On the Zero Slice of the Sphere Spectrum

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We prove the motivic analogue of the statement saying that the zero stable homotopy group of spheres is **Z**. In topology, this is equivalent to the fact that the fiber of the obvious map from the sphere S^n to the Eilenberg–MacLane space $K(\mathbf{Z}, n)$ is $(n+1)$-connected. We prove our motivic analogue by an explicit geometric investigation of a similar map in the motivic world. Since we use the model of the motivic Eilenberg–MacLane spaces based on the symmetric powers, our proof works only in zero characteristic.

1. INTRODUCTION

In [3], we introduced the slice filtration on the motivic stable homotopy category, which is a motivic analogue of the filtration by the subcategories of n-connected spectra in topology. Since the inclusion functors between different terms of the filtration have right adjoints, it makes sense to speak of the projection $s_n(E)$ of a spectrum E to the nth quotient of this filtration. This projection, which is again an object of the motivic stable homotopy category, is called the nth slice of E. Its topological analogue is the spectrum $\Sigma^n H_{\pi_n(E)}$, where $\pi_n(E)$ is the nth stable homotopy group of E and H_A is the Eilenberg–MacLane spectrum corresponding to the abelian group A. In this sense, slices provide a motivic replacement of stable homotopy groups.

The goal of this paper is to prove, over fields of characteristic zero, the main conjecture of [3], which asserts that the zero slice of the sphere spectrum **1** is the motivic Eilenberg–MacLane spectrum $H_{\mathbf{Z}}$. Using the analogy between slices and stable homotopy groups, one may interpret this result as a motivic version of the statement that $\pi_0^s(S^0) = \mathbf{Z}$. As an immediate corollary, one gets that the slices of any spectrum are modules over $H_{\mathbf{Z}}$.

We obtain our main result from an unstable statement about the motivic Eilenberg–MacLane spaces K_n. We introduce the notion of a (homotopically) n-thick space and show that, on the one hand, the suspension spectrum of any n-thick space belongs to the nth stage of the slice filtration and, on the other hand, for $n > 0$, the cone of the natural map $T^n \to K_n$ is $(n+1)$-thick.

The restriction on the characteristic of the base field k appears in our approach twice. Firstly, we use the model of K_n based on the symmetric powers of $T^n = \mathbf{A}^n/(\mathbf{A}^n - \{0\})$, which is only valid over fields of characteristic zero. Secondly, in order to analyze the structure of the dth symmetric power, we need the invertibility of $d!$ in k. We expect that the main result of this paper remains valid in any characteristic, even though the intermediate steps we use do not.

The first draft of this paper (with some mistakes) was announced at the Great Lakes K-theory meeting in Evanston in 2001, and I would like to thank all the listeners for pointing out gaps in the original argument.

2. THOM SPACES OF LINEAR REPRESENTATIONS AND THEIR QUOTIENTS

We fix a base field k and let Sch/k denote the category of quasi-projective schemes over k. For a finite group scheme G over k, we denote by G-Sch/k the category of G-objects in Sch/k. In [2],

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we defined an analogue of the Nisnevich topology on G-Sch/k. We let $(G\text{-Sch}/k)_{\mathrm{Nis}}$ denote the corresponding site and $\mathrm{Spc}_\bullet(G)$ denote the category of pointed simplicial sheaves on $(G\text{-Sch}/k)_{\mathrm{Nis}}$. Following [2], one introduces the class of $\mathbf{A}^1$-weak equivalences on $\mathrm{Spc}_\bullet(G)$ and defines the corresponding $\mathbf{A}^1$-homotopy category as the localization with respect to this class.

We let Σ_s^1 and Σ_T^1 denote the suspensions by the simplicial circle and the sphere $T = \mathbf{A}^1/(\mathbf{A}^1 - \{0\})$, respectively, and $\widetilde{\Sigma}_s$ denote the unreduced s-suspension:

$$\widetilde{\Sigma}_s F = \mathrm{cone}(F_+ \to S^0).$$

For a G-equivariant vector bundle E over X, let $\mathrm{Th}(E)$ denote the Thom space $E/(E - s_0(X))$, where s_0 is the zero section. For $X = \mathrm{Spec}(k)$, equivariant vector bundles are linear representations V of G and $\mathrm{Th}(V)$ is the object $V/(V - \{0\})$. Note that $\mathrm{Th}(V \oplus W) \cong \mathrm{Th}(V) \wedge \mathrm{Th}(W)$ and $\mathrm{Th}(0) = S^0$. Note also that, since V is $\mathbf{A}^1$-contractible as a G-scheme, there is a natural $\mathbf{A}^1$-equivalence of the form

$$\mathrm{Th}(V) \cong \widetilde{\Sigma}_s(V - \{0\}).$$

We let $\mathrm{Quot}_G \colon \mathrm{Spc}_\bullet(G) \to \mathrm{Spc}_\bullet$ denote the functor that commutes with colimits and such that $\mathrm{Quot}(X_+) = (X/G)_+$ (see [2, Section 5.1]).

In this section, we prove several general results about the structure of the quotients $\mathrm{Quot}_G(\mathrm{Th}(V))$. For simplicity, we assume in this section that G is a *finite group*. The following lemma is straightforward.

