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Homology of schemes and covariant motives

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Harvard University, 1992

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Homology of schemes and covariant motives

A thesis presented
by

Vladimir Voevodsky

to

The Department of Mathematics

in partial fulfilment of the requirements
for the degree of

Doctor of Philosophy

in the subject of

Mathematics

Harvard University
Cambridge, Massachusetts

April 1992

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1 Preface

In the present paper I will suggest a construction which assigns to the scheme S a tensor triangle category $DM(S)$ and a *covariant* functor M from the category of schemes over S to $DM(S)$, which satisfies the usual properties of homology theories. I hope that it gives us an appropriate theory of covariant mixed motives (except, that I have no idea how to prove the existence of the t-structure in $DM(S)$). This construction was inspired by topological analogs. The "homology theory of schemes" we obtain this way is related to the would-be homotopy theory of schemes in the same way as usual singular homologies of topological spaces are related to classical homotopy theory.

A part of this work was done in collaboration with M.Kapranov together with whom we started to think about these motivic matters two years ago in Moscow. I am also very grateful to David Kazhdan, Sasha Beilinson and Sasha Goncharov for their interest and very inspiring discussions.

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Contents

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2 Generalities.

2.1 Freely generated sheaves of abelien groups.

This section is devoted to the very useful construction one has for arbitrary site. Namely, like in the category of sets one can define the free group (or free abelien group) generated by a set in the category of sheaves of sets on a site one can define the free group object (resp. free abelien group object) generated by a sheaf of sets.

For a site T denote by $Sets(T)$ and $Ab(T)$ the categories of sheaves of sets and abelien groups on T respectively.

Proposition 2.1 *Let $f : Ab(T) \longrightarrow Sets(T)$ be the forgetful functor. Then there exists a functor $\mathbf{Z} : Sets(T) \longrightarrow Ab(T)$ left adjoint to F.*

Proof: For a sheaf X of sets on T we defined the sheaf $\mathbf{Z}(\mathrm{X})$ of abelien groups as the sheaf associated with the presheaf $U \longrightarrow \mathbf{Z}(X(U))$, where $\mathbf{Z}(X(U))$ is the free abelien group generated by the set $X(U)$. The proof of the adjointness property is trivial.

The sheaf $\mathbf{Z}(X)$ is called the sheaf of abelien groups freely generated by X. I shall also use a notation $\tilde{\mathbf{Z}}$ for the functor which takes X to the kernel of the natural map $\mathbf{Z}(\mathrm{X}) \longrightarrow \mathbf{Z}$ which is induced by the canonical morphism from X to the finite object in $Sets(T)$.

Following proposition summarize main properties of the functor $\mathbf{Z}(*)$.

Proposition 2.2 *1. The functor $\mathbf{Z}(*)$ is right exact.*

2. The functor $\mathbf{Z}(*)$ preserves monomorphisms.

3. For any $X \in Sets(T)$ and $U \in ob(T)$ the group $\mathbf{Z}(X)(U)$ has no torsion.

4. Sheaves $\mathbf{Z}(X)$ are flat (as sheaves of abelien groups).

5. For $X, Y \in Sets(T)$ one has $\mathbf{Z}(X \coprod Y) = \mathbf{Z}(X) \oplus \mathbf{Z}(Y)$ and $\mathbf{Z}(X \times Y) = \mathbf{Z}(X) \otimes \mathbf{Z}(Y)$.

Proof:

1. It follows from general properties of adjoint functors.

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2. It is obviously, that the functor which takes a sheaf X of sets to the presheaf of the form $U \longrightarrow \mathbf{Z}(\mathrm{X}(\mathrm{U}))$ preserves monomorphisms. Since $\mathbf{Z}$ is a composition of this functor with the functor of associated sheaf our statment follows from the exactness of this last functor.

3. This is obviously equivalent to the injectivity of the maps $\mathbf{Z}(\mathrm{X}) \xrightarrow{m} \mathbf{Z}(\mathrm{X})$ where $''m''$ is the multiplication by m. In this form our statment follows from the construction of the sheaf $\mathbf{Z}(\mathrm{X})$ in the same way as a second item of our proposition.

4.It is well known, that a sheaf of abelien groups is flat if and only if its fibers are flat as $\mathbf{Z}$-modules, which is equivalent to that they have no torsion. Our statment follows now from the previous item.

5. It follows directly from the definitions of direct sums and tensor products of sheaves.

Examples:

1. One can easily see that the presheave of the form $X \longrightarrow \mathbf{Z}(\mathrm{F}(\mathrm{X}))$ for some sheaf of sets F will never be a sheaf in any reasonable topology. The cause is that it takes the disjoint union of open sets U, V to the free abelien group generated by the product of $F(U)$ and $F(V)$, i.e. to the tensor product of the groups corresponding to each open set instead of their direct sum.

2. Let X be the spectrum of a strictly local ring and X_{ff} be the site whose objects are schemes flat and finite over X and coverings are the flat coverings. Then for any sheaf F of sets on X_{ff} the group $\mathbf{Z}(F)(U)$ is isomorphic to the direct sum $\oplus \mathbf{Z}(F(U_i))$, where U_i are the connected components of U. Therefore in this example the presheave $U \longrightarrow \mathbf{Z}(F(U))$ is very close to be a sheaf. This fact was used by Shatz [] to prove some results on the flat cohomological dimension of such schemes X.

3. Let T be the category of simplicial sets which we consider as a site with the weakest topology (i.e. the topology with respect to which all presheaves are sheaves). Then every sheaf on T is representable by some simplicial set and the corresponding freely generated sheaf of abelien groups is representable by the free abelien simplicial group

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generated by this simplicial set. Opposite to the previous example the group $\mathbf{Z}(F)(U)$ in this case is in general much biger than $\mathbf{Z}(F(U))$.

4. Let G be a profinite group and T_G be the site of finite G-sets with the topology topology defined by means of surjective families of morphisms. A sheaf of abelien groups (resp. of sets) on T_G is a discrete G-module (resp. discrete G-set).The functor of freely generated sheaf of abelien groups corresponds on this language to the functor which takes a G-set to the corresponding freely generated G-module.

5. Let T be a topological space and U be its open subset. Denote by $\mathbf{U}$ the corresponding represerntable sheaf of sets on T. Then $\mathbf{Z}(\mathbf{U}) \cong i_!(\mathbf{Z}_U)$ where $\mathbf{Z}_U$ is the constant sheaf on U and $i : U \longrightarrow T$ is the inclusion. (See the end of this section for a generalization of this example.)

From this point I suppose that our site has sufficiently many points. It means that there exists a family of morphisms of sites $x_i : Sets \longrightarrow T$ such, that a morphism $f : X \longrightarrow Y$ of sheaves of sets on T is surjective (resp. injective) if and only if all the morphisms $x_i^*(f)$ are surjective (resp. injective). This condition holds in particular for any site T such that topology on T is generated by a pretopology where coverings are finite families of morphisms (see [?, 6.9.0]).

Proposition 2.3 *Let $f : X \longrightarrow Y$ be a surjection of sheaves of sets on T. Then the sequence of sheaves of abelien groups*

$$\ldots \longrightarrow \mathbf{Z}(X \times_Y X \times_Y X) \longrightarrow \mathbf{Z}(X \times_Y X) \longrightarrow \mathbf{Z}(X) \longrightarrow \mathbf{Z}(Y) \longrightarrow 0$$

wehre the differencial is defined as alternated sum of the maps correspondings to the partial projections is exact.

Proof: Since T has sufficiently many points we can reduce our problem to the case $T \cong Sets$ where it is trivial.

We denote the exact sequence of sheaves which corresponds to a surjection $f : X \longrightarrow Y$ by the proposition above by $C_*(f)$.

For an object U of a site T we denote by $\mathbf{Z}(\mathrm{U})$ the sheaf of abelien groups freely generated by the sheaf of sets representable by U.

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Proposition 2.4 *Let T be a site and $U \longrightarrow V$ be a covering in T. Then the sequence of sheaves*

$$... \longrightarrow \mathbf{Z}(U \times_V U \times_V U) \longrightarrow \mathbf{Z}(U \times_V U) \longrightarrow \mathbf{Z}(U) \longrightarrow \mathbf{Z}(V) \longrightarrow 0$$

wehre the differencial is defined as alternated sum of the maps correspondings to the partial projections is exact.

Proof: It is a direct corollary of the proposition ??.

Proposition 2.5 *Let $g : X' \longrightarrow X$ be a morphism of sheaves of sets on T and $f : Y \longrightarrow X$ be a surjection. Then both the kernel and the cokernel of the natural morphism of complexes $C_*(f \times_X X') \longrightarrow C_*(f)$ are exact.*

Proof: Consider the decomposition of g of the form $g = g_1 g_0$ where g_0 is a surjection and g_1 is an injection. It defines a factorization of our morphism of complexes into the composition of surjection and injection in the full subcategory of exact complexes, which implies our result.

Proposition 2.6 *Let $f : X \longrightarrow Y$ be a morphism of sheaves of sets. Then one has a natural isomorphism*

$$Im(\mathbf{Z}(\mathrm{f})) \cong \mathbf{Z}(\mathrm{Im(f)}).$$

Proof: It follows easily from ??.1 and ??.2.

Proposition 2.7 *Let $f_1 : X_1 \longrightarrow Y$, $f_2 : X_2 \longrightarrow Y$ be morphisms of sheaves of sets on T. Then one has*

$$Im(\mathbf{Z}(\mathrm{f_1})) \cap \mathrm{Im}(\mathbf{Z}(\mathrm{f_2})) = \mathrm{Im}(\mathbf{Z}(\mathrm{X_1} \times_{\mathrm{Y}} \mathrm{X_2})).$$

Proof: One can easily reduce our problem to the case $T \cong Sets$, where it is trivial.

The following proposition is a generalization of the adjointness property of the functor $\mathbf{Z}(*)$.

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Proposition 2.8 *Let T be a site and $U \in ob(T)$. Then for any sheaf F of abelien groups on T and any $i \geq 0$ one has a natural isomorphism:*

$$H^i(U, F) = Ext^i(\mathbf{Z}(U), F)$$

Proof: It follows immediately from the adjointness property of $\mathbf{Z}(*)$ and the description of Ext-groups by means of an injective resolution of F.

Following construction provides a different approach to the definition of freely generated sheaves of abelien groups, which is sometimes more convenient than the one we gave above.

Let U be an object of a site T. Denote by U/T the relative category of objects of T over U which we consider as a site with the topology induced in an obvious way by the topology on T. There is a natural morphism of sites $p : T/U \longrightarrow T$ which corresponds to the functor p^{-1} of the form $p^{-1}(V) = (V \times U \longrightarrow U)$. Let p_*, p^* be the corresponding functors of the direct and inverse images of sheaves respectively.

Proposition 2.9 *There exists a functor $p_! : Ab(T/U) \longrightarrow Ab(T)$ left adjoint to p^*.*

Proof: Let F be a sheaf of abelien groups on T/U. Consider the presheaf $p_\#$ on T of the form

$$p_\#(F)(V) = \oplus_{f \in Hom(V,U)} F(f : V \longrightarrow U).$$

We define $p_!$ to be the sheaf associated with a presheaf $p_\#$. To prove that the functor $p_!$ defined by means of this construction is left adjoint to p^* we have to show that for any pair of sheaves $F \in ob(Ab(T/U))$ and $G \in ob(Ab(T))$ there exists a natural bijection

$$Hom_{Ab(T/U)}(F, p^*G) = Hom_{Ab(T)}(p_!F, G).$$

By the adjointness property of the functor of associated sheaf a right hand side is naturally isomorphic to $Hom_{Ab(T/U)}(p_\#F, G)$. Therefore a morphism $a : p_!F \longrightarrow G$ is just a natural family of morphisms

$$a_{f:V \longrightarrow U} : F(f : V \longrightarrow U) \longrightarrow G(V).$$

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From the other hand a one has

$$p^*G(f : V \longrightarrow U) = G(V)$$

and therefore a morphism $F \longrightarrow p^*G$ is a natural family of morphisms of exactly the same form. Proposition is proven.

Proposition 2.10 *Functor* $p_! : Ab(T/U) \longrightarrow Ab(T)$ *is exact.*

Proof: Since $p_!$ is left adjoint to p^* it is right exact by the general properties of the adjoint functors. It is sufficient to prove therefore, that $p_!$ preserves monomorphisms. By our construction $p_!$ is a composition of the functor $p_\#$ with the functor of associated sheaf. It follows immediately from an explicit description of $p_\#$ that it preserves monomorphism and the functor of associated sheaf is known to be exact. Proposition is proven.

A connection between functor $p_!$ and functor of freely generated abelien group is given by the following proposition.

Proposition 2.11 *Let* $\mathbf{Z}_{\mathrm{U}}$ *be a constant sheaf on* T/U. *Then one has*

$$p_!\mathbf{Z}_{\mathrm{U}} \cong \mathbf{Z}(\mathrm{U}).$$

Proof: It follows immediately from the constructions of the functors $p_!$ and $\mathbf{Z}$ given above.

Denote by $\mathbf{Z}_{\mathrm{U}} : \mathrm{Sets(T/U)} \longrightarrow \mathrm{Ab(T/U)}$ a functor of freely generated sheaf of abelien groups on T/U. The above proposition implies in particular, that for any $(V \longrightarrow U) \in ob(T/U)$ one has a natural isomorphism $p_!(\mathbf{Z}_{\mathrm{U}}(\mathrm{V} \longrightarrow \mathrm{U})) = \mathbf{Z}(\mathrm{V})$.

