The homotopy properties of the twisted product functors

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Contents

1 Introduction

2 The standard cd-structures on the category of G-schemes

Let S be a Noetherain scheme and G be a group scheme over S. We define the standard cd-structures on the category of G-schemes through the forgetful functor to the category of schemes. That is, a square of G-schemes is called upper distinguished, lower distinguished, plain upper distinguished or plain lower distinguished if it is of the corresponding type when considered as a square of schemes without a G-action. Since the forgetful functor commutes with the fiber products Lemma **??** implies that the standard cd-structures are complete and Lemma **??** that they are regular. Define the standard density structure on G-schemes through the forgetful functor as well. The equivariant dimension of a G-scheme is always less or equal to the non equivariant one. If Z is an invariant subset of a G-scheme X and $p : X \to Y$ is a G-morphism then the image of A is invariant. If Z is an invariant subset of a G-scheme X then the closure $cl(Z)$ of Z is invariant. Finally if Z is an invariant closed subset of a G-scheme X then there exists a unique G-action on the complement $X-Z$ such that $X-Z \to X$ is a morphism of G-schemes. These statements imply that the proof of Proposition **??** can be transfered without

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change to the equivariant case. To transfer the proof of Proposition **??** we need to know that for an equivariant morphism $f : Y \to X$ of finite type the scheme-theoretic image $Spec(ker(\mathcal{O}_X \to f_*\mathcal{O}_Y))$ of f has a canonical structure of a G-scheme. This follows immediately from the definition if G is flat over S. We proved the following result.

Proposition 2.1 *[**eqcase1**] The standard cd-structures on the category of G-schemes over S are complete and regular. The upper and plain upper cd-structures are bounded by the standard density structure for any G. The lower and plain lower cd-structures are bounded by the standard density structure if G is flat over S.*

Remark 2.2 We do not know of any example of a groups scheme G such that the lower cd-structure on G-schemes is not bounded by the standard density structure.

We will need a criterion for an etale G-morphism to be a covering in the upper cd-topology and for a proper morphism to be a covering in the lower cd-topology.

Definition 2.3 *[**uppersplit**] Let $f : \tilde{X} \to X$ be a morphism of G-schemes. A splitting sequence for f is a sequence of closed embeddings of G-schemes*

$$\emptyset = Z_{n+1} \to Z_n \to \ldots \to Z_1 \to Z_0 = X$$

such that for any $i = 0, \ldots, n$ the projection

$$(Z_i - Z_{i+1}) \times_X \tilde{X} \to (Z_i - Z_{i+1})$$

has a section (in the category of G-schemes).

Proposition 2.4 *[**upperchar**] Let G be a flat group scheme. Then an etale G-morphism $f : \tilde{X} \to X$ is a covering in the upper cd-topology if and only if it has a splitting sequence.*

Proof: We start with the following lemma. Its proof was suggested by P. Deligne.

Lemma 2.5 *[**complement**] Let G be a flat group scheme and Z be an invariant closed subset of a G-scheme X. Then there exists a G-scheme Z' and a closed embedding $i : Z' \to X$ such that $Im(i) = Z$.*

Proof: Consider Z as a scheme with the reduced structure. The morphism

$$f : G \times_S Z \to G \times_S X \to X$$

where the last arrow is the action is a G-equivariant morphism with respect to the action of G on $G \times Z$ through the first factor. We can now take Z' to be the scheme-theoretic image of f i.e. $Z' = Spec(ker(\mathcal{O}_X \to f_* \mathcal{O}_{G\times_S Z}))$. Since G is flat Z' has a canonical action of G.

The proof of the "only if" part is parallel, modulo Lemma 2.5, to the proof of the similar result in the non equivariant case given in [1, Lemma 3.1.5].