Lemma 2.1. *Let V_1 and V_2 be linear representations of G_1 and G_2. Then, there is a natural isomorphism*

$$\mathrm{Quot}_{G_1 \times G_2}(\mathrm{Th}(V_1 \times V_2)) = \mathrm{Quot}_{G_1}(V_1) \wedge \mathrm{Quot}_{G_2}(V_2).$$

In particular, if $\mathcal{O}^n$ is the trivial representation of G of dimension n, then

$$\mathrm{Quot}_G(\mathrm{Th}(V \oplus \mathcal{O}^n)) = \Sigma_T^n \mathrm{Quot}_G(\mathrm{Th}(V)).$$

For a subgroup H in G, we denote by $V^{\geq H}$ the closed subset of H-invariant elements in V and by $V^{>H}$ the subset of elements whose stabilizer is strictly greater than H. Note that, for $H_1 \neq H_2$, one has $V^{\geq H_1} \cap V^{\geq H_2} \subset V^{>H_1} \cap V^{>H_2}$ and, in particular,

$$\left(V^{\geq H_1} - V^{>H_1}\right) \cap \left(V^{\geq H_2} - V^{>H_2}\right) = \varnothing.$$

For a closed subset Z of V, we let GZ denote the orbit of Z, i.e., the closed subset $\mathrm{Im}(Z \times G \to V \times G \to V)$, where the first map is the closed embedding and the second one is the action.

Lemma 2.2. *Let H be a closed subgroup in G and $N(H)$ be the normalizer of H. Then, one has*

$$\mathrm{Quot}_G\left((V - GV^{>H})/(V - GV^{\geq H})\right) = \mathrm{Quot}_{N(H)}\left((V - V^{>H})/(V - V^{\geq H})\right). \qquad (2.1)$$

Proof. Let $A = G/N(H)$ be the set of subgroups adjoint to H. The scheme $GV^{\geq H} - GV^{>H}$ is (noncanonically) isomorphic to $(V^{\geq H} - V^{>H}) \times A$, and there is a G-equivariant map $GV^{\geq H} - GV^{>H} \to A$. Consider the section of the projection $(GV^{\geq H} - GV^{>H}) \times A \to (GV^{\geq H} - GV^{>H})$ defined by this map. Since this projection is etale, the image of this section is open, and we may consider the closed complement C to this image. Consider the square

$$\begin{array}{ccc}
(V - GV^{\geq H}) \times A & \longrightarrow & (V - GV^{>H}) \times A - C \\
\downarrow & & \downarrow{\scriptstyle p} \\
V - GV^{\geq H} & \xrightarrow{\ \ j\ \ } & V - GV^{>H}
\end{array}$$

where the vertical morphisms come from the obvious projections and the horizontal morphisms, from the obvious embeddings. Let us show that this is an upper distinguished square in G-Sch$/k$. It is clearly a pullback square, p is etale, and j is an open embedding. It remains to verify that $p^{-1}(GV^{\geq H} - GV^{>H}) \to GV^{\geq H} - GV^{>H}$ is an isomorphism, which follows from our choice of C. We conclude that

$$\left(V - GV^{>H}\right)/\left(V - GV^{\geq H}\right) = \left(V - GV^{\geq H}\right) \times A/\left(\left(V - GV^{>H}\right) \times A - C\right)$$

and therefore

$$\mathrm{Quot}_G\left(\left(V - GV^{>H}\right)/\left(V - GV^{\geq H}\right)\right) = \mathrm{Quot}_G\left(\left(V - GV^{\geq H}\right) \times A/\left(\left(V - GV^{>H}\right) \times A - C\right)\right).$$

Further, we have

$$\mathrm{Quot}_G\left(\left(V - GV^{\geq H}\right) \times A\right) = \mathrm{Quot}_{N(H)}\left(V - GV^{\geq H}\right)$$

and

$$\mathrm{Quot}_G\left(\left(V - GV^{>H}\right) \times A - C\right) = \mathrm{Quot}_{N(H)}\left(V - GV^{>H} - \left(GV^{\geq H} - V^{\geq H}\right)\right).$$

Since $GV^{\geq H} - GV^{>H} = (V^{\geq H} - V^{>H}) \coprod (GV^{\geq H} - V^{\geq H} - GV^{>H})$, we have

$$\left(V - GV^{>H} - \left(GV^{\geq H} - V^{\geq H}\right)\right)/\left(V - GV^{\geq H}\right) = \left(V - V^{>H}\right)/\left(V - V^{\geq H}\right),$$

and (2.1) follows.

Remark 2.3. An analogue of Lemma 2.2 holds for any finite etale group scheme G and any X in G-Sch$/k$.

Let H be a normal subgroup of G. Then, we may consider the relative analogue of the Quot functor

$$\mathrm{Quot}_{G,H} \colon \mathrm{Spc}_\bullet(G) \to \mathrm{Spc}_\bullet(G/H)$$

that commutes with colimits and is such that $\mathrm{Quot}(X_+) = (X/H)_+$, where X/H is considered with the natural action of G/H. One easily verifies that $\mathrm{Quot}_G = \mathrm{Quot}_{G/H} \mathrm{Quot}_{G,H}$.