2.2 Homological category of site with interval

Let T be a site. An interval in T is by the definition an object I^+, such that there exists a triple of morphisms $(\mu : I^+ \times I^+ \longrightarrow I^+, i_0, i_1 : pt \longrightarrow I^+)$ satisfying the following conditions

$$\mu(i_0 \times Id) = \mu(Id \times i_0) = i_0 p$$

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$$\mu(i_1 \times Id) = \mu(Id \times i_1) = Id,$$

where $p : I^+ \longrightarrow pt$ is a canonical morphism.

The goal of this section is to assign to any site with interval a tensor triangle category $H(T, I^+)$ (or just $H(T)$) which is called a homological category of T and to prove its elementary properties.

There are two most important examples of sites with interval. First is a category $\Delta^{op} - Sets$ of the simplicial sets with a weakest topology and a standard simplicial interval as I^+. In this case category $H(T)$ is equivalent to derived category of abelien groups. Another example is a category of scheme considering as a site with respect to some topology on it and with an affine line as an interval.

Let I^1 be a kernel of a canonical morphism $\mathbf{Z}(\mathrm{I}^+) \longrightarrow \mathbf{Z}$. Denote by $D(T)$ a derived category of the category $Ab(T)$ of sheaves of abelien groups on T constructed by means of bounded complexes. It is known to be a tensor triangle category.

Definition 2.12 *Homological category $H(T)$ of site with interval I^+ is defined as a strong localization of the category $D(T)$ with respect to a thick subcategory generated by objects of the form $X \otimes I^1$ where $X \in ob(D(T))$ (see Appendix B for definition of strong localization).*

It follows from the results of Appendix B that $H(T)$ has a natural tensor triangle structure. I shall also use a notation $H_0(T)$ for a category defined in the same way as $H(T)$ but by means of usual localization.

Definition 2.13 *A functor $M : Sets(T) \longrightarrow H(T)$ is defined as a composition of functor $\mathbf{Z}$ with a natural functor from $Ab(T)$ to $H(T)$. In the same way are defined functors $\tilde{M}, M_0, \tilde{M}_0$.*

Denote by $i : \mathbf{Z} \longrightarrow \mathrm{I}^1$ morphism induced by the difference $\mathbf{Z}(\mathrm{i}_0) - \mathbf{Z}(\mathrm{i}_1) : \mathbf{Z} \longrightarrow \mathbf{Z}(\mathrm{I}^+)$. It is easy to see that i is a monomorphism. Let S^1 be its cokernel and $j : S^1 \longrightarrow \mathbf{Z}[1]$ a corresponding morphism in $D(T)$. Denote $S^{\otimes n}$ (resp. $I^{\otimes n}$) by S^n (resp. I^n).

Proposition 2.14 *Let $X, Y \in obD(T)$ then one has:*

$$Hom_{H_0(T)}(X, Y) = lim_{n \to \infty} Hom_{D(T)}(X \otimes S^n, Y[n])$$

where the direct limit is defined by tensor multiplication of morphisms with $j : S^1 \longrightarrow \mathbf{Z}[1]$.

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Proof: Note first of all that the morphism $j : S^1 \longrightarrow \mathbf{Z}[1]$ represents isomorphism in $H_0(T)$ and therefore there is a canonical morphism:

$$lim_{n\to\infty} Hom_{D(T)}(X \otimes S^n, Y[n]) \longrightarrow Hom_{H_0(T)}(X, Y)$$

One can see that for any exact functor $F : D(T) \longrightarrow D'$ from $D(T)$ to a triangle category D'. such that $F(g)$ is an isomorphism for any such g that it cone lies in the thick subcategory generated by objects of the form $X \otimes I^1$ there exists the unique extension of the map $Hom_{D(T)}(X, Y) \longrightarrow Hom_{D'}(F(X), F(Y))$ to the map

$$lim_{n\to\infty} Hom_{D(T)}(X \otimes S^n, Y[n]) \longrightarrow Hom_{D'}(F(X), F(Y)).$$

Using universal properties of localization we see that to prove our theorem it is sufficient to show, that for any Y from the thick subcategory generated by objects of the form $X \otimes I^1$ there exists n such that $Id_Y \otimes j^{\otimes n} = 0$. It is sufficient to show that class of objects satisfying this property contains objects of the form $X \otimes I^1$ and is thick.

Let $Y = X \otimes I^1$. Then $Id_Y \otimes j : Y \otimes S^1 \longrightarrow Y[1]$ can be included in exact triangle:

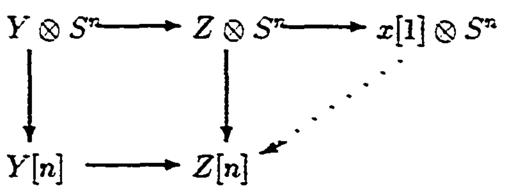
$$Y \longrightarrow Y \otimes I^1 \longrightarrow Y \otimes S^1 \longrightarrow Y[1]$$

The morphism $\mu : I^1 \otimes I^1 \longrightarrow I^1$ induces a splitting of the morphism $Y \longrightarrow Y \otimes I^1$ and, therefore $Id_Y \otimes j = 0$.

Let us show now, that our class of object is indeed thick, i.e. that it satisfies the axioms of the definition ??

1. Fist axioms satisfies by trivial reason.

2. Let $X \longrightarrow Y \longrightarrow Z \xrightarrow{f} X[1]$ be an exact triangle such that for some m and n one has $Id_X \otimes j^{\otimes m} = 0$ and $Id_Y \otimes j^{\otimes n} = 0$ (we can restrict ourself to this case because if $Id_U \otimes j^{\otimes n} = 0$ for some n then the same holds for any U[k]). Let us show that $Id_Z \otimes j^{\otimes(m+n)} = 0$. Consider a diagram:



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Doted arrow exists because the upper string is exact and $Y \otimes S^n \longrightarrow Y[n]$ is equal to zero. Denote it by α. One obviously has

$$Id_Z \otimes j^{\otimes m+n} = (Id_Z \otimes j^{\otimes n}) \otimes j^{\otimes m} = (\alpha \otimes j^{\otimes m})(f \otimes Id_{S^m})$$

and

$$\alpha \otimes j^{\otimes m} = \alpha[m](Id_{X[1]\otimes S^n} \otimes j^m) = 0.$$

3. Similary.

Corollary 2.15 *Let X, Y be a pair of objects of the $D(T)$ such that for any n and m one has*

$$Hom_{D(T)}(X \otimes I^n, Y[m]) = 0$$

then

$$Hom_{H_0(T)}(X, Y[m]) = Hom_{D(T)}(X, Y[m]).$$

Proof: We should show that morphisms

$$Hom_{D(T)}(X, Y[m]) \longrightarrow Hom(X \otimes S^n, Y[m+n])$$

are isomorphisms for all n. We shall prove it by the induction on n. For $n = 0$ our statment is trivial. To make an inductive step consider the exact triangle

$$X \otimes S^{n-1} \longrightarrow X \otimes I^1 \otimes S^{n-1} \longrightarrow X \otimes S^n \longrightarrow X \otimes S^{n-1}[1].$$

It is sufficient to show that $Hom_{D(T)}(X \otimes I^1 \otimes S^{n-1}, Y[m]) = 0$. Obviously, if X satisfies the conditions of our proposition so does $X \otimes I^1$. Therefore, by the induction we have

$$Hom_{D(T)}(X \otimes I^1 \otimes S^{n-1}, Y[m]) = Hom_{D(T)}(X \otimes I^1, Y[m-n]) = 0.$$

Definition 2.16 *An object $Y \in ob(D(T))$ is called strictly homotopy invariant if for any $X \in ob(D(T))$ one has $Hom(X \otimes I^1, Y) = 0$.*

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Proposition 2.17 *Let $Y \in ob(D(T))$ be a strictly homotopy invariant object. Then for any X one has*

$$Hom_{H(T)}(X,Y) = Hom_{D(T)}(X,Y)$$

Proof: Follows from the proposition ??

We call an object X of $D(T)$ an object of finite dimension if there exists N such that for any $F \in ob(Ab(T))$ and any $n > N$ one has

$$Hom_{D(T)}(X, F[n]) = 0.$$

Proposition 2.18 *Suppose, that $\mathbf{Z}, \mathrm{I}^1$ and X are objects of finite dimension. Then for any $Z \in ob(D(T))$ one has*

$$Hom_{H(T)}(X,Z) = Hom_{H_0(T)}(X,Z)$$

Proof: It is sufficient to show that for unbounded (with respect to the objects of the form $Y \times I^1$) object $Z \in ob(D(T))$ one has

$$Hom_{H_0(T)}(X,Z) = 0.$$

It follows immediately from our assumptions and proposition ??.

Let $(T_1, I_1^+), (T_2, I_2^+)$ be a pair of sites with interval. A morphism $F : (T_1, I_1^+) \longrightarrow (T_2, I_2^+)$ is by definition a morphism of sites $F : T_1 \longrightarrow T_2$ such, that $F^{-1}(I_2^+)$ is isomorphic to I_1^+. For example if T_1, T_2 have the same underlying categories and topology of T_1 is stronger than that of T_2 and $I_1^+ \cong I_2^+$ then an identity functor is a morphism of sites with interval.

Proposition 2.19 *Let $F : (T_1, I_1^+) \longrightarrow (T_2, I_2^+)$ be a morphism of sites with interval, then it induces a tensor triangle functor*

$$H(F) : H(T_2) \longrightarrow H(T_1).$$

Proof: There is a functor $D(F) : D(T_2) \longrightarrow D(T_1)$ which is induces by the inverse image of sheaves. One can easily see, using universal properties of localization, that it can be descended to a functor $H(F) : H(T_2) \longrightarrow H(T_1)$ which obviously satisfies all the properties we need.

There is an obvious analogue of this proposition for the categories $H_0(T_1), H_0(T_2)$. We denote the corresponding functor by $H_0(F)$.

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Proposition 2.20 *Let $(T_1, I_1^+), (T_2, I_2^+)$ be sites with interval such, that underlying categories of T_1 and T_2 coincide, $I_1^+ \cong I_2^+$ and topology on T_1 is stronger than topology on T_2. Denote by $F : T_1 \longrightarrow T_2$ a morphism which corresponds to the identity functor on underlying categories. Then $H_0(F) : H_0(T_2) \longrightarrow H_0(T_1)$ is a localization of $H_0(T_2)$ with respect to thick subcategory generated by the objects which correspond to the sheaves F of abelien groups on T_2 such, that T_1-sheaf associated with F is isomorphic to zero.*

Proof: Obviously.

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3 H-topology on the category of schemes.

3.1 H-topology.

This section is devoted to the definition of the new Grothendieck topology on the category of schemes over the base which I call h-topology because it seems to be the suitable for the developing of the homotopy theory of schemes.

All through this section by the scheme I mean the separated netherien scheme over a fixed base S and all the morphisms of schemes are morphisms over S. I shall omit S in all the notations below where it is possible.

The most important cause why the topologies usually used are not satisfactory for our purpose is that there are all *too weak*. Let me explain what I mean. The only cause why we are to use the topologies and sheaves in the homotopy theory is the absence of the direct limits in the category of schemes which we need to define such objects as a cone of a morphism, suspension or realization of a simplicial scheme. From the other hand there are several situations when the direct limits in the category of schemes exist. The most important examples are the symmetric powers and the objects like $\partial\Delta^n$ or ∂I^n which can be considered as the direct limits of the suitable diagrams of affine spaces. Therefore it is natural to try to find the topology such that the direct limits of such kind would be representable by the corresponding direct limits of sheaves. (Note that the functor which takes an object of the category to the corresponding representable sheaf of sets preserves inverse limits but not in general direct ones.)

Let C be a site, i.e. the category with a Grothendieck topology on it. Denote by $Shv(C)$ a category of sheaves of sets on C and by $L : C \longrightarrow Shv(C)$ natural functor which takes an object to the corresponding representable sheaf. Consider an equalizer of the two morphisms

$$X \rightrightarrows Y \longrightarrow Z.$$

It is easy to see that for $L(Z)$ be the equalizer of the morphisms $L(X) \rightrightarrows L(Y)$ it is necessary (but not sufficient) that the morphism $Y \longrightarrow Z$ is the covering in C. Therefore if we want $L(\partial\Delta^n)$, to be a direct limit of the diagram of the sheaves $L(\mathbf{A}^k)$ we must admit in our topology the coverings like a covering of the scheme by its irreducible components. Considering the easiest example of such kind, say the covering of the scheme

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$X = Spec\, k[x,y]/(xy)$ by two affine lines we notice that the corresponding morphism, being the strict epimorphism in the category of schemes, is not the *universal* epimorphism. To see it it is sufficient to consider the base change over the morphism $Spec\, k[t]/(t^2) \longrightarrow X$ corresponding to the tangent vector in the singular point of X which does not lie along any of the irreducible components. Our would be covering after this base change become the morphism $Spec(k \oplus k) \longrightarrow Spec\, k[t]/(t^2)$ which obviously is not the epimorphism of schemes. The main conclusion we can make from this example is that the topology for which the functor L preserves the direct limits of such type can not be subcanonical, i.e. that the presheaf representable by the scheme will not be in general the sheaf in it.