Lemma 2.6 [**hasdense**] *If $f : \tilde{X} \to X$ is an upper cd-covering then there exists an open embedding $j : U \to X$ with dense image and a section of f over U.*

Proof: We work in the category of G-schemes. Using the fact that the upper cd-structure is complete we may assume that $f = \coprod f_i$ where $\{\tilde{X}_i \to X\}$ is a simple covering. By induction we may assume further that our covering is of the form $\{\tilde{Y}_i \stackrel{p_i}{\to} Y \to X, \tilde{A}_j \stackrel{q_j}{\to} A \to X\}$ where $\coprod p_i$ and $\coprod q_j$ have sections over dense open subschemes Y_0, A_0 of Y and A respectively and $Y \to X$ and $A \to X$ are two sides of an upper distinguished square of the form (**??**). An open subset of X is dense if it belongs to $D_1(X)$ defined by the standard density structure. Using the fact that any upper square is reducing and applying the definition of a reducing square for Y_0, A_0 and $B_0 = \emptyset$ we conclude that there is a dense open subset U of X such that f has a section over U.

Let $f : \tilde{X} \to X$ be an upper cd-covering. To find a splitting sequence for f take a dense open subset U of X such that f has a section over U. Let $Z_1 \to Z_0 = X$ be a closed embedding of G-schemes whose image is $X - U$. Consider the pull-back of f to Z_1 and apply again Lemma 2.6. One gets a sequence of closed embeddings $Z_{i+1} \to Z_i \to \ldots \to X$ such that $Z_{i+1} - Z_i$ is dense in Z_{i+1} and in particular non empty. Since X is Noetherain this sequence must stabilize giving a finite splitting sequence for f. This proofs the "only if" part.

To prove the "if" part consider an etale morphism $f : \tilde{X} \to X$ with a splitting sequence $Z_n \to \ldots \to Z_0 = X$. We will construct an upper distinguished square of the form (**??**) based on X such that the pull-back of

f to Y has a section and the pull-back of f to A has a splitting sequence of length less than n. The result then follows by induction on n. We take $A = X - Z_n$. To define Y consider the section s of $f_n : \tilde{X} \times_X Z_n \to Z_n$ which exists by definition of a splitting sequence. Since f is etale and in particular unramified the image of s is an open subscheme. Let W be its complement. The morphism $\tilde{X} \times_X Z_n \to \tilde{X}$ is a closed embedding thus the image of W is closed in $\tilde{X}$. We take $Y = \tilde{X} - W$. One verifies immediately that the pull-back square defined by $A \to X$ and $Y \to X$ is upper distinguished. The pull-back of f to Y has a section and the pull-back of f to A has a splitting sequence of length $n-1$. This finishes the proof of the proposition.

Proposition 2.7 *[***lowerchar***] Let G be a flat group scheme. A proper morphism of G-schemes $f : \tilde{X} \to X$ is a lower cd-covering if and only if it has a splitting sequence.*

Proof: The proof of the "only if" part is parallel to the proof given for the upper case in Proposition 2.4 with the following lemma replacing Lemma 2.6.

Lemma 2.8 *[***hasdenselow***] Let $f : \tilde{X} \to X$ be a lower cd-covering. Then there exists an open embedding $U \to X$ with a dense image and a section of f over U.*

Proof: Same argument as in the proof of Lemma 2.6.

To prove the "if" part consider a proper morphism $f : \tilde{X} \to X$ with a splitting sequence $Z_n \to \ldots \to Z_0 = X$. We will construct a lower distinguished square of the form (**??**) based on X such that the pull-back of f to Y has a section and the pull-back of f to A has a splitting sequence of length less than n. The result then follows by induction on n. We take $A = Z_1$. To define Y consider the section s of $f_n : \tilde{X} \times_X (X - Z_1) \to (X - Z_1)$ which exists by definition of a splitting sequence. Since f is proper and in particular separated, the image of s is a closed subscheme. Let W be its complement. The morphism $\tilde{X} \times_X (X - Z_1) \to \tilde{X}$ is an open embedding thus the image of W is open in $\tilde{X}$. We take $Y = \tilde{X} - W$. One verifies immediately that the pull-back square defined by $A \to X$ and $Y \to X$ is lower distinguished. The pull-back of f to Y has a section and the pull-back of f to A has a splitting sequence of length $n-1$. This finishes the proof of the proposition.