Lemma 2.4. *If H is a normal subgroup of G of order prime to* $\mathrm{char}(k)$, *then one has*

$$\mathrm{Quot}_G\left(\left(V - V^{>H}\right)/\left(V - V^{\geq H}\right)\right) = \mathrm{Quot}_{G/H}\left(\left(V^{\geq H} - V^{>H}\right)_+ \wedge \mathrm{Quot}_{G,H}\left(\mathrm{Th}(V/V^{\geq H})\right)\right).$$

Proof. Since

$$\mathrm{Quot}_G = \mathrm{Quot}_{G/H} \mathrm{Quot}_{G,H},$$

it is sufficient to show that $\mathrm{Quot}_{G,H}((V - V^{>H})/(V - V^{\geq H}))$ is isomorphic to $(V^{\geq H} - V^{>H})_+ \wedge \mathrm{Quot}_{G,H}(\mathrm{Th}(V/V^{\geq H}))$ as a G/H-space. Since the order of H is prime to $\mathrm{char}(k)$, there is an isomorphism $V = V^{\geq H} \oplus (V/V^{\geq H})$. Using this isomorphism, we get an isomorphism

$$\left(V - V^{>H}\right)/\left(V - V^{\geq H}\right) = \left(V^{\geq H} - V^{>H}\right)_+ \wedge \mathrm{Th}(V/V^{\geq H}).$$

Since the action of H on $V^{\geq H}$ is trivial, we get

$$\mathrm{Quot}_{G,H}\left(\left(V - V^{>H}\right)/\left(V - V^{\geq H}\right)\right) = \left(V^{\geq H} - V^{>H}\right)_+ \wedge \mathrm{Quot}_{G,H}\left(\mathrm{Th}(V/V^{\geq H})\right).$$

Combining Lemmas 2.2 and 2.4, we get the following result.

Proposition 2.5. *Let G be a finite group of order prime to* $\operatorname{char}(k)$. *Let, further, V be a linear representation of G over k such that $V^{\geq G} = 0$. Then,* $\operatorname{Quot}_G(\operatorname{Th}(V))$ *belongs to the smallest class that is closed under cones, finite coproducts, and* $\mathbf{A}^1$-*equivalences and contains the following objects*:

(1) $\widetilde{\Sigma}_s(\operatorname{Quot}_G(V_0))$, *where V_0 is the open subscheme in V where G acts freely*;

(2) $\operatorname{Quot}_{N(H)/H}\big((V^{\geq H} - V^{>H})_+ \wedge \operatorname{Quot}_{N(H),H}(\operatorname{Th}(V/V^{\geq H}))\big)$ *for all subgroups H in G such that $H \neq e, G$.*

Proof. Consider the sequence of open embeddings $V_0 \to V_1 \to \ldots \to V_N = V$, where V_i is the subscheme of points with the stabilizer of order at most $i + 1$ and $N = |G| - 1$. In particular, V_0 is the open subscheme where G acts freely. Under our assumption, $V_{N-1} = V - \{0\}$, and hence $\operatorname{Th}(V) = \widetilde{\Sigma}_s(V_{N-1})$.

Lemma 2.6. *Let $X_0 \to X_1 \to \ldots \to X_m$ be a sequence of monomorphisms in* $\operatorname{Spc}_\bullet(G)$. *Then,* $\widetilde{\Sigma}_s(X_m)$ *belongs to the smallest class that is closed under simplicial weak equivalences and cones and contains* $\widetilde{\Sigma}_s(X_0)$ *and* X_i/X_{i-1} *for* $i = 1, \ldots, m$.

Proof. Denote the smallest class satisfying the conditions of the lemma by B. Let us show by induction on m that $\widetilde{\Sigma}_s(X_m)$ is in B. For $m = 0$, the statement is obvious. By induction, we may assume that $\widetilde{\Sigma}_s X_{m-1}$ is in B. The diagram

$$
\begin{array}{ccccc}
\operatorname{pt} & \longrightarrow & S^0 & \longrightarrow & S^0 \\
\downarrow & & \downarrow & & \downarrow \\
X_m/X_{m-1} & \longrightarrow & \Sigma_s X_{m-1,+} & \longrightarrow & \Sigma_s X_{m,+} \\
\downarrow & & \downarrow & & \downarrow \\
X_m/X_{m-1} & \longrightarrow & \widetilde{\Sigma}_s X_{m-1} & \longrightarrow & \widetilde{\Sigma}_s X_m
\end{array}
$$

whose rows and columns are cofibration sequences, shows that $\widetilde{\Sigma}_s X_m = \operatorname{cone}(X_m/X_{m-1} \to \widetilde{\Sigma}_s X_{m-1})$ and therefore $\widetilde{\Sigma}_s X_m$ is in B.

In view of Lemma 2.6, it is sufficient to show that the quotients V_i/V_{i-1} for $i = 1, \ldots, N - 1$ belong to the class we consider. This follows directly from Lemmas 2.2 and 2.4.

3. THICK SPACES

Definition 3.1. The class of n-thick objects is the smallest class A_n such that

(1) A_n is closed under $\mathbf{A}^1$-equivalences;

(2) A_n is closed under filtering colimits;

(3) if (F_i) is a simplicial object in $\Delta^{\operatorname{op}} \operatorname{Spc}_\bullet(G)$ such that $F_i \in A_n$ for all i and Δ is the diagonal functor, then $\Delta((F_i)) \in A_n$;

(4) for any smooth X in G-Sch/k and Z closed in X everywhere of codimension $\geq n$, $X/(X - Z)$ is in A_n.

We say that an object is thick if it is 1-thick.

Remark 3.2. Note that we have constructed our definition in such a way that an object of $\operatorname{Spc}_\bullet(G)$ is 0-thick if and only if it can be build out of smooth schemes by means of homotopy colimits. In particular, unless k has resolution of singularities, it is not clear that any object is 0-thick.