There is also another class of the coproducts in the category of schemes which I want to be preserved by the functor taking a scheme to the corresponding representable sheaf. Namely, consider a blowing up $p : X_Z \longrightarrow X$ of the closed subscheme Z of X. Then in the category of schemes X is a coproduct of X_Z and Z with respect to the natural morphisms $p^{-1}(Z) \longrightarrow Z$ and $p^{-1}(Z) \longrightarrow X$.

Gathering all these examples together we come to the following definition.

Let $p : X \longrightarrow Y$ be the morphism of schemes. It is called *topological epimorphism* if the underlying topological space of Y is the quotient space of the underlying topological space of X, i.e. if p is surjective and the subset A in Y is open if and only if $p^{-1}(A)$ is open in X. One can easily see that any open or closed surjective morphism is the topological epimorphism in this sense. The topological epimorphism $p : X \longrightarrow Y$ is called *universal topological epimorphism* if for any morphism $f : Z \longrightarrow Y$ the projection $Z \times_Y X \longrightarrow Y$ is the topological epimorphism. Note that any surjective proper or flat morphism is the universal topological epimorphism as well as any composition of such morphisms.

Definition 3.1 *H-topology on the category of schemes is defined as the topology associated with the pretopology with the coverings of the form $\{p_i : U_i \longrightarrow X\}$, where $\{p_i\}$ is the finite family of the morphisms of finite type such that the morphism $\coprod p_i : \coprod U_i \longrightarrow X$ is the universal topological epimorphism.*

I shall also use qfh-topology, which corresponds to the coverings of the same type, but only for the quasifinite morphisms p_i.

Proposition 3.2 *The class of the coverings defined above forms the Grothendieck pretopology on the category of schemes.*

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Proof: Obviously.

Examples:

1. Any flat covering is, obviously an h-covering. Moreover, since any flat surjective morphism of the finite type admit a section over the quasi-finite surjective flat morphism, even qfh-topology is stronger then the flat one.

2. Any surjective proper morphism is an h-covering. It follows almost immediately from this remark, that for the closed subscheme Z of X, the sheaf representable by X is a suitable coproduct of the sheaves representable by Z and the blow-up X_Z.

3. Let X be a scheme and G be a finite group acting on X. Suppose that there exist a categorical factor X/G (see [?, ex.5 n.1]). then the corresponding representable qfh-sheaf (and, a fortiori, an h-sheaf) is a quotient sheaf of the sheaf representable by X with respect to the induced G-action. Note, that if the action of G is not free, then even in the easiest cases this statment is false for any standard topology like etale or flat.

4. Consider a blow up $p : X_x \longrightarrow X$ of the surface X with the center in the closed point $x \in X$ and let $U = X_x - \{x_0\}$ where x_0 is a closed point over x. Then the natural morphism $p_U : U \longrightarrow X$ *is not* an h-covering. To see it consider a curve C in X such that $p^{-1}(C) = p^{-1}(\{x\}) \cup \tilde{C}$ and $\tilde{C} \cap p^{-1}(\{x\}) = \{x_0\}$. Obviously, $p_U^{-1}(C - \{x\})$ is closed in U but $C - \{x\}$ is not. Therefore p_U is not a topological epimorphism.

5. The condition of the finiteness of the family of morphisms in the definitions of both h- and qfh-topologies is essential. Consider a surface X and a closed point $x \in X$. Let

$$a = (a_1, ..., a_n, ..) \in \varprojlim m/m^n$$

be an element in the completion of the maximal ideal m of the local ring of x which does not correspond to any element of m. let $X_n \longrightarrow X$ be a blow-up of an ideal $(a_n)+m^n$. It is easy to see that X_{n+1} is a blow-up of X_n with the center in the point $x_n \in X_n$ over $x \in X$. Let $U_n = X_n -$

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$\{x_n\}$. It can be shown, that the morphism $\coprod_{n>0} U_n \longrightarrow X$ is a universal topological epimorphism, though the morphisms $\coprod_{N>n>0} U_n \longrightarrow X$ are not for any $N < \infty$.

It is clear from the above examples that it is not easy, in the general case, to say whether or not a given surjective morphism is a universal topological epimorphism. I am going to define now some special class of h-coverings which I call coverings of the normal form. The main result of this paragraph is the theorem, that any h-covering of an excellent scheme admit a refinitement which is an h-covering of the normal form.

Proposition 3.3 *Let $\{U_i \xrightarrow{p_i} X\}$ be an h-covering of the scheme X, denote by $\coprod V_j$ the disjoint union of the irreducible components of $\coprod U_i$ such that for any j there exists an irreducible component of X_i of X over which V_j is dominant. Then the morphism $q : \coprod V_j \longrightarrow X$ is surjective.*

Proof: Suppose first that X is irreducible. Let $x \in X$. We want to prove that x lies in the image of q. Considering a base change along the natural morphism $Spec(\mathcal{O}_x) \longrightarrow X$ we may suppose that X is a spectrum of the local ring and x is a closed point of X.

Denote by Z the closure of the image of the irreducible components of $\coprod U_i$ which are not dominant over X. Since this image is a constructible set which does not contain the generic point of X one has $Z \neq X$. Let $x \in X$ be a closed point of X. Considering the base change along the natural morphism $Spec\mathcal{O}_x \longrightarrow X$ we may restrict ourself to the case $X = Spec\mathcal{O}_x$. It follows from [?, 10.5.5 and 10.5.3], that the set of the points of the dimension one is dense in X. Therefore, there exists a point $y \in X$ of the dimension one which does not lie in Z. If x does not lie in the image of q then the preimage $q^{-1}(y)$ is closed which implies that $p_i^{-1}(y)$ are closed as well, which gives us a contradiction with the condition that $\{p_i\}$ is an h-covering.

Suppose now, that X is a general scheme and $X_{red} = \cup X_k$ be the composition of the maximal reduced subscheme of X into the union of its irreducible components. Consider the natural morphisms $X_k \longrightarrow X$ and let $\{U_i \times_X X_i \longrightarrow X_i\}$ be the preimages of our h-covering. Then the morphisms $\coprod V_{jk} \longrightarrow X_k$, where V_{jk} are the irreducible components of $\coprod U_i \times_X X_k$ which are dominant over X_k are surjective, which implies that $\coprod V_j \longrightarrow X$ is surjective, since $\coprod V_j = \coprod \coprod V_{jk}$.

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Remark: This proposition leads to the following generalization of the example 4 above. Let Z be a closed subscheme of the integral scheme X and $X_Z \longrightarrow X$ be a blow-up with the center in Z. Suppose, that for an open subscheme $U \subset X_Z$ the composition $U \longrightarrow X_Z \longrightarrow X$ is an h-covering. Then $U = X_Z$. To show it, let me consider a base change along the projection $X_Z \longrightarrow X$. Then $U \times_X X_Z$ is an open subscheme in $X_Z \times_X X_Z$. This last scheme is a union of the diagonal Δ and a component, which is not dominant over X_Z. According to our proposition $(U \times_X X_Z) \cap \Delta \longrightarrow X_Z$ is a surjection, which implies that $U = X_Z$.

Proposition 3.4 *Let $\{p_i : U_i \longrightarrow X\}$ be a finite family of the quasi-finite morphisms over the normal connected scheme X. Then $\{p_i\}$ is a qfh-covering if and only if the subfamily $\{q_j\}$ consisting of those p_i which are dominant over X is surjective. In that case $\{q_j\}$ is also a qfh-covering of X.*

Proof: The "only if" part follows immediately from the previous proposition.

To prove the "if" part it is sufficient to notice that in the case of the normal connected scheme X each dominant quasi-finite morphism is universally open [?, p.24] and therefore each surjective family of such morphisms is an h-covering.

Remark: The statment of the proposition above is false for the schemes which are not normal. To see it, consider a surface X over an algebraically closed field and let $x, y \in X$ be two different closed points of X. Let Y be a scheme obtained from X by glueing the point x, y together. Let $U = X - \{x\}$. The natural morphism $p : U \longrightarrow Y$ is surjective but it is not a qfh-covering. To show it, consider a curve $C \subset X$ in X, which contains x and does not contain y. Then $p^{-1}(C - \{x\})$ is closed in U, while $C - \{x\}$ is not closed in Y.

Definition 3.5 *The finite family of the morphisms $\{U_i \xrightarrow{p_i} X\}$ is called an h-covering of the normal form if the morphisms p_i admit a factorization of the form $p_i = s \circ f \circ in_i$, where $\{in_i : U_i \longrightarrow \bar{U}\}$ is an open covering, $f : \bar{U} \longrightarrow X_Z$ is a finite surjective morphism and $s : X_Z \longrightarrow X$ is a blowing up of the closed subscheme of X.*

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Beginning from this point, I restrict my considerations to the excellent schemes (see [?, 7.8]). I am almost sure, that this restriction can be omited, but since it leads to the significant simplifications of the proofs, it seems to be reasonable.

Let me recall several properties of the excellent schemes, which I shall use below without additional references. Any scheme of the form $X = Spec(A)$ where A is a field or a Dedekind domain with the field of fractions of the characteristic zero is excellent. If scheme X is excellent and $Y \longrightarrow X$ is a morphism of the finite type, then Y is excellent. Any localization of the excellent scheme is excellent.

The most important for our purposes property of the excellent schemes is that for any excellent integral scheme X and the finite extension L of the field of functions on X, the normalization of X in L is finite over X.

Lemma 3.6 *Let $f : Y \longrightarrow X$ be a finite morphism, then the underlying topological space of the diagonal $Y \subset Y \times_X Y$ is an irreducible component of $Y \times_X Y$.*

Proof: Obviously.

Lemma 3.7 *Let X be an excellent normal connected scheme and L ba a finite purely inseparable extension of the field of functions $K(X)$ of X. Then the normalization $f : Y \longrightarrow X$ of X in L is a universal homeomorphism.*

Proof: Since X is excellent, the morphism f is finite and surjective , which implies that it is universally surjective. It is sufficient to show, that f is universally injective. According to [?, 3.7.1] this is equivalent to the surjectivity of the diagonal morphism $\Delta : Y \longrightarrow Y \times_X Y$. Since X is normal our morphism f is universally open ([?, p.24]). In particular, considering a base change along f we see that the projection $Y \times_X Y \longrightarrow Y$ is an open morphism. It implies that each irreducible component of $Y \times_X Y$ is dominant over Y. According to the previous lemma our statment would follow if we prove that the general fiber of the projection $Y \times_X Y \longrightarrow Y$ is connected. This fiber is a scheme $Z = Spec(L) \times_{Spec(K(X))} Spec(L)$ and since our extension is purely inseparable one has $Z_{red} = Spec(L)$ which finish the proof.

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To prove that any h-covering admit a refinitement which is an h-covering of the normal form, we need first to introduce some notations.

Let Z be a closed subscheme of the scheme X. I denote by $p_Z : X_Z \longrightarrow X$ the blow-up of X with the center in Z. For the scheme $Y \longrightarrow X$ over X, denote by $\tilde{p}_Z(Y)$ the closure $Y \times_X X_Z$ of the open subscheme $Y \times_X X_Z - pr_2^{-1}(p_Z^{-1}(Z))$. The scheme $\tilde{p}_Z(Y)$ over X_Z is called a strict transform of Y.

Theorem 3.8 (platification par eclatement) *Let $f : Y \longrightarrow X$ be a morphism of the finite type, which is flat over an open subset $U \subset X$. Then there exists a closed subscheme Z disjoint with U such that the strict transform $\tilde{p}_Z(Y)$ is flat over X_Z.*

Proof: See [?, 5.2].

Theorem 3.9 *Let $\{U_i \xrightarrow{p_i} X\}$ be an h-covering of the excellent reduced scheme X which has the finite number of the irreducible components , then there exists an h-covering of the normal form, which is a refinitement of $\{p_i\}$.*

Proof: Suppose first, that X is a normal connected scheme and all the morphisms p_I are dominant and quasi-finite. Considering the normalizations of the schemes U_i we may suppose, that U_i are normal and connected as well. Let $\bar{p}_i : \bar{U}_i \longrightarrow X$ be the finite morphisms such that $\bar{U}_i$ are normal and connected and there exist the factorizations of $U_i \xrightarrow{in_i} \bar{U}_i \xrightarrow{\bar{p}_i} X$, where in_i are the open immersions ([?, 1.1.8]).

There exists a connected normal scheme $\bar{V}$ and a finite surjective morphism $\bar{q} : \bar{V} \longrightarrow X$ such that it can be factorized through all the morphisms $\bar{p}_i$ and there exists the factorization of $\bar{q}$ of the form $\bar{V} \xrightarrow{\bar{g}} \bar{W} \xrightarrow{\bar{r}} X$ where $\bar{W}$ is a connected normal scheme and $\bar{r}$,$\bar{g}$ correspond to the purely inseparable and Galois extensions of the fields of functions respectively. Let $V_i = V \times_{\bar{U}_i} U$. The compositions $\{q_i : V_i \longrightarrow \bar{V} \longrightarrow X\}$ define an h-covering which is a refinitement of our initial one. Let G be a Galois group of the extension of the fields which corresponds to the morphism g. Then G acts on $\bar{V}$. Consider the open subsets $\sigma(V_i)$ for $\sigma \in G$. Since $\cup q_i(V_i) = X$ and the morphism $\bar{r}$ defines a homeomorphism of the underlying topological spaces (lemma ??), we have $\cup\sigma(V_i) = V$. The covering $\{\sigma(V_i) \longrightarrow X\}$ is of the normal form and I claim that it is a refinitement of the covering $\{V_i \longrightarrow X\}$. To see it it is

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sufficient to define a morphism from one to another as a family of morphisms $\sigma^{-1} : \sigma(V_i) \longrightarrow V_i$.