Let G be a finite flat group scheme over S and W be a finite flat scheme over S with G-action. If X is a quasi-projective scheme over S the functor

$S' \mapsto Hom(S' \times_S W, X)$ on the category of schemes over S is represented by a quasi-projective scheme which we denote by X^W. The evaluation morphism

$$X^W \times W \to X$$

corresponding to the identity morphism of X^W with respect to the identification

$$Hom_S(X^W, X^W) = Hom_S(X^W \times W, X)$$

composed with the action $G \times W \to W$ of G on W gives a morphism $X^W \times G \times W \to X$ corresponding to a morphism

$$[\mathbf{act}] X^W \times G \to X^W \tag{1}$$

One verifies easily that (1) is an action of G on X^W. Thus $X \mapsto X^W$ is a functor from the category of (quasi-projective) schemes over S to the category of quasi-projective G-schemes over S.

Proposition 2.9 *[**takestocov**] Let $f : \tilde{X} \to X$ be an upper cd-covering of quasi-projective schemes. Then $f^W : \tilde{X}^W \to X^W$ is an upper cd-covering.*

Proof: Since the upper cd-structure is complete f has a simple refiniment. Thus we may assume that f is an etale morphism.

Lemma 2.10 *[**ettoet**] Let $f : \tilde{X} \to X$ be an etale morphism of quasi-projective schemes. Then $f^W : \tilde{X}^W \to X^W$ is an etale morphism.*

Proof: It follows immediately from the fact that a morphism $g : X_1 \to X_2$ of finite type is etale if and only if for any morphism $Z \to X_2$ and any closed embedding $Z_0 \to Z$ defined by a nilpotent ideal the map

$$Hom_{X_2}(Z, X_1) \to Hom_{X_2}(Z_0, X_1)$$

is bijective ([,]).

By Lemma 2.4 we may assume that f has a splitting sequence and need to verify that f^W has a splitting sequence. By a simple inductive argument we can reduce the problem to the case when f has a splitting sequence of length one i.e. when there exists an open embedding $j : U \to X$ and a closed embedding $i : Z \to U$ such that $X = j(U) \cap i(Z)$ and f has a section over $U \coprod Z$.

Consider the evaluation morphism $ev : X^W \times W \to X$, let $U' = ev^{-1}(U)$ and let V_i be the set of points v in X^W such that

$$dim_{k_v}(\mathcal{O}(U' \times_{X^W} Spec(k_v))) \geq i$$

where k_v is the residue field of v and $Speck_v \to X^W$ is the canonical morphism. Then $V_0 = X^W$ and $V_i = \emptyset$ for $i > deg(W/S)$. Let

$$[\mathbf{splseq}]\emptyset = Z_0 \to Z_1 \to \ldots \to Z_d \to X^W \tag{2}$$

be a sequence of closed embeddings of G-schemes such that $X^W - Z_i = Z_i$ which can be constructed inductively starting with Z_d by Lemma 2.5. We claim that any such sequence is a splitting sequence for f^W (note that we have the numbering of closed subschemes reversed compared to Definition 2.3). The proof is based on the following lemma.

Lemma 2.11 *[**finmor**] Let $p : T \to S$ be a finite flat morphism of schemes and U an open subset in T. Define V_i as the set of points v of S such that*

$$dim_{k_v}(\mathcal{O}(U \times_S Spec(k_v))) \geq i$$

Then one has:

1. *V_i are open*
2. *the subscheme $U \times_S (V_i - V_{i+1})$ is a connected component of $T \times_S (V_i - V_{i+1})$.*

Proof: The first statement follows from [,]. To prove the second one it is sufficient to show that $U \times_S (V_i - V_{i+1})$ is closed in $T \times_S (V_i - V_{i+1})$. Since it is constructible it is sufficient to show that for any henselian local scheme S' and a morphism $S' \to S$ which lands in $V_i - V_{i+1}$ the subscheme $U' = U \times_S S'$ is closed in $T' = T \times_S S'$. Let g be the generic point of S' and c the closed point. The condition that the image of S' is contained in $V_i - V_{i+1}$ implies that

$$dim_{k_g}(\mathcal{O}(U' \times_{S'} Speck_g)) = dim_{k_c}(\mathcal{O}(U' \times_{S'} Speck_c)) = i$$