Lemma 3.3. *The class A_n of n-thick objects has the following properties*:

(1) *A_n is closed under coproducts*;

(2) *for a morphism $f\colon X \to Y$, where X and Y are in A_n, one has $\mathrm{cone}(f) \in A_n$*;

(3) *for X in A_n, one has $\Sigma_T^1 X \in A_{n+1}$.*

Proof. Since A_n is closed under filtering colimits and any coproduct is a filtering colimit of finite coproducts, it is sufficient to show that if X and Y are in A_n, then $X \vee Y$ is in A_n. Let us show first that, for any space X in A_n, any smooth scheme U, and any Z of codimension at least $n+1$ in U, $X \vee U/(U-Z)$ is in A_n. Indeed, the class of X for which this holds clearly satisfies the conditions of Definition 3.1 and therefore contains A_n. Similarly, the class of all Y such that $X \vee Y$ is in A_n clearly satisfies the first three conditions of Definition 3.1, and we have just shown that it satisfies the fourth. Hence, it contains A_n.

The cone $\mathrm{cone}(f\colon X \to Y)$ is the diagonal of a bisimplicial object with terms $Y, Y \vee X, Y \vee X \vee X$, etc. that one obtains if one writes the definition of the cone using the second dimension for the simplicial interval Δ_+^1. Hence, any class closed under finite coproducts and diagonals is also closed under the cones.

To verify the last condition, observe that the class of X such that $\Sigma_T^1 X$ is in A_{n+1} satisfies the first three conditions of Definition 3.1. It satisfies the fourth one since

$$\Sigma_T^1(X/(X-Z)) = X \times \mathbf{A}^1/(X \times \mathbf{A}^1 - Z \times \{0\})$$

and $\mathrm{codim}\, Z \times \{0\} = \mathrm{codim}\, Z + 1$.

For a finite etale G-scheme W, consider the functor $F \mapsto F^{\wedge W}$ introduced in [2].

Proposition 3.4. *Let W be a finite etale G-scheme of degree d. Then, the functor $F \mapsto F^{\wedge W}$ takes n-thick objects to nd-thick objects.*

Proof. It is sufficient to show that the class of F such that $F^{\wedge W}$ is nd-thick satisfies the conditions of Definition 3.1. The first condition follows from the fact that $F \mapsto F^{\wedge W}$ preserves $\mathbf{A}^1$-equivalences (see [2, p. 63, Proposition 5.2.11]). The second condition follows from the fact that $F \mapsto F^{\wedge W}$ commutes with filtering colimits. The third condition is obvious.

To see that it satisfies the fourth condition, consider X in $G\text{-Sch}/k$ and Z closed in X everywhere of codimension $\geq n$. An easy generalization of [2, p. 63, Remark 5.2.8] shows that

$$(X/(X-Z))^{\wedge W} = X^W/(X^W - Z^W),$$

and it remains to note that if W is etale, then $\mathrm{codim}\, Z^W = d\,\mathrm{codim}\, Z$.

Lemma 3.5. *Let X be a scheme with a free action of G and F be a pointed solid G-sheaf that is n-thick. Then, $\mathrm{Quot}_G(X_+ \wedge F)$ is n-thick.*

Proof. Let $R\colon \mathrm{Spc}_\bullet(G) \to \mathrm{Spc}_\bullet(G)$ be the resolution functor introduced in [2, p. 43, Construction 3.6.3], which takes a simplicial sheaf to a weakly equivalent one with terms being coproducts of representable sheaves. By [2, p. 57, Proposition 5.1.4], the functor $\mathrm{Quot} \circ R$ respects $\mathbf{A}^1$-equivalences, and for a solid F, the natural morphism $\mathrm{Quot}(R(F)) \to \mathrm{Quot}(F)$ is a weak equivalence. Hence, it is sufficient to show that, for F satisfying the conditions of the lemma, $\mathrm{Quot}(R(X_+ \wedge F))$ is n-thick.

Since F is assumed to be n-thick, it is sufficient to verify that the class of F such that $\mathrm{Quot}(R(X_+ \wedge F))$ is n-thick satisfies the conditions of Definition 3.1. The first condition holds since the functor $\mathrm{Quot}(R(X_+ \wedge \cdot))$ preserves $\mathbf{A}^1$-equivalences. To see the second one, recall that R and Quot commute with filtering colimits. Similarly, the third condition follows from the fact that both Quot and R commute with the diagonal functor.

To check the fourth condition, we have to verify that $\mathrm{Quot}(R(X_+ \wedge (Y/(Y-Z))))$ is n-thick for any closed pair (Y,Z), where $\mathrm{codim}\, Z \geq n$. Using again [2, p. 57, Proposition 5.1.4], we see that

$$\mathrm{Quot}\big(R(X_+ \wedge (Y/(Y-Z))))\big) \to \mathrm{Quot}\big(X_+ \wedge (Y/(Y-Z)))\big)$$

is an $\mathbf{A}^1$-equivalence. Further, we have

$$\mathrm{Quot}\big(X_+ \wedge (Y/(Y-Z)))\big) = \mathrm{Quot}(X \times Y)/\big(\mathrm{Quot}(X \times Y) - \mathrm{Quot}(X \times Z)\big).$$

Since the action of G on X is free, the scheme $\mathrm{Quot}(X \times Y)$ is smooth, and since the projection $X \times Y \to \mathrm{Quot}(X \times Y)$ is finite, $\mathrm{codim}(\mathrm{Quot}_G(X \times Z)) = \mathrm{codim}(X \times Z) = \mathrm{codim}\, Z$.

In the following two results, G is the trivial group.

Proposition 3.6. *Let X be an n-thick space over a perfect field k. Then, the suspension spectrum $\Sigma_T^\infty(X)$ is in $\Sigma_T^n SH^{\mathrm{eff}}$.*

Recall (see [3]) that SH^{eff} is the localizing subcategory in the motivic stable homotopy category generated by the suspension spectra of smooth schemes. The following result is an immediate corollary to our definitions and the motivic homotopy purity theorem.