Let now X be a general reduced scheme and p_i be the flat quasi-finite morphisms. Consider a normalization $X_{norm} \longrightarrow X$ of X. It is a finite morphism and X_{norm} is a disjoint union of the connected normal schemes X_j. Applying the above construction to the covering $U_i \times_X X_j \longrightarrow X_j$ we obtain in this case the refinitement we need.

Consider now the case of the general h-covering $\{p_i : U_i \longrightarrow X\}$ of the reduced scheme X with the finite number of the irreducible components. It follows from [?, 11.1.1] that there exists a dense open subscheme X_0 of X such that all the morphisms p_i are flat over X_0. Let Z be a closed subscheme disjoint with X_0 such that the morphism $f : \tilde{p}_Z(\coprod U_i) \longrightarrow X_Z$ is flat (theorem ??). Since $X_Z \times_X (\coprod U_i) \longrightarrow X_Z$ is an h-covering and the closure of the complement $X_Z \times_X (\coprod U_i) - \tilde{p}_Z(\coprod U_i)$ lies over $p_Z^{-1}(Z)$ and, therefore is not dominant over any irreducible component of X_Z the proposition ?? implies that f is a surjection. There exists then a quasi-finite flat surjective morphism $U' \longrightarrow X_Z$ which can be factorized through f. The normal refinitement for such type of coverings was constructed above.

3.2 Representable sheaves.

Let L be a functor $Sch/S \longrightarrow Shv_h(S)$ which takes a scheme X/S to the corresponding representable sheaf, i.e $L(X)$ is the h-sheaf associated with the presheaf $Y \longrightarrow Mor_S(X,Y)$. I shall also use a notation L_{qfh} for the corresponding functor with respect to the qfh-topology.

The question we are interisting about in this paragraph is what can be said about the morphisms $L(X) \longrightarrow L(Y)$? Note, that the set $Mor(L(X), L(Y))$ coincides with the set of sections of the sheaf $L(Y)$ over X. Therefore, to answer our question we just need to describe a sheaf $L(Y)$ which is associated with the presheaf representable by Y.

Let me recall first a general construction of the sheaf associated with the presheaf [?, 2.2],[?]. Let P be a presheaf. For any scheme X define an equivalence relation on the set $P(X)$, setting the sections $a, b \in P(X)$ to be equivalent if there exists a covering $\{p_i : U_i \longrightarrow X\}$ of X such that for any i one has $p_i^*(a) = p_i^*(b)$. Denote by P' the presheaf such that $P'(X)$ is a set

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of the equivalence classes of $P(X)$.

For any covering $\mathcal{U} = \{p_i : U_i \longrightarrow X\}$ denote by $H^0(\mathcal{U}, P')$ the equalizer of the maps $\coprod P'(U_i) \rightrightarrows \coprod P'(U_i \times_X U_j)$ which are induced by the projections. For any refinitement $\mathcal{U}'$ of $\mathcal{U}$ there is defined an obvious map $H^0(\mathcal{U}, P') \longrightarrow H^0(\mathcal{U}', P')$. We set

$$aP(X) = \varinjlim H^0(\mathcal{U}, P').$$

It can be shown that aP is indeed a sheaf associated with P. Note that the natural morphism of presheaves $P' \longrightarrow aP$ is injective.

I am going now to apply this construction to the representable presheaves.

Lemma 3.10 *Let X be a scheme and X_{red} be its maximal reduced subscheme. Then the natural morphism $L_{qfh}(i) : L_{qfh}(X_{red}) \longrightarrow L_{qfh}(X)$ is an isomorphism.*

Proof: Since the morphism $i : X_{red} \longrightarrow X$ is a monomorphism in the category of schemes and the functor L is left exact, so is $L(i)$. From the other hand, i is a qfh-covering which implies that $L(i)$ is an epimorphism. Therefore $L(i)$ is an isomorphism.

Lemma 3.11 *Let X be a reduced scheme and $U \longrightarrow X$ be an h-covering, then it is epimorphism in the category of schemes. In particular for any reduced X and any Y the natural map $Mor_S(X, Y) \longrightarrow Mor(L(X), L(Y))$ is injective.*

Proof: It follows immediately from the fact that h-coverings are surjective on the underlying topological spaces of schemes. For a scheme X denote by $L_0(X)$ a presheaf we obtain on the first step of the construction of the sheaf L(X) which is described above. Two previous lemmas shows that for any scheme Y one has $L_0(X)(Y) = Mor_S(Y_{red}, X)$.

Lemma 3.12 *Let $X = Spec(K)$, where K is a field, then for any scheme Y one has $Mor(L(X), L(Y)) = Mor(L_{qfh}(X), L_{qfh}(Y)) = Y(K')$, where K' is a maximal purely inseparable extension of the field K.*

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Proof: It follows immediately from the previous lemma and the remark that the extension L of K is purely inseparable if and only if the diagonal $\Delta : Spec(L) \longrightarrow Spec(L) \times_{Spec(K)} Spec(L)$ induces the isomorphism of $Spec(L)$ with $(Spec(L) \times_{Spec(K)} Spec(L))_{red}$.

Definition 3.13 *Let $f : X \longrightarrow Y$ be a morphism of the finite type. It is called radicial (resp. universal homeomorphism) if for any scheme $Z \longrightarrow Y$ over Y the morphism $X \times_Y Z \longrightarrow Z$ induces an immersion (resp. homeomorphism) of the underlying topological spaces.*

Proposition 3.14 *Let $f : X \longrightarrow Y$ be a morphism of the finite type. Then*

1. *$L(f)$ (resp. $L_{qfh}(f)$) is a monomorphism if and only is f is radicial.*
2. *$L(f)$ is an epimorphism if and only if f is a universal topological epimorphism.*
3. *$L(f)$ (resp. $L_{qfh}(f)$) is an isomorphism if and only if f is a universal homeomorphism.*

Proof: It follows from lemma ?? that we may suppose X, Y to be the reduced schemes.

1. The "if" part follows from the trivial observation that any radicial morphism with the reduced source is a monomorphism in the category of schemes and the left exactness of the functor L. The "only if" part follows from the proposition ?? and the criterion that the morphism is radicial if and only if it induces the monomorphisms on the sets of the geometrical points (see [?]).
2. It is easy to show that the morphism of schemes $f : X \longrightarrow Y$ induces an epimorphism on the corresponding representable sheaves if and only if there exists a covering $U \longrightarrow Y$ which can be factorized through f. It implies the result we need, since if there exists a universal topological epimorphism which can be factorized through f then f is a universal topological epimorphism itself.

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3. Suppose that f is a universal homeomorphism. Then it is a qfh-covering, and, therefore, $L_{qfh}(f)$ is a surjection. From the other hand any universal homeomorphism is a radicial morphism which implies, according to (1), that $L(f)$ is a monomorphism as well. Suppose now, that $L(f)$ is an isomorphism, then by (1) and (2), f is a radicial universal topological epimorphism, which obviously implies that f is a universal homeomorphism.

Let X, Y be the schemes and $f \in Mor(L(X), L(Y))$. I say that the h-covering $\{p_i : U_i \longrightarrow X\}$ *realises* f if there exist the morphisms $f_i : U_i \longrightarrow Y$ such that $L(f_i) = f \circ L(p_i)$. It follows from lemma ?? that in that case one has $f_i \circ pr_1^{red} = f_j \circ pr_2^{red}$, where pr_i^{red} are the restrictions of the projections $U_i \times_X U_j \longrightarrow U_i$ and $U_i \times_X U_j \longrightarrow U_j$ to the maximal reduced subscheme $(U_i \times_X U_j)_{red}$ of the scheme $U_i \times_X U_j$. Note, that if $\{V_{ij} \longrightarrow U_i \longrightarrow X\}$ is a refinitement of the h-covering $\{p_i : U_i \longrightarrow X\}$ and $\{V_{ij} \longrightarrow X$ realises f, then the coverings $\{V_{ij} \longrightarrow U_i\}$ realize $f \circ L(p_i)$.

Lemma 3.15 *Let X be a reduced scheme and $f \in Mor(L(X), L(Y))$ be such a morphism, that it can be realised on the open covering of X, then there exists a morphism $\tilde{f} \in Mor_S(X, Y)$ such, that $L(\tilde{f}) = f$.*

Proof: It is sufficient to notice, that since for the open subschemes U, V of the reduced scheme X one has $U \times_X V = U \cap V = (U \times_X V)_{red}$ and the open coverings are the effective epimorphisms in the category of schemes, one can descended the morphism which realises f to the morphism $\tilde{f} : X \longrightarrow Y$.

Lemma 3.16 *Let $p : X' \longrightarrow X$ be an h-covering such that $p_*(\mathcal{O}_{X'}) = \mathcal{O}_X$. Then for any $f \in Mor(L(X), L(Y))$ which can be realised by p there exists a morphism $\tilde{f} \in Mor_S(X, Y)$ such that $L(\tilde{f}) = f$.*

Proof: Denote by $f' : X' \longrightarrow Y$ the morphism such that $L(f') = f \circ L(p)$. Then there obviously exists a *continuous map* $\tilde{f}$ from the underlying topological space of X to the underlying topological space of Y such that $f' = \tilde{f} \circ p$ as the continuous map. But since $p_*(\mathcal{O}_{X'}) = \mathcal{O}_X$, the morphism of sheaves $\mathcal{O}_Y \longrightarrow f'_*(\mathcal{O}_{X'})$ defines the morphism of sheaves $\mathcal{O}_Y \longrightarrow f'_*(\mathcal{O}_{X'}) =$

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$\tilde{f}_*(p_*(\mathcal{O}_{X'})) = \tilde{f}_*(\mathcal{O}_X)$, and, therefore $\tilde{f}$ corresponds to the morphism of schemes, which obviously satisfies the condition we need.

Proposition 3.17 *Let $\tilde{f} \in Mor(L(X), L(Y))$ be a morphism of the representable h-sheaves, then there exists a finite surjective morphism $p : X' \longrightarrow X$ such that $\tilde{f} \circ L(p) = L(f')$ for a morphism $f' : X' \longrightarrow Y$.*

Proof: Let $\{p_i : U_i \longrightarrow X\}$ be an h-covering which realises $\tilde{f}$ and $f_i : U_i \longrightarrow Y$ be the corresponding morphisms. According to the theorem ?? we may suppose that our covering has a normal form. Let $U_i \xrightarrow{in_i} \bar{U} \xrightarrow{s} X_Z \xrightarrow{r} X$ be a normal decomposition of p_i. Consider a morphism $r \circ s$. Since it is proper there exists a Stein decomposition of it of the form $r \circ s = r' \circ s'$ where s' is a proper surjective morphism $\bar{U} \longrightarrow X'$ such that $s'_*(\mathcal{O}_{\bar{U}}) = \mathcal{O}_{X'}$ and r' is a finite surjective morphism. Our proposition follows now from the lemmas ??, ??.

Theorem 3.18 *The category $L(Sch/S)$ (resp. $L_{qhf}(Sch/S)$) of the representable h-sheaves (resp. qhf-sheaves) is a localization of the category Sch/S of the schemes over S with respect to the universal homeomorphisms.*

Proof: It follows from the proposition ??(3) that it is sufficient to show, that for any schemes X, Y and a morphism $f \in Mor(L(X), L(Y))$, there exists a universal homeomorphism $X_0 \longrightarrow X$ which realises f. Let $p : X' \longrightarrow X$ be a finite morphism such that there exists a morphism $f' : X' \longrightarrow Y$ satisfying $L(f') = f \circ L(p)$. Let us define a sheaf $\mathcal{R}$ of the finite $\mathcal{O}_X$-algebras over X as follows. Let U be an open subset of X. Then $\mathcal{R}(U)$ is a subalgebra in $\mathcal{O}_{X'}(f'^{-1}(U))$ which consists of those functions $g \in \mathcal{O}_{X'}(f'^{-1}(U))$ that there exists an element $\tilde{g} \in Mor(L(X), L(\mathbf{A}^1))$, such that $L(g) = \tilde{g} \circ L(p)$. One can easily see, that the morphism $Spec(\mathcal{R}) \longrightarrow X$ is a finite surjective morphism, which realize f. To finish the prove it is sufficient to show that it is a universal homeomorphism. It is almost obviously.

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Proposition 3.19 *Let S be a scheme of the characteristic zero, then there is a functor $R : L(Sch/S) \longrightarrow Sch/S$ left adjoint to L. For the scheme X the scheme $R(L(X))$ is a semi-normalization of X (see [?]).*

In particular for any seminormal scheme X and any scheme Y one has

$$Mor(L(X), L(Y)) = Mor_S(X, Y).$$

Proof: Let X be a normal scheme of the characteristic zero. Suppose, that $p : Y \longrightarrow X$ is the universal homeomorphism. Then considering the base change along the immersion of the generic point of X we conclude, that p is birational. From the other hand p is universally closed and quasi-finite, which implies that it is finite. Then p is an isomorphism by [?, 4.4.9].