The scheme T' is finite over S' and therefore it is a disjoint union of connected components which are henselian local schemes. Let T_0 be the union of components which are contained in U and T_1 the union of the rest of the components. Since T' and therefore all its components are flat over S' we have

$$dim_{k_g}(\mathcal{O}(T_0 \times_{S'} Speck_g)) = dim_{k_c}(\mathcal{O}(T_0 \times_{S'} Speck_c))$$

and since $U = T_0 \coprod (U \times_S T_1)$ we have

$$dim_{k_g}(\mathcal{O}(U \times_{S'} T_1 \times_{S'} Speck_g)) = dim_{k_c}(\mathcal{O}(U \times_{S'} T_1 \times_{S'} Speck_c))$$

By construction the closed points of T_1 do not lie in U which implies that the right hand side is zero. Thus the left hand side is zero i.e. $U \times_{S'} T_1 \times_{S'} Speck_g = \emptyset$ and since $T_1 \times_{S'} Speck_g$ is dense in T_1 we conclude that $U \cap T_1 = \emptyset$ i.e. $U = T_0$.

To show that (2) is a splitting sequence we need to construct G-equivariant sections of f^W over $Z_{i+1} - Z_i$. By our assumption f has a section over $U \coprod Z$. Thus f^W has an equivariant section over $(U \coprod Z)^W$. Consider the morphism

$$(Z_{i+1} - Z_i) \times W \to X^W \times W \stackrel{ev}{\to} X.$$

By Lemma 2.11 the pull-back of U to $(Z_{i+1} - Z_i) \times W$ is a closed embedding. Therefore the pull-back of $U \coprod Z \to X$ is a closed subcheme A which is given by a nilpotent sheaf of ideals. Since f is etale its section over A extends uniquely to a section over $(Z_{i+1} - Z_i) \times W$ which is also equivariant. By adjunction we get an equivariant section of f^W over $Z_{i+1} - Z_i$.

Example 2.12 The analog of Proposition 2.9 for the plain upper topology is false. Let $W = S \coprod S$ and $G = \mathbf{Z}/2$. Then $X^W = X^2$ with the permutation action of $\mathbf{Z}/2$. Let $X = U_1 \cup U_2$ be a covering of X by two open subsets such that $f : U_1 \coprod U_2 \to X$ is a plain upper covering. Consider $f^2 : (U_1 \coprod U_2)^2 \to X^2$. For f^2 to be an upper covering in the category of schemes with $\mathbf{Z}/2$-action there should exists a collection of invariant open subsets V_i of X^2 such that $X^2 = \cup V_i$ and f^2 has equivariant sections over each V_i. Let z_1 be a point of $X - U_1$ and z_2 a point of $X - U_2$ and V an invariant open neighborhood of (z_1, z_2). Assume that X^2 is irreducible. Then V is connected and the section of f^2 over V must land in one of the connected components of $(U_1 \coprod U_2)^2$. But one verifies easily that neither one of the components maps surjectively to V which implies that no such section exists.

In the rest of this section we analize the "exactness" properties of the functor $X \mapsto X/G$ with respect to the standard cd-structures in the case of a finite flat group scheme G.

We consider the upper cd-structure first. Let C be a subcategory of the category of G-schemes which satisfies the following conditions

1. for any X in C and any etale morphism $U \to X$ one has $U \in C$

2. for any X in C the categorical quotient X/G exists in Sch/S, the morphism $p : X \to X/G$ is finite and surjective, and for any etale morphism $V \to X/G$ the morphism
$$(V \times_{X/G} X)/G \to V$$
is an isomorphism.

Lemma 2.13 [**isopenem**] *Let C be as above, X be an object of C and $A \to X$ be an equivariant open embedding and $Y \to X$ an equivariant etale morphism. Then the morphism $A/G \to X/G$ is an open embedding and the square*

$$[\mathbf{qg0}] \quad \begin{array}{ccc} (A \times_X Y)/G & \to & Y/G \\ \downarrow & & \downarrow \\ A/G & \to & X/G \end{array} \qquad (3)$$

is a pull-back square.