Lemma 3.7. *Let U be a smooth rational scheme over k. Then, $\widetilde{\Sigma}_s(U)$ is thick.*

Proof. Since U is rational, there is a dense open subscheme V in U such that V is an open subscheme in $\mathbf{A}^n$ for some n. Then, $\widetilde{\Sigma}_s(V) \cong \mathbf{A}^n/V$ is thick and U/V is thick. We conclude by Lemma 2.6 and Lemma 3.3(2) that $\widetilde{\Sigma}_s U$ is thick.

4. SYMMETRIC POWERS OF T

Let S_n be the symmetric group. In this section, we assume that $\mathrm{char}(k) = 0$ or $n < \mathrm{char}(k)$. Consider the symmetric power

$$\mathrm{Symm}^n(T^m) := \mathrm{Quot}_{S_n}((T^m)^{\wedge n}).$$

Our goal is to prove the following result.

Theorem 4.1. *For any $n \geq 2$ and $m \geq 1$, the space $\mathrm{Symm}^n(T^m)$ is $(m+1)$-thick.*

Consider the linear representations $P(m,n)$ of S^n defined by the permutation action on $(\mathbf{A}^m)^n$. Then,

$$\mathrm{Symm}^n(T^m) = \mathrm{Quot}_{S_n}(\mathrm{Th}(P(m,n))).$$

Let $V(m,n)$ be the reduced version of $P(m,n)$,

$$V(m,n) = \ker(p\colon (\mathbf{A}^m)^n \to \mathbf{A}^m),$$

where $p(x_1, \ldots, x_n) = x_1 + \ldots + x_m$. Under our assumptions on $\mathrm{char}(k)$, there is an isomorphism

$$P(m,n) \cong V(m,n) \times \mathbf{A}^m,$$

where the action on $\mathbf{A}^m$ is trivial. Therefore, by Lemma 2.1,

$$\mathrm{Symm}^n(T^m) = \mathrm{Quot}_{S_n}(\mathrm{Th}(P(m,n))) = \Sigma_T^m \mathrm{Quot}_{S_n}(\mathrm{Th}(V(m,n))),$$

and Theorem 4.1 follows from Lemma 3.3(3) and Theorem 4.3 below.

Let (n) be the standard set of n-elements and $j_1, \ldots, j_n$ be integers ≥ 0 such that

$$\sum i j_i = n. \tag{4.1}$$

Then, there is a bijection

$$\phi \colon (n) \cong (1)^{j_1} \amalg (2)^{j_2} \amalg \ldots \amalg (n)^{j_n}, \tag{4.2}$$

and any such bijection defines an embedding

$$e_\phi \colon S_1^{j_1} \times S_2^{j_2} \times \ldots \times S_n^{j_n} \to S_n.$$

The stabilizer in S_n of a point x in $P(m,n)$ is determined by the set of diagonals $x_i = x_j$ that contain x, and subgroups of the form $\mathrm{Im}(e_\phi)$ are exactly the subgroups that occur as stabilizers of different points.

The adjunction class of a subgroup of the form $\mathrm{Im}(e_\phi)$ is determined by the integers $j_1, \ldots, j_n$. The case $j_1 = n$ and $j_i = 0$ for $i \neq 1$ corresponds to a point with all components being different, and the case $j_n = 1$ and $j_i = 0$ for $i \neq n$, to a point with all components being the same. We choose one isomorphism of the form (4.2) for each collection $j_1, \ldots, j_n$ satisfying (4.1) and let $H_{j_1, \ldots, j_n}$ denote the corresponding subgroup.

The same classification of stabilizers applies to $V(m,n)$ since $V(m,n)$ is a subspace of $P(m,n)$.

The normalizer of $H = H_{j_1, \ldots, j_n}$ is of the form $G_{j_1,1} \times \ldots \times G_{j_n,n}$, where $G_{j,i}$ is the semidirect product of S_j and S_i^j with respect to the obvious permutational action. The quotient $N(H)/H$ is of the form $S_{j_1} \times \ldots \times S_{j_n}$.

Consider now the quotient $V(m,n)/V(m,n)^{\geq H}$. One can easily see that it is isomorphic to the sum $\bigoplus V(m,i)^{\oplus j_i}$, the action of $N(H)$ on it is the direct product of actions of $G_{j_i,i}$ on $V(m,i)^{\oplus j_i}$, and the action of $G_{j_i,i}$ is given by the product of the standard actions of j_i copies of S_i on $V(m,i)$ and the permutation action of S_{j_i}. Summing this up, we get the following result.

Lemma 4.2. *One has an isomorphism of $N(H)/H$ spaces of the form*

$$\mathrm{Quot}_{N(H),H}\big(\mathrm{Th}(V(m,n)/V(m,n)^{\geq H})\big) = \bigwedge_{i=1}^{n} \mathrm{Quot}_{S_i}(\mathrm{Th}(V(m,i)))^{j_i}.$$

Theorem 4.3. *For $n \geq 2$ and $m > 0$, the space $\mathrm{Quot}_{S_n}(\mathrm{Th}(V(m,n)))$ is thick.*

Example 4.4. For $n = 2$, $V(m,n)$ is $\mathbf{A}^m$ with the sign action of $S_2 = Z/2$. Over a field of odd characteristic, one has

$$\mathrm{Quot}_{S_2}(V(m,2) - \{0\}) = \mathcal{O}(-2)_{\mathbf{P}^{m-1}} - z(\mathbf{P}^{m-1}),$$

where z is the zero section. In particular, it is a rational variety, and therefore

$$\mathrm{Quot}_{S_2}(\mathrm{Th}(V(m,2))) = \widetilde{\Sigma}_s\big(\mathrm{Quot}_{S_2}(V(m,2) - \{0\})\big)$$

is thick by Lemma 3.7.