Therefore, for any scheme X of the characteristic zero and any $f \in Mor(L(X), L(Y))$ there exists a finite morphism $p : X' \longrightarrow X$ which realize f such that p is a universal homeomorphism and the normalization of X can be factorized through p. It follows easy from the results of [?], that the seminormalization of X is exactly the universal morphism satisfying this property, which finish the proof.

The situation in the positive characteristic is a bit more complicated. Roughly speaking, there exists the analog of the functor R in that case. Namely $R(L(X))$ for the integral scheme X should be a seminormalization of X in the maximal purely inseparable extension of its field of functions. The problem is that this scheme is not in general a Netherien scheme, and, therefore we can not construct R in the category of the Netherien schemes.

The following proposition provides us all the information we shall really need about the sets $Mor(L(X), L(Y))$ in the general case.

Proposition 3.20 *Let X be a normal connected scheme. Then for any scheme Y one has:*

$$Mor(L(X), L(Y)) = \varinjlim_{L} Mor_S(X_L, Y)$$

where the limit is defined over the category of the purely inseparable extensions of the field of functions of X and X_L denotes a normalization of X in the extension L.

Proof: It follows almost automaticly from the above results.

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Proposition 3.21 *Let Y be a scheme of the finite type over S, then the natural morphism*

$$Mor_S(X,Y) \longrightarrow Mor(L(X),L(Y))$$

is a bijection for any X if and only if Y is etale over S.

Proof: It follows from the valuative criterion for etale morphisms (see [?, ex.17])

3.3 Sheaves $\mathbf{Z}(X)$ in h-topology.

Let X be a scheme over S. I denote by **Z(X)** (resp. $\mathbf{Z}_{qfh}(X)$) an h-sheaf (resp. a qfh-sheaf) of the abelien groups freely generated by the sheaf of sets $L(X)$. I shall also use the notations **N(X)**,$\mathbf{N}_{qfh}(X)$ for the corresponding freely generated sheaves of the abelien semi-groups.

For an abelien semi-group A I denote by A^+ an abelien group associated with A in an obvious way.

Proposition 3.22 *For any schemes X, Y over S and a section $a \in \mathbf{Z}_{qfh}(X)(Y)$ there exists a finite surjective morphism $\bar{p} : \bar{U} \longrightarrow Y$ such, that $\bar{p}^*(a) = \sum a_j^+ - \sum a_k^-$, where a_j^+, a_k^- correspond to the morphisms $\bar{U} \longrightarrow X$.*

Proof: According to the construction of the associated sheaf and a theorem ?? above for any $a \in \mathbf{Z}_{qfh}(X)(Y)$ there exists a covering $\{U_i \xrightarrow{in_i} \bar{U} \xrightarrow{\bar{p}} Y$ of the normal form such, that $in_i^*\bar{p}^*(a) = \sum a_{ij}^+ - \sum a_{ik}^-$ where $a_{ij}^+, a_{ik}^- \in Mor_S(U_i, X)$ are such elements that $a_{ij}^+ \neq a_{ik}^-$ for any j,k.

For a pair i_1, i_2 of indexes we have

$$pr_1^*(\sum a_{i_1j}^+ - \sum a_{i_1k}^-) = pr_2^*(\sum a_{i_2j}^+ - \sum a_{i_2k}^-)$$

in $\mathbf{Z}_{qfh}(X)(U_{i_1} \times_U U_{i_2})$. Since $U_{i_1} \times_U U_{i_2} = U_{i_1} \cap U_{i_2}$ is reduced it implies, that this equality also holds on the level of the formal sums of morphisms $U_i \longrightarrow X$. It means, that with respect to some order on the set of indexes one has

$$pr_1^* a_{i_1j}^+ = pr_2^* a_{i_2j}^+$$

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$$pr_1^* a_{i_1k}^- = pr_2^* a_{i_2k}^-$$

There exists then a family of morphisms $a_{ij}^+, a_{ik}^- \in Mor_S(U_{i_1} \cup U_{i_2}, X)$ such, that

$$a_{ij|U_{i_1}}^+ = a_{i_1j}^+$$

$$a_{ij|U_{i_2}}^+ = a_{i_2j}^+$$

$$a_{ik|U_{i_1}}^- = a_{i_1k}^-$$

$$a_{ik|U_{i_2}}^- = a_{i_2k}^-$$

The statment of our proposition follows now by the induction on the number of open subschemes U_i of $\bar{U}$.

Proposition 3.23 *Let X be a normal connected scheme and $p : Y \longrightarrow X$ be a normalization of X in a Galois extension of its field of functions. Then for any qfh-sheaf F of the abelien semi-groups an image of $p^* : F(X) \longrightarrow F(Y)$ coincide with a subsemigroup $F(Y)^G$ of the Galois invariant elements of $F(Y)$*

Proof: Obviously $Im(p^*)$ lies in $F(Y)^G$. Let $a \in F(Y)^G$ be a Galois invariant element of $F(Y)$. Consider a scheme $Y \times_X Y$. It is a union of the irreducible components

$$Y \times_X Y = \cup_{g \in G} Y_g$$

and Y_g can be identified with Y in such a way that a restriction of the first projection $Y \times_X Y \longrightarrow Y$ becomes an identity and a restriction of the second one is an isomorphism $Y \longrightarrow Y$ induced by $g \in G$. To prove, that $a \in Im(p^*)$ it is sufficient to show, that $pr_1^*(a) = pr_2^*(a)$ in $F(Y \times_X Y)$. Since a decomposition of $Y \times_X Y$ in the union of its irreducible components is a qfh-covering it is sufficient that for any $g \in G$ one has $pr_1^*(a)_{|Y_g} = pr_2^*(a)_{|Y_g}$, which means exactly, that a is a Galois invariant.

Theorem 3.24 *Let X be a scheme and Y be a normal scheme, then one has:*

$$\mathbf{Z}_{qfh}(X)(Y) = \mathbf{N}_{qfh}(X)(Y)^+.$$

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Proof: Denote by F a presheaf of the form

$$Y \longrightarrow \mathbf{N}_{qfh}(X)(Y)^+$$

Obviously a qfh-sheaf associated with F is isomorphic to $\mathbf{Z}_{qfh}(X)$. In particular, there is a natural map

$$\phi : \mathbf{N}_{qfh}(X)(Y)^+ \longrightarrow \mathbf{Z}_{qfh}(X)(Y)$$

and we have to prove that it is a bijection for normal Y. Let us show first that ϕ is an injection. It follows immediately from the construction of the associated sheaf, that it is sufficient to show that for any qfh-covering $\{U_i \longrightarrow Y\}$ a natural map

$$F(Y) \longrightarrow \oplus_i F(U_i)$$

is injective. Note, that according to the axioms of sheaf a map

$$\mathbf{N}_{qfh}(X)(Y) \longrightarrow \oplus_i \mathbf{N}_{qfh}(X)(U_i)$$

is injective. Our statment now follows easily from the following lemma:

Lemma 3.25 *Let $a, b \in \mathbf{N}_{qfh}(X)(Y)$ be a pair of sections such that $a + x = b + x$ for some $x \in \mathbf{N}_{qfh}(X)(Y)$, then $a = b$.*

Proof: There exists a covering $\{p_i : U_i \longrightarrow Y\}$ of Y such, that

$$p_i^*(x) = \sum x_{ij}$$

$$p_i^*(a) = \sum a_{ik}$$

$$p_i^*(b) = \sum x_{il}$$

where $x_{ij}, a_{ik}, b_{il} \in L(X)(U_i)$.

Since $\mathbf{N}_{qfh}(X)$ is a sheaf it is sufficient to show that $p_i^*(a) = p_i^*(b)$. An equality

$$\sum a_{ik} + \sum x_{ij} = \sum b_{il} + \sum x_{ij}$$

in $\mathbf{N}_{qfh}(X)(U_i)$ means that there is a covering $\{q_{im} : V_{im} \longrightarrow U_i\}$ such that for any m one has

$$\sum q_{im}^* a_{ik} + \sum q_{im}^* x_{ij} = \sum q_{im}^* b_{il} + \sum q_{im}^* x_{ij}$$

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as a formal sum of sections of the sheaf $L(X)$ over V_{im}. It implies, that

$$\sum q_{im}^* a_{ik} = \sum q_{im}^* b_{il}$$

and, therefore, $p_i^*(a) = p_i^*(b)$.

Note that an injectivity of the map ϕ which we just proved is still valid in the context of a general site.

Let me prove now that in our case a map ϕ is also surjective. By the proposition ?? for any $a \in \mathbf{Z}_{qfh}(X)(Y)$ there exists a finite surjective morphism $\bar{p} : \bar{U} \longrightarrow Y$ such that $\bar{p}^*(a) = \sum a_j^+ - \sum a_k^-$. We may suppose, that Y is connected. Since Y is normal we may suppose, that $\bar{p}$ admit a decomposition of the form

$$\bar{U} \xrightarrow{\bar{p}_0} \bar{U}_0 \xrightarrow{\bar{p}_1} Y$$

where $\bar{p}_1$ is a normalization of Y in a purely inseparable extension of its field of functions and $\bar{p}_0$ is a normalization of $\bar{U}_0$ is a Galois extension of its field of functions with a Galois group G. For any $g \in G$ we have

$$\sum a_j^+ - \sum a_k^- = \sum g a_j^+ - \sum g a_k^-$$

in $\mathbf{Z}_{qfh}(X)(\bar{U})$ and, since $\bar{U}$ is reduced the same equality holds on the level of the formal sums of morphisms $\bar{U} \longrightarrow X$. It implies, that

$$\sum a_j^+ = \sum g a_j^+$$

$$\sum a_k^- = \sum g a_k^-$$

in $\mathbf{N}_{qfh}(X)(\bar{U})$ and, according to the proposition ??, that there exist a pair ak, a^- of elements of $\mathbf{N}_{qfh}(X)(\bar{U}_0)$ such, that $\bar{p}_0^*(a^+) = \sum a_j^+$ and $\bar{p}_0^*(a^-) = \sum a_k^-$. By lemma ?? we have $\mathbf{N}_{qfh}(X)(\bar{U}_0) = \mathbf{N}_{qfh}(X)(Y)$which finish the proof.

Theorem 3.26 *Let X be an affine scheme over S, then one has*

$$\mathbf{Z}(\mathrm{X}) = \mathbf{Z}_{\mathrm{qfh}}(\mathrm{X}).$$

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Proof: It is sufficient to show, that for an affine scheme X a qfh-sheaf $\mathbf{Z}_{qfh}(X)$ is an h-sheaf. By the theorem ?? we have to prove only, that $\mathbf{Z}_{qfh}(X)$ satisfies the axioms of sheaf for h- coverings of normal form Let Y be a scheme over S and $\{U_i \longrightarrow \bar{U} \longrightarrow Y_Z \longrightarrow Y\}$ be its covering of the normal form. Let us show first that a map $u : \mathbf{Z}_{qfh}(X)(Y) \longrightarrow \oplus_i \mathbf{Z}_{qfh}(X)(U_i)$ is injective. Let $a \in \mathbf{Z}_{qfh}(X)(Y)$ be an element such, that $u(a) = 0$. By the proposition ?? there exists a finite surjective morphism $\bar{q} : \bar{V} \longrightarrow Y$ such, that $\bar{q}^*(a) = \sum a_j^+ - \sum a_k^-$ where a_j^+, a_k^- correspond to the morphisms $\bar{V} \longrightarrow X$. denote a morphism $Y_Z \longrightarrow Y$ by s. Since $\{U_i \longrightarrow \bar{U} \longrightarrow Y_Z\}$ is a qfh-covering an equality $u(a) = 0$ implies, that $s^*(a) = 0$ as an element of $\mathbf{Z}_{qfh}(X)(Y_Z)$. Consider a fiber product $Y_Z \times_Y \bar{V}$ and let pr_1, pr_2 be the projection to Y_Z and $\bar{V}$ respectively. We have $pr_2^* \bar{q}^*(a) = pr_1^* s^*(a) = 0$ in $Y_Z \times_Y \bar{V}$. It implies that with respect to a suitable order on the index set we have $a_j^+ \circ pr_2 = a_j^- \circ pr_2$ as morphisms $(Y_Z \times_Y \bar{V})_{red} \longrightarrow X$. Therefore, since $(Y_Z \times_Y \bar{V})_{red} \longrightarrow \bar{V}_{red}$ is an epimorphism in the category of schemes we have $a_j^+ = a_j^-$ on $\bar{V}_{red}$ which implies, that $a = 0$.