Proof: Let us show first that our conditions on C imply that the square

$$[\mathbf{qg1}] \quad \begin{array}{ccc} A & \to & X \\ \downarrow & & \downarrow \\ A/G & \to & X/G \end{array} \qquad (4)$$

is a pull-back square. Let $A' = (A/G) \times_{X/G} X$. We have an open embedding $A \to A'$ and $A/G = A'/G$. Since the morphisms $A \to A/G$ and $A' \to A'/G$ are finite we conclude that $A \to A'$ is a closed embedding. Then $A' = A \coprod A''$ and there exists a function f on A' which is 0 on A and 1 on A''. Since A is invariant in A' this function is invariant and thus factors through $A' \to A'/G$. Since $A \to A'/G$ is surjective it implies that $f = 0$ i.e. $A = A'$.

Since the morphism $X \to X/G$ is finite and surjective it is in particular universally closed which implies that $A/G \to X/G$ is also an open embedding. The same reasoning implies that $(A \times_X Y)/G \to Y/G$ is an open embedding and therefore $(A/G) \times_{X/G} (Y/G)$ and $(A \times_X Y)/G$ are two open subsets of Y/G and $(A \times_X Y)/G \to (A/G) \times_{X/G} (Y/G)$. To check the opposite inclusion take a geometric point $\tilde{y}$ of Y/G whose image in X/G lies in A/G. Let y be its lifting to Y which exists since $Y \to Y/G$ is surjective. Then y lies in $A \times_X Y$ since the composite square

$$\begin{array}{ccc} A \times_X Y & \to & Y \\ \downarrow & & \downarrow \\ A & \to & X \\ \downarrow & & \downarrow \\ A/G & \to & X/G \end{array}$$

is a pull-back square.

Let Q be an upper distinguished square of the form (**??**) in C and S be the henselian local scheme of a point x of X/G. The morphism

$$S_Y = S \times_{X/G} (Y/G) \to S$$

is quasi-finite and thus S_Y is a disjoint union of the form $S_Y = S_{Y,fin} \coprod S_{Y,0}$ where $S_{Y,fin}$ is finite over S and the image of $S_{Y,0}$ does not contain the closed point of S.

Lemma 2.14 *[**ll1**] If x does not lie in A/G then the map $S_{Y,fin} \to S$ is an isomorphism.*

Proof: Let

$$[\mathbf{qg2}] \quad \begin{array}{ccc} \tilde{S}_Y & \to & \tilde{S} \\ \downarrow & & \downarrow \\ S_Y & \to & S \end{array} \tag{5}$$

be the pull-back of the square

$$[\mathbf{qg3}] \quad \begin{array}{ccc} Y & \to & X \\ \downarrow & & \downarrow \\ Y/G & \to & X/G \end{array} \tag{6}$$

along the morphism $S \to X$. The right vertical arrow is a finite morphism and therefore $\tilde{S}$ is the disjoint union of a finite number of henselian local schemes. Let $\tilde{s}_{Y,fin}$ be the union of the connected components of $\tilde{S}_Y$ which are finite over $\tilde{S}$. Since the closed point of S lies outside of A/G the closed points of $\tilde{S}$ lie outside of A. Together with the fact that Q is an upper distinguished square this implies that the morphism $\tilde{s}_{Y,fin} \to \tilde{S}$ is an isomorphism. On the other hand the fact that the vertical arrows in (5) are finite implies that the square

$$[\mathbf{qg4}] \quad \begin{array}{ccc} \tilde{S}_{Y,fin} & \to & \tilde{S} \\ \downarrow & & \downarrow \\ S_{Y,fin} & \to & S \end{array} \tag{7}$$

is pull-back. Thus $S_{Y,fin} = \tilde{S}_{Y,fin}/G = \tilde{S}/G = S$.

Proposition 2.15 *[**upqu**] Let C be as above and Q be an upper distinguished square in C of the form (**??**). Consider the square*

$$[\mathbf{qg}]Q/G = \left(\begin{array}{ccc} B/G & \to & Y/G \\ \downarrow & & \downarrow \\ A/G & \to & X/G \end{array} \right) \tag{8}$$

The the corresponding square $\rho(Q/G)$ of the representable sheaves in the upper cd-topology on Sch/S is push-forward.