Proof of Theorem 4.3. We proceed by induction on $n \geq 2$. Let V_0 be the open subscheme of $V(m,n)$ where S_n acts freely. By Lemma 4.5 below, $\widetilde{\Sigma}_s \mathrm{Quot}_{S_n}(V_0)$ is thick. In view of Proposition 2.5 and Lemma 3.3(1) and (2), it remains to show that, for $H = H_{j_1, \ldots, j_n}$, $H \neq e, S_n$, the space

$$\mathrm{Quot}_{N(H)}\Big(\big(V^{\geq H} - V^{>H}\big)_+ \wedge \mathrm{Quot}_{N(H),H}\big(\mathrm{Th}(V/V^{\geq H})\big)\Big)$$

is thick. For $n = 2$, any subgroup of S_n is e or S_n, and our statement is trivial. Hence, we may assume inductively that $n \geq 3$ and the theorem is proved for $n-1$. Since the action of $N(H)$ on $V^{\geq H} - V^{>H}$ is free and the space $F = \mathrm{Quot}_{N(H),H}(\mathrm{Th}(V/V^{\geq H}))$ is solid, it is sufficient by Lemma 3.5 to see that F is thick as an $N(H)/H$-space. Since $H \neq e, S_n$, there exists $n > i \geq 2$ such that $j_i \neq 0$.

By the inductive assumption, $\mathrm{Quot}_{S_i}(\mathrm{Th}(V(m,i)))$ is thick, and therefore $\mathrm{Quot}_{S_i}(\mathrm{Th}(V(m,i)))^{j_i}$ is thick as an S_{j_i}-space by Lemma 3.4. By Lemma 4.2, F is the smash product of the form $F' \wedge \mathrm{Quot}_{S_i}(\mathrm{Th}(V(m,i)))^{j_i}$ and $N(H)/H = G \times S_{j_i}$, where G acts on the first factor and S_{j_i} on the second, and we conclude that F is thick.

Lemma 4.5. *The object* $\mathrm{Quot}_{S_n}(\widetilde{\Sigma}_s V_0)$ *is thick.*

Proof. By Lemma 3.7, it is sufficient to show that $\mathrm{Quot}_{S_n}(V_0)$ is rational. Let $V_{0,0}$ be an open subset in V_0 that consists of $x_1, \ldots, x_n$ such that the first components of all $x_i \in \mathbf{A}^m$ are different. The projection to the first component,

$$V_{0,0}(m,n) \to V_0(1,n), \tag{4.3}$$

makes $V_{0,0}(m,n)$ into an equivariant vector bundle over $V_0(1,n)$. Since the action of S_n on $V_0(1,n)$ is free, the map of quotient schemes defined by (4.3) is a vector bundle as well. Hence, it is sufficient to show that $\mathrm{Quot}(V_0(1,n))$ or, equivalently, $\mathrm{Quot}(V(1,n))$ is rational.

The quotient $\mathrm{Quot}(P(1,n)) = \mathbf{A}^n/S_n$ can be identified in the standard way with $\mathbf{A}^n$, where the first coordinate is given on $P(1,n)$ by the sum of components. It follows that $\mathrm{Quot}(V(1,n)) = \mathbf{A}^{n-1}$ and, in particular, that it is rational.

5. REFORMULATION FOR SMOOTH SCHEMES

Denote, for a moment, the (pointed, simplicial) sheaves on smooth schemes by $\mathrm{Spc}_\bullet(\mathrm{Sm})$ and sheaves on all quasi-projective schemes by $\mathrm{Spc}_\bullet(\mathrm{Sch})$. Let A_n be the class of n-thick spaces in $\mathrm{Spc}_\bullet(\mathrm{Sch})$ and A_n^{Sm} be the similarly defined class in $\mathrm{Spc}_\bullet(\mathrm{Sm})$. The functor

$$\pi_* \colon \mathrm{Spc}_\bullet(\mathrm{Sch}) \to \mathrm{Spc}_\bullet(\mathrm{Sm})$$

that takes a sheaf on Sch/k to its restriction on Sm/k respects limits, colimits, and $\mathbf{A}^1$-equivalences. This immediately implies that π_* takes A_n to A_n^{Sm}. Therefore, all the results about thickness proved above remain valid if we work in the context of sheaves on smooth schemes.

6. MOTIVIC EILENBERG–MACLANE SPACES

In this section, k is a field of characteristic zero and we work in the context of sheaves on smooth schemes. Let $K_n = \mathbf{Z}_{\mathrm{tr}}(\mathbf{A}^n)/\mathbf{Z}_{\mathrm{tr}}(\mathbf{A}^n - \{0\})$ be the nth motivic Eilenberg–MacLane space over k. Consider the obvious morphism $T^n \to K_n$, where $T^n = T^{\wedge n} = \mathbf{A}^n/(\mathbf{A}^n - \{0\})$. The goal of this section is to prove the following theorem.

Theorem 6.1. *For* $n > 0$, *the space* $\Sigma_s \mathrm{cone}(T^n \to K_n)$ *is* $(n+1)$-*thick.*

Example 6.2. Consider the case $n = 1$. It is easy to see, using Lemma 6.3 below, that K_1 is $\mathbf{A}^1$-equivalent to $(\mathbf{P}^\infty, *)$, where $*$ is a rational point and the morphism $T \to K_1$ corresponds to the standard embedding $(\mathbf{P}^1, *) \to (\mathbf{P}^\infty, *)$. The cone is given by $\mathbf{P}^\infty/\mathbf{P}^1$, and its s-suspension is 2-thick by the reduced analogue of Lemma 2.6 and Lemma 3.3(2).