Now let $a_i \in \mathbf{Z}_{qfh}(X)(U_i)$ be a family of sections such, that $pr_1^*(a_i) = pr_2^*(a_j)$ in $\mathbf{Z}_{qfh}(X)(U_i \times_Y U_j)$ where $pr_1 : U_i \times_Y U_j \longrightarrow U_i, pr_2 : U_i \times_Y U_j \longrightarrow U_j$ are projections. We have to prove, that there exists an element $a \in \mathbf{Z}_{qfh}(X)(Y)$ such that its restrictions on U_i is equal to a_i. passing to a refinitement we may suppose, that $a_i = \sum a_{ij}^+ - \sum a_{ik}^-$ where a_{ij}^+, a_{ik}^- correspond to the morphisms $U_i \longrightarrow X$. as in the proof of the proposition ?? we see, that there exist a family of morphisms $a_j^+, a_k^- \in Mor_S(\bar{U}, X)$ such, that

$$a_{j|U_i}^+ = a_{ij}^+$$

$$a_{k|U_i}^- = a_{ik}^-.$$

Consider a Stein decomposition $\bar{U} \xrightarrow{f} W \xrightarrow{g} Y$ of the morphism $\bar{U} \longrightarrow Y_Z \longrightarrow Y$. Since $f_* \mathcal{O}_{\bar{U}} = \mathcal{O}_W$ and X is affine over S one has $Mor_S(\bar{U}, X) = Mor_S(W, X)$. Therefore our family a_j^+, a_k^- can be descended to the family b_j^+, b_k^- of morphisms $W \longrightarrow X$. Since

$$pr_1^*(\sum a_j^+ - \sum a_k^-) = pr_2^*(\sum a_j^+ - \sum a_k^-)$$

in $\mathbf{Z}_{qfh}(\bar{U} \times_Y \bar{U})$ and a natural morphism $\bar{U} \times_Y \bar{U} \longrightarrow W \times_Y W$ is an h-covering it follows from the injectivity result proved above, that the same

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equality holds in $\mathbf{Z}_{qfh}(W \times_Y W)$. Since $W \longrightarrow Y$ is a finite surjective morphism and, therefore a qfh-covering, it implies that there exists an element $a \in \mathbf{Z}_{qfh}(X)(Y)$ such, that $g^*(a) = \sum b_j^+ - \sum b_k^-$ in $\mathbf{Z}_{qfh}(X)(W)$, which finish the proof.

Proposition 3.27 *Let X be a scheme over S such that there exist symmetric powers S^nX of X over S. Then the sheaves $\mathbf{N(X)}$,$\mathbf{N}_{qfh}(X)$ are representable by the (ind-) scheme $\coprod_{n\geq 0} S^nX$.*

Proof: It is obviously sufficient to prove our proposition in the case of qfh-topology. Note first that a sheaf representable by $\coprod_{n\geq 0} S^nX$ is a sheaf of abelien semi-group. To prove the proposition it is sufficient to show that it satisfies a universal property of $\mathbf{N}_{qfh}(X)$. It means that for any qfh-sheaf of abelien semi-groups G and any section $a \in G(X)$ of G over X there should exists the unique element $f \in Hom(L(\coprod_{n\geq 0} S^nX), G) = G(\coprod_{n\geq 0} S^nX)$ which is a homomorphism of sheaves of abelien groups and which restriction on $X = S^1X$ is equal to a.

Consider a natural morphism $q : X^n \longrightarrow S^nX$ and let $y_n = \sum pr_i^*(a) \in G(X^n)$. It is obviously invariant with respect to the action of the symmetric group S_n. Exactly in the same way as in the proof of the proposition ?? one can show that there exists an element $f_n \in G(S^nX)$ such, that $q^*(f_n) = y_n$.

It is very easy to see that an element $1 \oplus y_1 \oplus ... \oplus y_n \in \oplus_{n\geq 0} G(S^nX) = G(\coprod_{n\geq 0} S^nX)$ satisfies our conditions and is unique.

Proposition 3.28 *Let Z be a closed subscheme of scheme X. Denote by PN_Z a projectivization of the normal cone to Z in X. Then a kernel of the map*

$$\mathbf{Z}_{qfh}(p_Z) : \mathbf{Z}_{qfh}(X_Z) \longrightarrow \mathbf{Z}_{qfh}(X)$$

is naturely isomorphic to a kernel of the map

$$\mathbf{Z}_{qfh}(p) : \mathbf{Z}_{qfh}(PN_Z) \longrightarrow \mathbf{Z}_{qfh}(Z).$$

Proof: There is a morphism $ker(\mathbf{Z}_{qfh}(p)) \longrightarrow ker(\mathbf{Z}_{qfh}(p_Z))$ which is obviously an injection. We have to prove that it is also a surjection. Let us

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consider first the case when X is our base scheme. It is sufficient to show that for normal connected scheme Y over X and an element $f \in \mathbf{Z}_{qfh}(X_Z)(Y)$ such that $\mathbf{Z}_{qfh}(p_Z)(f) = 0$ there exists a lifting of f to an element of $\mathbf{Z}_{qfh}(PN_Z)(Y)$. Let me suppose for the simplicity that we are working in the case of characteristic zero. Then by the proposition ?? and theorem ?? we have

$$\mathbf{Z}_{qfh}(X_Z)(Y) = Hom_X(Y, \coprod_{n\geq 0} S^n_X X_Z)^+.$$

Let $f = \sum f_i^+ - \sum f_j^-$ be a decomposition of f into the sum of indecomposable morphisms. It is easy to see that the condition $\mathbf{Z}_{qfh}(p_Z)(f) = 0$ implies, that with respect to some identification of the index sets one has

$$\mathbf{Z}_{qfh}(p_Z)(f_i^+) = \mathbf{Z}_{qfh}(p_Z)(f_i^-).$$

It implies that we may restrict our considerations to the case $f = f^+ - f^-$.

Consider a scheme $S^n_X X_Z$. One can easily see, that it is a union of closed subschemes one of which is isomorphic to X_Z and another one to $S^n_Z PN_Z$. Since Y is irreducible and a morphism is birational our condition on f implies, that $f^+(Y), f^-(Y) \subset S^n_Z PN_Z$ for some n, i.e. f can be lifted to an element of $\mathbf{Z}_{qfh}(PN_Z)$.

A statment of our proposition for a general base scheme follows now from the propositions ??,??.

Theorem 3.29 *Let X be a normal connected scheme and $f : Y \longrightarrow X$ be a finite surjective morphism of the separable degree d. Then there is defined a morphism*

$$tr(f) : \mathbf{Z}_{qfh}(X) \longrightarrow \mathbf{Z}_{qfh}(Y)$$

such, that $\mathbf{Z}_{qfh}(f) tr(f) = d Id_{\mathbf{Z}_{qfh}(X)}$

Proof: We may suppose, that Y is a normalization of X in a finite extension of the field of functions on X. There is a decomposition $f = f_0 f_1$, where f_1 corresponds to the separable and f_0 to purely inseparable extensions respectively. By lemma ?? and proposition ?? a morphism f_0 induced an isomorphism on the qfh-sheaves. It implies that we may restrict our considerations to the case $f_0 = Id$. Let $\tilde{f} : \tilde{Y} \longrightarrow X$ be a normalization of X in the

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Galois extension which contains $K(Y)$. A morphism $\mathbf{Z}_{qfh}(X) \longrightarrow \mathbf{Z}_{qfh}(Y)$ is nothing but a section of the sheaf $\mathbf{Z}_{qfh}(Y)$ over X. Let $G = Gal(\tilde{Y}/X)$ be a Galois group of $\tilde{Y}$ over X and $H = Gal(\tilde{Y}/Y)$ be its subgroup which corresponds to Y. By the proposition ?? it is sufficient to construct such a section to find a section a of $\mathbf{Z}_{qfh}(Y)$ over $\mathbf{Z}_{qfh}(\tilde{Y})$ which is G-invariant. We set

$$a = \sum_{x \in G/H} x(g),$$

where $g : \tilde{Y} \longrightarrow Y$ is a natural morphism. It is easy to see, that the corresponding section of $\mathbf{Z}_{qfh}(Y)$ over X satisfy all the properties we need.

3.4 Comparison results and cohomological dimension.

Theorem 3.30 *Let X be a normal scheme and F be a qfh-sheaf of $\mathbf{Q}$-vector spaces, then one has*

$$H^i_{qfh}(X, F) = H^i_{et}(X, F).$$

Proof: It follows from the Leray spectral sequence, that to prove our theorem it is sufficient to show that for any normal strictly local ring R one has

$$H^i_{qfh}(Spec(R), F) = 0$$

for $i > 0$. It is easy to see that we actually need only to consider a case $i = 1$. Let $a \in H^1_{qfh}(Spec(R), F)$ be a cohomological class. then there exists a qfh-covering $\{U_i \longrightarrow Spec(R)\}$ and a Čech cocycle $\{a_ij\} \in \oplus F(U_i \times_{Spec(R)} U_j)$ which represents a. To prove, that $a = 0$ it is sufficient to show, that a natural surjection of sheaves of $\mathbf{Q}$-vector spaces $\mathbf{Z}(\coprod \mathrm{U_i}) \otimes \mathbf{Q} \longrightarrow \mathbf{Z}(\mathrm{Spec(R)}) \otimes \mathbf{Q}$ has a splitting. It follows from the theorem ?? above and the next lemma.

Lemma 3.31 *Let X be a spectrum of the strictly local ring and $\{U_i \longrightarrow X\}$ be a qfh-covering. Then there exists the finite surjective morphism $V \longrightarrow X$ and the splitting $V \longrightarrow \coprod U_i$.*

Proof: It is well known that we may assume that $U_1 \longrightarrow X$ is finite and the image of all other U_i does not contain the closed point of X. We should

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prove that if our family of morphisms was a qfh-covering then $U_1 \longrightarrow X$ is surjective. Let us do it by the induction on the dimension of X. The result is obvious for $dimX < 2$. Let $x \in X$ be a point of the dimension one. Considering the base change over the embedding $Z_x \longrightarrow X$, where Z_x is the closure of x we conclude that x lies in the image of U_1. Therefore the image of U_1 contains all the points of the dimension 1 of X, but it is closed and therefore coincide with all X.

Our theorem is proven.

Theorem 3.32 *Let X be a scheme over S and F be a locally constant in the etale topology sheaf on Sch/S, then F is an h-sheaf and one has*

$$H^i_h(X, F) = H^i_{et}(X, F).$$

Proof: I Shall prove only the case of h-topology. The proof for qfh-topology is similar. By the Leray spectral sequence it is sufficient to show that for a strictly local ring R one has $H^i_h(Spec(R), F) = 0$ for $i > 0$. We need first a following technical result.

Lemma 3.33 *Let $p : T_1 \longrightarrow T_2$ be a morphism of sites and F be a sheaf of abelien groups on T_1 such, that the sheaves on T_2 associated with the presheaves $U \longrightarrow \check{H}^i_{T_1}(p^{-1}(U), F)$ are isomorphic to zero for $i > 0$, then $R^i p_*(F) = 0$ for $i > 0$.*

Proof: It is well known that $R^i p_*(F)$ is a sheaf associated with a presheaf $U \longrightarrow H^i_{T_1}(p^{-1}(U), F)$. I am going to prove the result we need by the induction on i. for $i = 1$ one has $\check{H}^1_{T_1}(p^{-1}(U), F) = H^1_{T_1}(p^{-1}(U), F)$ and, therefore, $R^1 p_*(F) = 0$ by our assumption. Suppose, that everything is proved for $i \leq n$. there is a spectral sequence

$$\check{H}^p_{T_1}(p^{-1}(U), \underline{H}^q(F)) \Rightarrow H^{p+q}(p^{-1}(U), F)$$

where $\underline{H}^q(F)$ denote a presheaf of cohomological groups. Since, this spectral sequence is natural with respect to U it is sufficient to show, that for any $a \in \check{H}^p_{T_1}(p^{-1}(U), \underline{H}^q(F))$ such that $p + q = n + 1$ there exists a T_2-covering $\{U_i \longrightarrow U\}$ such that the restrictions of a to each of the objects U_i are zero.

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It is obviously.

I define p-topology (resp. f-topology) as a Grothendieck topology associated to the pretopology of in which coverings are proper (resp. finite) surjective families of morphisms. There is a following sequence of morphisms of sites:

$$h-topology \longrightarrow p-topology \longrightarrow f-topology.$$

I am going to show now that for any scheme X over S and a locally constant in the etale topology sheaf F one has natural isomorphisms:

$$H^i_h(X,F) = H^i_p(X,F) = H^i_f(X,F).$$

To prove the first isomorphism it is sufficient to show by lemma ??, that for any $a \in \check{H}^i_h(X,F), i > 0$ there exists a proper surjective morphism $p : Y \longrightarrow X$ such, that $p^*(a) = 0$. let $\{U_i \longrightarrow \bar{U} \longrightarrow X_Z \longrightarrow X\}$ be an h-covering of the normal form which realises a. then a restriction of a to $\bar{U}$ can be realised by the covering $\{U_i \longrightarrow \bar{U}\}$. since $\bar{U}$ is a disjoint union of the irreducible schemes one has $\check{H}^i_{Zar}(\bar{U},F) = H^i_{Zar}(\bar{U},F) = 0$, which implies that this restriction is equal to zero.

Let us now prove the second isomorphism. Let $a \in \check{H}^i_p(X,F), i > 0$ be a class which can be realised by the covering $p : Y \longrightarrow X$, where p is a proper morphism. Considering a Stein decomposition $Y \xrightarrow{p_0} Y_0 \xrightarrow{p_1} X$ of p, where p_1 is a finite morphism and $(p_0)_*(\mathcal{O}_Y) = \mathcal{O}_{Y_0}$ one can see, that it is sufficient to show (using, once more lemma ??), that if p_1 is an isomorphism, i.e. $p_*(\mathcal{O}_Y) = \mathcal{O}_X$, then $a = 0$. We may suppose, that X is connected and F is a constant sheaf associated with a group A. Then for any Z one has $F(Z) = \oplus_{i=1}^N A$, where N is a number of the connected components of Z. Now, our statment is trivial, since if X is connected and $p : Y \longrightarrow X$ is such a morphism, that $p_*(\mathcal{O}_Y) = \mathcal{O}_X$, then $Y \times_X ... \times_X Y$ are connected schemes.

