X)$ are in $Mor(CC')$ one has $g \in Mor(CC')$. We have $ft(g) = gp_{f^*(X)}$, therefore by definition of $Mor(CC')$ it remains to check that $s_g \in \widetilde{C}$. The diagram of canonical pull-back squares

$$\begin{array}{ccccc}
(f^*Y)_g & \longrightarrow & f^*Y & \xrightarrow{q(f,X)} & X \\
\downarrow & & \downarrow & & \downarrow \\
Z & \xrightarrow{ft(g)} & Y & \xrightarrow{f} & ft(X)
\end{array}$$

shows that $s_g = s_{gq(f,X)}$ and therefore $s_g \in Mor(CC')$.

To finish the proof of the proposition it remains to show that $Ob(CC') = C$ and $\widetilde{Ob}(CC') = \widetilde{C}$. The first assertion is tautological. The second one follows immediately from the fact that for $(s : ft(X) \to X) \in \widetilde{Ob}(CC)$ one has $ft(s) = Id_{ft(X)}$ and $s_s = s$.

5 Regular congruence relations on C-systems

Definition 5.1 *Let CC be a C-system. A regular congruence relation on CC is a pair of equivalence relations $\sim_{Ob}, \sim_{Mor}$ on $Ob(CC)$ and $Mor(CC)$ respectively such that:*

[4] Vladimir Voevodsky. Notes on type systems. *https://github.com/vladimirias/old_notes_on_type_systems*, 2009-2012.

[5] Vladimir Voevodsky. B-systems. *In preparation.*, 2014.

Notes

1-2 Wrong reference. [3] is the Palmgren-Vickers paper referred to in Remark 2.5. This one should be "Semantics of type theory".

1-3 Not completely new. s_f is introduced in Cartmell's thesis (p. 2.19) as 'f'.

2-1 What do pt and ft stand for?

3-1 Beware! ft is not a functor.

3-2 X and Y in diagram (1) have now switched their roles, slightly confusing.

4-1 is there

4-2 f*(X,i)

5-1 It seems to me that s_{p_X} should be s_{Id_X}.

Remark 42 refers to δ_X (not δ(X)) explicitly as the second of these.

In 4.7 δ_X is implicitly referred to as s_Id.

If is_{p_X} is right, then the codomain is p_{X,2}*(ft(X)).

5-4 Ob?

6-2 The notation is not systematic. C is got from CC by removing a C, CC' is got by adding a prime.

7-2 This notation seems to be new, X_f = ft(f)*(X).

7-3 Are these meant to be the same?

7-4 in

8-1 p*_X(X)

8-5 X

11-1 Streicher "Semantics of type theory"