Consider the sheaf K_n^{eff} associated to the presheaf of the form

$$K_n^{\mathrm{eff}} \colon U \mapsto c^{\mathrm{eff}}(U \times \mathbf{A}^n/U)/c^{\mathrm{eff}}(U \times (\mathbf{A}^n - \{0\})/U), \tag{6.1}$$

where $c^{\mathrm{eff}}(X/U)$ is the monoid of effective finite cycles on X over U and the quotient on the right-hand side means that two cycles Z_1 and Z_2 on $U \times \mathbf{A}^n$ are identified if $Z_1 - Z_2$ is in $U \times (\mathbf{A}^n - \{0\})$.

Lemma 6.3. *The obvious map* $K_n^{\mathrm{eff}} \to K_n$ *is an* $\mathbf{A}^1$-*equivalence for* $n \geq 1$.

Proof. Let F^{eff} be the presheaf (6.1) and let F be the presheaf given by

$$U \mapsto \mathbf{Z}_{\mathrm{tr}}(\mathbf{A}^n)(U)/\mathbf{Z}_{\mathrm{tr}}(\mathbf{A}^n - \{0\})(U)$$

such that the sheaf associated with F is K_n. Consider the singular simplicial presheaves $C_*(F^{\mathrm{eff}})$ and $C_*(F)$. The standard argument shows that the morphisms

$$K_n^{\mathrm{eff}} = a_{\mathrm{Nis}} F^{\mathrm{eff}} \to a_{\mathrm{Nis}} C_*(F^{\mathrm{eff}}) \qquad \text{and} \qquad K_n = a_{\mathrm{Nis}} F \to a_{\mathrm{Nis}} C_*(F)$$

are $\mathbf{A}^1$-equivalences. For any smooth U, the abelian group associated with the monoid $F^{\mathrm{eff}}(U)$ coincides with $F(U)$. Therefore, $a_{\mathrm{Nis}} C_*(F)$ is the sheaf of simplicial abelian groups associated with the sheaf of simplicial monoids $a_{\mathrm{Nis}} C_*(F^{\mathrm{eff}})$. Let us show that the natural map

$$a_{\mathrm{Nis}} C_*(F^{\mathrm{eff}}) \to a_{\mathrm{Nis}} C_*(F)$$

is a local equivalence in the Nisnevich topology, i.e., that, for any Henselian local S, the map of simplicial sets

$$C_*(F^{\mathrm{eff}})(S) \to C_*(F)(S)$$

is a weak equivalence.

For any i, $C_i(F^{\mathrm{eff}})(S) = F^{\mathrm{eff}}(\mathbf{A}_S^i)$ is the free commutative monoid generated by irreducible closed subsets in $\Delta_S^i \times \mathbf{A}^n$ that are finite and equidimensional over Δ_S^i and have a nontrivial intersection with $\Delta_S^i \times \{0\}$.

Therefore, by Lemma 6.4 below, it is sufficient to show that

$$\pi_0(C_*(F^{\mathrm{eff}})(S)) = 0.$$

An element in $C_0(F^{\mathrm{eff}})(S)$ may be represented by a cycle on $\mathbf{A}_S^n$ of the form $Z = \sum n_i Z_i$, where Z_i are closed irreducible subsets of $\mathbf{A}_S^n$ and $n_i > 0$. Since Z_i are finite over S, they are local (since S is Henselian), and therefore we may assume that the closed points of all Z_i lie in $\{0\}_S$. Consider now the cycle H on $\mathbf{A}_S^1 \times \mathbf{A}^n$ obtained from Z by the pullback with respect to the map $(t, x) \mapsto x - t$. This cycle is finite over $\mathbf{A}_S^1$. Its restriction to $t = 0$ is Z, and the restriction to $t = 1$ has all the closed points in $\{1\}_S$ and therefore lies in $(\mathbf{A}^n - \{0\})_S$. Hence, the image h of H in $C_1(F^{\mathrm{eff}})(S)$ has the property $\partial_0 h = Z$ and $\partial_1 h = 0$, and we conclude that $\pi_0 = 0$.

Lemma 6.4. *Let M be a commutative simplicial monoid and $f \colon M \to M^+$ be the canonical map from M to the associated simplicial group. If, for any n, the monoid M_n is a free commutative monoid and $\pi_0(M)$ is a group, then f is a weak equivalence.*

Proof. If $\pi_0(K)$ is a group, the topological monoid $|M|$ is a grouplike associative H-space. For such H-spaces, the canonical map $|M| \to \Omega B(|M|)$ is a weak equivalence (this follows from [1]). The same argument shows that $|M^+| \to \Omega B(|M^+|)$ is a weak equivalence. It remains to show that $B(|M|) \to B(|M^+|)$ is a weak equivalence. By construction, $B(|M|)$ is the same as $|B(M)|$, where $B(M)$ is the diagonal of the bisimplicial object obtained by applying the classifying space construction to each term of M. Since the class of weak equivalences of simplicial sets is closed under taking diagonals, it remains to show that, for a free commutative monoid M, the map $B(M) \to B(M^+)$ is a weak equivalence. A free commutative monoid is a direct product of copies of natural numbers, and the result follows from the obvious equivalence $B(\mathbf{N}) \cong B(\mathbf{Z})$.