Now we are ready to finish the prove of the theorem.

let X be a spectrum of the strictly local ring. We have to prove, that $H^i_h(X,F) = 0$ for $i > 0$. according to the discussion above it is sufficient to prove, that $H^i_f(X,F) = 0$ for $i > 0$. Since any scheme which is finite over X is a disjoint union of spectrums of the strictly local rings it follows from the spectral sequence, which connected usual and Čech cohomologies, that it is sufficient to prove, that $\check{H}^i_f(X,F) = 0$ for $i > 0$. It follows immediately from the next lemma.

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Lemma 3.34 *Let $U \longrightarrow X$ be a finite morphism over X such, that U is connected, then the schemes $U \times_X ... \times_X U$ are connected.*

Proof: See [?, p.33].

Our theorem is proven.

Theorem 3.35 *Let X be a scheme of the (absolute) dimension N, then for any h-sheaf of abelien groups and any $i > n$ one has:*

$$H^i_h(X, F) \otimes \mathbf{Q} = 0.$$

Proof: We need first the following lemma

Lemma 3.36 *Let X be a scheme of the absolute dimension N, then for any etale sheaf of the abelien groups F and any $i > N$ one has:*

$$H^i_{et}(X, F) \otimes \mathbf{Q} = 0.$$

Proof: (cf. [?, p.221]) we use an induction on N. For $N = 0$ our statment is obvious. Let $x_1, ..., x_k$ be a set of general points of X and $in_j : Spec(K_j) \longrightarrow X$ be the corresponding inclusions. Consider a natural morphism of sheaves on the small etale site over X:

$$F \longrightarrow \oplus_{j=1}^k (in_j)_*(in_j)^*(F).$$

Then kernel and cokernel of this morphism have a support in the codimension at least one and, therefore, there cohomologies vanish in the dimension greater then $N - 1$ by the inductive assumption. to finish the proof it is sufficient now to notice, that $H^i(X, (in_j)_*(in_j)^*(F)) \otimes \mathbf{Q} = 0$ by the Leray spectral sequence of the inclusions in_j.

It follows from this lemma and a theorem ?? above, that for the normal scheme X of the dimension N and any $i > 1$ one has $H^i_{qfh}(X, F) \otimes \mathbf{Q} = 0$.

According to the spectral sequence which is connected Čech and usual cohomologies it is sufficient to prove our theorem to show, that $\check{H}^i_h(X, F) \otimes \mathbf{Q} = 0$ for $i > N$. let $a \in \check{H}^i_h(X, F) \otimes \mathbf{Q}$ be a cohomological class and

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$\{U_i \longrightarrow \bar{U} \longrightarrow X_Z \longrightarrow X\}$ be an h-covering of the normal form which realize a. Passing to the refinitement we may suppose, that X_Z is normal. Since $\{U_i \longrightarrow \bar{U} \longrightarrow X_Z\}$ is a qfh-covering a restriction of a to X_Z is equal to zero. It follows from the propositions ?? and ??, that there are defined two long exact sequences:

$$... \longrightarrow Ext^{i-1}(G,F) \longrightarrow H^i_h(X,F) \longrightarrow H^i_h(X_Z,F) \longrightarrow Ext^i(G,F) \longrightarrow ...$$

and

$$... \longrightarrow Ext^{i-1}(G,F) \longrightarrow H^i_h(Z,F) \longrightarrow H^i_h(PN_Z,F) \longrightarrow Ext^i(G,F) \longrightarrow ...$$

and, since $dim(PN_Z) < dim(X)$ our result follows by the induction on $dim(X)$.

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4 Categories $DM(S)$.

4.1 Definition and general properties

Consider a category Sch/S of schemes over a base S as a site with either h- or qfh-topology. It has a structure of site with interval if we set $I^+ = \mathbf{A}^1_S$. Morphisms (μ, i_0, i_1) from the definition of site with interval are multiplication on $\mathbf{A}^1_S$ and points 0, 1 respectively.

Definition 4.1 *Category $DM(S)$ is a homological category $H(Sch/S, \mathbf{A}^1_S)$. When it is necessary I specify a topology writing $DM_h(S)$ or $DM_{qfh}(S)$ for h- and qfh-topologies respectively.*

In this section I summarize elementary properties of this categories and corresponding functors $M : Sch/S \longrightarrow DM(S)$. All of them follows easily from the properties of h- and qfh-topologies which were proven in previous chapter and general properties of the construction of the homological category of site with interval.

I usually identify sheaves of abelien groups on Sch/S with corresponding objects of $DM(S)$ and schemes with corresponding representable sheaves of sets. I also use a sign $=$ for canonical isomorphisms.

Proposition 4.2 *Categories $DM(S)$ are tensor triangle categories. For any morphism $f : S_1 \longrightarrow S_2$ there is defined tensor triangle functor $f^* : DM(S_2) \longrightarrow DM(S_1)$. If X is a scheme over S_2 then one has $f^*(M(X)) = M(X \times_{S_2} S_1)$.*

Proof: It follows immediately from the general properties of our construction and proposition ??

Proposition 4.3 *For any schemes X, Y over S one has*

$$M(X \coprod Y) = M(X) \oplus M(Y)$$

$$M(X \times_S Y) = M(X) \otimes M(Y)$$

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Proof: It follows from the corresponding properties of the functor **Z** (see proposition ??).

Proposition 4.4 *Let $X = U \cup V$ be an open or close covering of X. Then there is defined a natural exact triangle in $DM(S)$ of the form*

$$M(U \cap V) \longrightarrow M(U) \oplus M(V) \longrightarrow M(X) \longrightarrow M(U \cap V)[1]$$

Proof: It follows from the proposition ??.

Proposition 4.5 *Let $p : Y \longrightarrow X$ be a locally trivial in Zariski topology fibration which fibers are affine spaces, then morphism $M(p) : M(Y) \longrightarrow M(X)$ is an isomorphism.*

Proof: It follows from the proposition ?? and an obvious remark that for any scheme X a natural morphism $M(pr_1) : X \times \mathbf{A}^n \longrightarrow M(X)$ is an isomorphism.

Proposition 4.6 *Let X be a scheme over S and F be a sheaf on Sch/S, then one has a natural map*

$$H^i(X, F) \longrightarrow DM(M(X), F[i])$$

Proof: It follows immediately from our construction.

Proposition 4.7 *Let F be a locally free in etale topology sheaf of torsion prime to characteristic of S , then for any scheme X one has a natural isomorphism*

$$DM(M(X), F[n]) = H^n_{et}(X, F)$$

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Proof: It follows from the proposition ??, theorem ?? and a homotopy invariance of etale cohomologies with locally constant coefficients (see [?, p.240]).

Proposition 4.8 *Let S be a scheme of characteristic $p > 0$, then category $DM(S)$ is* $\mathbf{Z}[1/p]$*-linear.*

Proof: One can easily see, that $\mathbf{Z}[1/p]$-linearity is equivalent to acyclicity of the sheaf $\mathbf{Z}/p\mathbf{Z}$. Consider Artin-Shrier exact sequence

$$0 \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathbf{G_a} \xrightarrow{F-1} \mathbf{G_a} \longrightarrow 0$$

where $\mathbf{G}_a$ is a sheaf of abelien groups representable by $\mathbf{A}^1$ and F is a geometrical Frobenious. Since $\mathbf{G}_a$ is obviously acyclic it implies the result we need.

Proposition 4.9 *Let $f : Y \longrightarrow X$ be a finite surjective morphism of the normal connected schemes of the separable degree d. Then there is defined a morphism $tr(f) : M(X) \longrightarrow M(Y)$ such, that $M(f)tr(f) = dId_{M(X)}$.*

Proof: It follows from the theorem ??

At the end of this section I want to prove the result which describes the objects of the simplest type in $DM(SpecZ)$.

Let X be a simplicial set. We shall call X regular if it can be represented as the simplicial subset in Δ^n for some n. In this case one can define the schematic realization of X as follows. We realize the simplex Δ^n as a scheme

$$\Delta^n_{\mathbf{Z}} = SpecZ[x_0, ..., x_n]/(\sum x_i - 1).$$

To each face of the Δ^n corresponds the closed subscheme of $\Delta^n_{\mathbf{Z}}$, namely its intersection with the corresponding affine subspace. We define the realization $X_{\mathbf{Z}}$ of X as a union of this subschemes corresponding to the faces of $X \subset \Delta^n$. For general simplicial sets one can define another construction of the schematic realization. Consider the functor from the simplicial category Δ to the category $Sets(SpecZ)$ of the sheaves of sets in h-topology

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over *specZ*, which turns an object $[n]$ of Δ to the sheaf represented by $\Delta^n_{\mathbf{Z}}$. Using the standard construction (Kan extension) one can associate with that functor the pair of the adjoint functors between the category $\Delta^{op}Sets$ of the simplicial sets and $Sets(SpecZ)$. One of them, which we denote $|?|_{\mathbf{Z}}$, provides the schematic realisations for general simplicial sets.

Lemma 4.10 *If X is a regular simplicial set then the h-sheaf represented by $X_{\mathbf{Z}}$ is naturally isomorphic to $|X|_{\mathbf{Z}}$.*

Proof: It follows immediately from the fact that the covering of the scheme by its irreducible components is the h-covering.

For the simplicial set X denote by $C_*(X)$ its simplicial chain complex (i.e. the normalization of the free simplicial abelien group generated by X).

Proposition 4.11 *Let X be a finite dimensional simplicial set, then $M(|X|_{\mathbf{Z}})$ is isomorphic in the $DM(SpecZ)$ to the $C_*(X)$, which is considered as the complex of constant sheaves on $Sch/Spec\mathbf{Z}$.*

Proof: Using the fact that X is an inductive limit of the finite simplicial sets and that all our constructions are obviously compatible with the passing to the inductive limits, we can restrict our considerations to the case of the finite X. Let me suppose for the simplicity, that X is even a regular simplicial set. Consider the covering of X by its maximal faces. It is an h-covering and we obtain from it the long exact sequence which defines the resolvent of $\mathbf{Z}(X)$ which terms are the sums of the sheaves of the form $\mathbf{Z}(\mathbf{A}^k)$. Replacing all the sheaves $\mathbf{Z}(\mathbf{A}^k)$ by the constant sheaves $\mathbf{Z}$ we obtain the complex of sheaves which represents the same object in the category DM. It is sufficient to show now that considering as the complex of the abelien groups it is quasi-isomorphic to the $C_*(X)$. It follows immediately from the remark that it is naturally dual to the usual Cech complex of the constant sheaf $\mathbf{Z}$ on X with respect to its covering by the maximal faces.

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4.2 Motives of smooth proper schemes of the relative dimension ≤ 1

Let $p : X \longrightarrow S$ be a morphism of schemes. We denote by $Pic_{X/S}$ a qfh-sheaf of abelein groups associated with the flat sheaf $R^1p_*(\mathbf{G}_m)$ on Sch/S.

Theorem 4.12 *Let $p : X \longrightarrow S$ be a smooth projective morphism of the relative dimension one over a normal scheme S such that $p_*(\mathcal{O}_X) \cong \mathcal{O}_S$. Then there is defined a canonical exact triangle in $DM_{qfh}(S)$ of the form*

$$\mathbf{G}_m[1] \longrightarrow M(X) \longrightarrow Pic_{X/S} \longrightarrow \mathbf{G}_m[2].$$

Proof: Let me recall first several definitions (see [?, §3]). Effective Cartier divisor Z on X is called a relative Cartier divisor on X over S if the corresponding closed subscheme of X is flat over S. One can easily show (loc. cit.) that this condition is stable under base changes and multiplication of Cartier divisors. Denote abelien semi-group of the relative Cartier divisors of X over S by $Div^{\geq 0}_{X/S}$ and by $\mathbf{Div}_{X/S}$ a qfh-sheaf of abelien groups associated with the presheaf of the form

$$T/S \longrightarrow (Div^{\geq 0}_{X\times_S T/T})^+$$

Lemma 4.13 *One has a natural isomorphism of qfh-sheaves of abelien groups:*

$$\mathbf{Z}_{qfh}(X) \cong \mathbf{Div}_{X/S}.$$

Proof: Consider a morphism $\phi : \mathbf{Z}_{qfh}(X) \longrightarrow \mathbf{Div}_{X/S}$, which corresponds by the adjointness property of functor $\mathbf{Z}_{qfh}$ to the section of $\mathbf{Div}_{X/S}$ over X defined by the diagonal $X \longrightarrow X \times_S X$ (it is a relative Cartier divisor, since p is smooth and S is normal, see [?, 21.14.3]). We want to prove, that ϕ is an isomorphism.

Let us prove first, that ϕ is a monomorphism. Consider aq section $u \in \mathbf{Z}_{qfh}(X)(T)$ of $\mathbf{Z}_{qfh}(X)$ over T/S. There exists a qfh-covering $U \longrightarrow T$ such, that the corresponding section u' over U has a form

$$u' = \sum u_i^+ - \sum u_j^-,$$

where u_i^+, u_j^- are morphisms $U \longrightarrow U \times_S X$. It is sufficient to show, that $u' = 0$ if $\phi(u') = 0$. Since normalization is a qfh-covering, we may suppose

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that U is a disjoint union of normal connected schemes. It is obviously sufficient to consider a case of the connected U. Let Z_i^+ (resp. Z_j^-) be a Cartier divisor of $U \times_S X$ over U which corresponds to u_i^+ (resp. u_j^-). Since these divisors are irreducible an equality $\sum Z_i^+ - \sum Z_j^- = 0$ in $\mathbf{Div}_{X/S}(U)$ implies, that with respect to some identification of the index sets one has $Z_i^+ = Z_i^-$, which means, that $u' = 0$.