Proof: By Lemma 2.13 the square Q/G is a pull-back square and the horizontal arrows are open embeddings. By the same argument as in the proof of Lemma **??** it is enough to check that the morphisms

$$[\mathbf{firstm}]A/G \coprod Y/G \to X/G \tag{9}$$

and

$$[\mathbf{secondm}]Y/G \coprod (B/G \times_{A/G} B/G) \to Y/G \times_X /GY/G \tag{10}$$

define epimorphisms of the representable sheaves in the upper cd-topology. Lemma **??** implies that a morphism of finite type $Z \to W$ defines an epimorphism of the representable upper sheaves if and only if for any henselian local scheme S and any morphism $s : S \to W$ there exists a lifting of s to a morphism $S \to Z$. To prove that (9) defines an epimorphism let S_0 be a henselian local scheme and $x : S_0 \to X/G$ be a morphism. If the image of the closed point of S_0 lies in A/G then x lifts to A/G since $A/G \to X/G$ is an open embedding. If the image of the closed point of S_0 lies outside of A/G then x lifts to Y/G by Lemma 2.14.

To prove that (10) defines an epimorphism let S_0 be a henselian local scheme and $(y_1, y_2) : S_0 \to Y/G \times_{X/G} Y/G$ a morphism. Since Q/G is a pull-back square S_0 factors through $(B/G \times_{A/G} B/G)$ if and only if the corresponding morphism $S_0 \to X/G$ factors through A/G. Thus we may assume that the image of the closed point of S_0 lies outside of A/G. Then by Lemma 2.14 we have $y_1 = y_2$ i.e. (y_1, y_2) lifts to the diagonal.

Corollary 2.16 *[**exact**] Let C be as above and $\{p_i : U_i \to X\}$ be an upper cd-covering in C. Then the family of morphisms $\{U_i/G \to X/G\}$ is an upper cd-covering in Sch/S.*

Proof: The upper cd-structure is complete on C by Lemma **??**. Thus $\{p_i\}$ has a simple refiniment. It remains to show that for any simple covering $\{p_i : U_i \to X\}$ the family $U_i/G \to X/G$ is a covering. The class S of simple coverings for which it is true contains isomorphisms. Let us show that it satisfies the second condition of Definition **??**. Let Q be an upper distinguished square in C of the form (**??**) and $\{p_i : Y_i \to Y\}$ and $\{q_j : A_j \to A\}$ be simple coverings in S. The families $\{Y_i/G \to Y/G\}$ and $\{A_j/G \to A/G\}$ are upper cd-coverings by the assumptions. The pair of morphisms $\{A/G \to X/G, Y/G \to X/G\}$ is a covering by Proposition 2.15. Thus $\{A_j/G \to X/G, Y_i/G \to X/G\}$ is a covering.

Example 2.17 [**et**] The analog of Corollary 2.16 is false in the etale topology that is given an etale covering $U \to X$ the morphism $U/G \to X/G$ need not be an etale covering. Indeed let $G = \mathbf{Z}/2$, $X = \mathbf{A}^1$ and $U = \mathbf{A}^1 \coprod \mathbf{A}^1$ such that G acts on X by $z \mapsto -z$ and on U by the composition of the sign map on each component with the permutation of the components. Then $U/G = \mathbf{A}^1$, $X/G = /af$ and the map $U/G \to X/G$ is $z \mapsto z^2$. It has no section over the strictly henselian local scheme of the point zero and thus is not an etale covering.

Example 2.18 For an upper dsitingushed square Q the square Q/G does not have to be an upper dsitinguished square. Indeed let $X = A \coprod X_0$ and $Y = B \coprod X_0$ where $B \to A$ is isomorphic to the map $U \to X$ of the previous example. Then the map $Y/G \to X/G$ is not etale and in particular Q/G is not an upper distinguished square.

References

[1] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *www.math.uiuc.edu/K-theory/305*, 1998.