We are going now to analyze the structure of K_n^{eff}. Let $K_n^{\leq d}$ be the subsheaf in K_n^{eff} associated with the subpresheaf that takes U to the image in $K_n^{\mathrm{eff}}(U)$ of the set of finite cycles on $\mathbf{A}_U^n/U$ of degree everywhere $\leq d$ over U. Clearly, $K_n^{\mathrm{eff}} = \mathrm{colim}_d K_n^{\leq d}$. The sheaf $K_n^{\leq 1}$ coincides with $T^n = \mathbf{A}^n/(\mathbf{A}^n - \{0\})$ that is naturally embedded into K_n^{eff}. Consider the quotients $K_n^{\leq d}/K_n^{\leq d-1}$.

Lemma 6.5. *Let k be a field of characteristic zero. Then, there is an isomorphism*

$$K_n^{\leq d}/K_n^{\leq d-1} \cong \mathrm{Symm}^d(T^n), \tag{6.2}$$

where $\mathrm{Symm}^d(T^n) := \mathrm{Quot}_{S_d}((T^n)^{\wedge d})$.

Proof. Let $c^{\leq d}(\mathbf{A}^n)$ be the sheaf of finite cycles of degree $\leq d$ on $\mathbf{A}^n$. By [4], for k of characteristic zero, this sheaf is represented by

$$\coprod_{i \leq d} \mathrm{Symm}^i(\mathbf{A}^n) = \coprod_{i \leq d} \mathrm{Quot}_{S_i}(\mathbf{A}^{ni}).$$

Since the right-hand side of (6.2) is given by $\mathrm{Quot}_{S_d}(\mathbf{A}^{nd}/(\mathbf{A}^{nd} - \{0\}))$, it remains to show that the restriction of the map

$$c^{\leq d}(\mathbf{A}^n) \to K_n^{\leq d}/K_n^{\leq d-1}$$

to $\mathrm{Quot}_{S_d}(\mathbf{A}^{nd})$ is surjective and that two sections of $\mathrm{Quot}_{S_d}(\mathbf{A}^{nd})$ over a Henselian local scheme S project to the same section of $K_n^{\leq d}/K_n^{\leq d-1}$ if and only if they project to the same section of $\mathrm{Quot}_{S_d}(\mathbf{A}^{nd}/(\mathbf{A}^{nd} - \{0\}))$.

The surjectivity is clear. Let $Z = \sum n_i Z_i$ and $W = \sum m_j W_j$ be two sections of $c^{\leq d}(\mathbf{A}^n)$ over S of degree strictly d. By definition, they project to the same section of $K_n^{\leq d}/K_n^{\leq d-1}$ if both Z and W are equivalent, modulo the components lying in $\mathbf{A}^n - \{0\}$, to a cycle of degree $\leq d - 1$. Since Z_i and W_j are local, this means that not all of the closed points of the Z_i's (respectively, W_j's) lie in $\{0\}$. This is equivalent to the condition that Z and W lie in $\mathrm{Quot}_{S_d}(\mathbf{A}^{nd} - \{0\})$.

Proof of Theorem 6.1. By Lemma 6.3, it is sufficient to show that the space $\Sigma_s(K_n^{\mathrm{eff}}/T^n)$ is $(n + 1)$-thick. Since $T^n = K_n^{\leq 1}$, the space K_n^{eff}/T^n has a filtration that starts with $K_n^{\leq 2}/K_n^{\leq 1}$ and has quotients of the form $K_n^{\leq d}/K_n^{\leq d-1}$ for $d \geq 3$. Since the class of thick objects is closed under filtering colimits, by the obvious reduced analogue of Lemma 2.6 and Lemma 3.3(2), it is sufficient to show that $K_n^{\leq d}/K_n^{\leq d-1}$ are $(n + 1)$-thick for $d \geq 2$. This follows immediately from Lemma 6.5 and Theorem 4.1.

As a corollary to Theorem 6.1, we can prove the main conjecture of [3] over fields of characteristic zero.

Theorem 6.6. *Let k be a field of characteristic zero. Then, $s_0(\mathbf{1}) = H_{\mathbf{Z}}$.*

Proof. Consider the unit map $e \colon \mathbf{1} \to H_{\mathbf{Z}}$. Since $s_0(H_{\mathbf{Z}}) = H_{\mathbf{Z}}$, it is sufficient to show that $s_0(e)$ is an isomorphism, i.e., that $s_0(\mathrm{cone}(e)) = 0$. By the definition of s_0, this would follow if we could show that $\mathrm{cone}(e)$ belonged to $\Sigma_T^1 SH^{\mathrm{eff}}$. The map e is given on the level of spectra by the maps $T^n \to K_n$ considered above. In particular, we have

$$\mathrm{cone}(e) = \mathrm{hocolim}_n \Sigma_T^{-n} \Sigma_T^\infty(\mathrm{cone}(T^n \to K_n)).$$

Since $\Sigma_T^1 SH^{\mathrm{eff}}$ is closed under homotopy colimits, it remains to verify that

$$\Sigma_T^{-n} \Sigma_T^\infty(\mathrm{cone}(T^n \to K_n)) \in \Sigma_T^1 SH^{\mathrm{eff}},$$

i.e., that

$$\Sigma_T^\infty(\mathrm{cone}(T^n \to K_n)) \in \Sigma_T^{n+1} SH^{\mathrm{eff}}.$$

Since $\Sigma_T^{n+1} SH^{\mathrm{eff}}$ is stable under Σ_s, this follows from Theorem 6.1 and Proposition 3.6.

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