Let us show now that ϕ is epimorphism. Let $Z \in \mathbf{Div}_{X/S}(T)$ be a section of the sheaf $\mathbf{Div}_{X/S}$ over T. One can easily see, that there exists a qfh-covering $U \longrightarrow X$ such that the corresponding section Z' over U is of the form $Z' = \sum n_i^+ Z_i^+ - \sum n_j^- Z_j^-$, where Z_i^+, Z_i^- are the divisors, which correspond to U-points of $X \times_S U$. Therefore Z' is contained in the image of ϕ, which implies, that the same is true for Z because of the injectivity of ϕ.

For a scheme X denote by $\mathcal{K}^*(X)$ a multiplicative group of the invertible elements of the total quotient ring of X (see [?, p.140]).

Let $\mathcal{M}^*_{X/S}$ be a qfh-sheaf of abelien groups associated with a presheaf of the form:

$$T/S \longrightarrow \begin{array}{l} \text{subgroup of } \mathcal{K}^*(X \times_S T) \text{ which consists of the el-} \\ \text{ements } f \text{ such that the divisor } D(f) \text{ of } f \text{ is a rel-} \\ \text{ative Cartier divisor of } X \times_S T \text{ over } T \end{array}$$

Lemma 4.14 *There is defined a following exact sequence of the qfh-sheaves of abelien groups over S*

$$0 \longrightarrow \mathbf{G}_m \longrightarrow \mathcal{M}_{X/S} \longrightarrow \mathbf{Z}_{qfh}(X) \longrightarrow Pic_{X/S} \longrightarrow 0.$$

Proof: Definition of the morphisms in this sequence as well as exactness in all terms except last one is trivial. A surjectivity of the morphism $\mathbf{Z}_{qfh}(X) \cong \mathbf{Div}_{X/S} \longrightarrow Pic_{X/S}$ is proved in the case of flat topology in [?], which implies our result because of the exactness of the functor of associated sheaf.

To prove our theorem it is sufficient now to show, that $\mathcal{M}_{X/S}$ is acyclic. We are going first to construct a presheaf F of abelien semi-groups on Sch/S such, that qfh-sheaf of abelien groups associated with F is isomorphic to $\mathcal{M}_{X/S}$ and F is (ind-) representable.

Denote by $\mathcal{L}$ a very ample sheaf on X. We may suppose, that $R^i p_*(L) = 0$ for $i > 0$ and that there exists a section $s_0 : \mathcal{O}_X \longrightarrow calL$ of $calL$ over X

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such, that the corresponding divisor $Z \longrightarrow X$ is a relative Cartier divisor of X over S.

For any T over S define a set $F(T)$ as a direct limit with respect to s_0 of the sets $F_n(T)$ of nonzero sections of $pr_1^*(\mathcal{L}^{\otimes n})$ over $X \times_S T$ which correspond to the relative Cartier divisors on $X \times_S T$ over T. One can easily see that $F : T \longrightarrow F(T)$ is a presheaf of abelien semigroups on Sch/S with respect to the abelien semi-group structures on $F(T)$ given by the tensor multiplication of sections. As a presheaf of sets F is a direct limit of the presheaves $F_n : T \longrightarrow F_n(T)$.

Lemma 4.15 *A qfh-sheaf of abelien groups associated with F is isomorphic to $\mathcal{M}_{X/S}$.*

Proof: Let us first construct a morphism of presheaves $\phi : F \longrightarrow \mathcal{M}_{X/S}$. For any section f of F_n over T we define $\phi(f)$ as an element of $\mathcal{K}^*(X \times_S T)$ of the form $f/pr_1^*(s_0)^{\otimes n}$. One can easily see, that ϕ is well defined. Denote by F^+ a qfh-sheaf of abelien groups associated with F and by ϕ^+ a morphism

$$F^+ \longrightarrow \mathcal{M}_{X/S}$$

which corresponds to ϕ. we are going to show, that ϕ^+ is an isomorphism. It is obviously a monomorphism. To prove, that it is isomorphism it is sufficient, therefore to show, that for any section $g \in \mathcal{M}_{X\S}(T)$ of $\mathcal{M}_{X/S}$ over T there exists a qfh-covering $U \longrightarrow T$ such, that the corresponding section g' of $\mathcal{M}_{X/S}$ over U is of the form $g' = g^+ - g^-$ where $g^+, g^- \in F_n(U)$ for some $n \geq 0$. It is a direct corollary of the ampleness of $\mathcal{L}$.

Lemma 4.16 *For any $n \geq 0$ a direct image $p_*(\mathcal{L}^{\otimes n})$ is a locally free sheaf on S and F_n is represented by the complement to the zero section of the corresponding vector bundle on S.*

Proof: It follows from the results of [?] that $p_*(\mathcal{L}^{\otimes n})$ is locally free sheaf. let E_n be a corresponding vector bundle over S and E_n^* be a complement to the zero section of E_n. Denote by $L(E_n^*)$ qfh-sheaf of sets representable by E_n^*. There is defined an obvious morphism of presheaves $F_n \longrightarrow L(E_n^*)$, which is injective by trivial reasons. Let us show, that the corresponding morphism of associated sheaves is an isomorphism. Let g be a section of $L(E_n^*)$ over

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T. we may suppose, that T is reduced and g corresponds to the morphism $T \longrightarrow E_n^*$, i.e to the section of $pr_1^*(\mathcal{L}^{\otimes n})$ over $X \times_S T$ such, that the corresponding divisor does not contain fibers of the projection $X \times_S T \longrightarrow T$. to prove our lemma it is sufficient to show, that such a section is contained in $F_n(T)$, i.e., that the corresponding Cartier divisor is a relative Cartier divisor on $X \times_X T$ over T. Since T is reduced it follows from [?, 1.2.5 p.9].

Lemma 4.17 *There exists an open covering $S = \cup U_i$ of S such, that for any $n \geq 0$ restrictions of $p_*(\mathcal{L}^{\otimes n})$ to U_i are free sheaves.*

Proof: Let Z be a closed subscheme which corresponds to the divisor of the section s_0 of $\mathcal{L}$. Since a composition $p \circ i : Z \longrightarrow S$ is a flat finite morphism a direct image $(p \circ i)_*(i^*\mathcal{L})$ is a locally free sheaf on S. Let $S = \cup U_i$ be an affine open covering of S such that restrictions of both $p_*(\mathcal{L})$ and $(p \circ i)_*(i^*\mathcal{L})$ on U_i are free. I claim that this covering satisfies our condition. We prove it by the induction by n. Suppose, that we already proved, that $p_*(\mathcal{L}^{\otimes n})$ is free over U_i. Consider an exact sequence of coherent sheaves on X:

$$0 \longrightarrow \mathcal{L}^{\otimes n} \longrightarrow \mathcal{L}^{\otimes (n+1)} \longrightarrow \mathcal{L}/calO_X \otimes \mathcal{L}^{\otimes n} \longrightarrow 0$$

Since $R^ip_*(\mathcal{L}) = 0$ for $i > 0$ there is defined an exact sequence of coherent sheaves on S

$$0 \longrightarrow p_*\mathcal{L}^{\otimes n} \longrightarrow p_*\mathcal{L}^{\otimes (n+1)} \longrightarrow p_*(\mathcal{L}/calO_X \otimes \mathcal{L}^{\otimes n}) \longrightarrow 0.$$

One can easily see, that

$$p_*(\mathcal{L}/calO_X \otimes \mathcal{L}^{\otimes n}) \cong (p \circ i)_*(\mathcal{L})^{\otimes n}.$$

Therefore this sheaf is free over U_i and our exact sequence splits over U_i since U_i is affine. Lemma is proven.

To prove that $\mathcal{M}_{X/S}$ is acyclic it is sufficient to prove that its restrictions on U_i are acyclic. It follows from our lemmas, that over U_i sheaf $\mathcal{M}_{X/S}$ is isomorphic to the sheaf of abelien groups associated with a presheaf of abelien semigroups, which is representable as a presheaf of sets by the (ind-)scheme

$$\mathbf{A}^\infty - \{0\} = \lim_{n \to \infty} (\mathbf{A}^n - \{(0, ..., 0)\}).$$

We need now a following technical lemma.

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Lemma 4.18 *Let F be a presheaf of abelien semi-groups on Sch/S such, that for any regular simplicial set K and any T over S a natural morphism*

$$F(\Delta^n_{\mathbf{Z}} \times_{Spec(\mathbf{Z})} T) \longrightarrow F(K_{\mathbf{Z}} \times_{Spec(\mathbf{Z})} T)$$

is surjective. Then a sheaf of abelien groups associated with F is acyclic.

Proof: Denote by F^+ a *presheaf* of abelien groups associated with F. It is sufficient to show, that F^+ is isomorphic to zero in the category $DM_0(S)$ which is defined in the same way as $DM(S)$ but, with respect to weakest topology on Sch/S. Consider a simplicial presheaf $S_*(F)$ on Sch/S which terms are internal Hom-objects in the category of presheaves of the form

$$S_n(F) = \mathbf{Hom}(\Delta^n_{\mathbf{Z}} \times_{Spec(\mathbf{Z})} T, F)$$

and face and degeneration maps defined in an obvious way. It follows immediately from our assumptions, that $S_*(F)$ is a presheaf of Kan simplicial abelien semi-groups with trivial homotopy groups. It implies, that $S_*(F^+)$ is a presheaf of simplicial abelien groups, which homotopy groups are also trivial. Consider a normalization $N(S_*(F^+))$ of $S_*(F^+)$. Since homotopy groups of $S_*(F^+)$ are trivial it is exact complex of presheaves of abelien groups on Sch/S. There are natural monomorphisms of presheaves $F^+ \longrightarrow S_n(F^+)$ which assign to a section f of F^+ over T a section $(pr_2 : \Delta^n_{\mathbf{Z}} \times_{Spec(\mathbf{Z})} T\ T)^*(f)$ of $S_n(F^+)$ over T. They define a monomorphism of the complexes of presheaves of the form:

$$\begin{array}{ccccccccccc}
\dots \xrightarrow{0} & F^+ & \xrightarrow{Id} & F^+ & \xrightarrow{0} & F^+ & \xrightarrow{Id} & F^+ & \rightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots \rightarrow & S_4(F^+) & \rightarrow & S_3(F^+) & \rightarrow & S_2(F^+) & \rightarrow & S_1(F^+) & \rightarrow & S_0(F^+)
\end{array}$$

One can easily se, that $S_0(F^+) \cong F^+$ and all the vertical arrows except the last one are isomorphisms in $DM_0(S)$. Consider a cokernel of this monomorphism. Since both complexes are exact it is also exact. We obtained therefore a resolvent of F which consists of acyclic objects, which implies by ?? that F is also acyclic.

To finish the proof of our theorem it is sufficient now to show, that a presheaf representable by $\mathbf{A}^\infty - \{0\}$ satisfies a condition of the previous lemma. It is obviously.

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Theorem is proven.

Corollary 4.19 *A morphism $\tilde{M}(\mathbf{P}^1_S) \longrightarrow \mathbf{G}_m[1]$ which corresponds to cohomological class in $H^1(\mathbf{P}^1, \mathbf{G}_m)$ representable by line bundle $\mathcal{O}(1)$ on $\mathbf{P}^1_S$ is an isomorphism in $DM_{qfh}(S)$*

Proof: Obviously.

Corollary 4.20 *Let $p : X \longrightarrow S$ be a smooth projective morphism of the relative dimension one such, that $p_*(\mathcal{O}_X) \cong \mathcal{O}_S$ and n be a number prime to characteristic of S, then object $M(X) \times \mathbf{Z}/\mathrm{n}\mathbf{Z}$ is representable in $DM(S)$ by a complex of locally constant (in etale topology) sheaves of finite groups over S.*

Proof: Obviously.

4.3 Tate motives.

All through this section I work with categories $DM(S)$ with respect to qfh-topology. All the results below obviously hold for h-topology as well.

Since the results of this section do not depend of the base scheme S I omit S in all the notations below where it is possible.

Definition 4.21 *Tate motive $\mathbf{Z}(1)$ is an object of the category DM which corresponds to the sheaf $\mathbf{G}_m$ shifted by minus one, i.e.*

$$\mathbf{Z}(1) = \mathbf{G}_\mathrm{m}[-1]$$

We denote by $\mathbf{Z}(\mathrm{n})$ n-tensor power of $\mathbf{Z}(1)$ and for any object X of DM by $X(n)$ a tensor product $X \times \mathbf{Z}(\mathrm{n})$.

Proposition 4.22 *For any n and k there exists a following exact triangle*

$$\mathbf{Z}(\mathrm{n}) \xrightarrow{\mathrm{k}} \mathbf{Z}(\mathrm{n}) \longrightarrow \mu_\mathrm{k}^{\otimes \mathrm{n}} \longrightarrow \mathbf{Z}(\mathrm{n})[1]$$

where $\mu_k^{\otimes n}$ denote an object of the category DM which corresponds to the n-th tensor power of the sheaf μ_k of k-th roots of unit.

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Proof: It is sufficient to show that one has a natural isomorphism $Z(n) \otimes \mathbf{Z}/k\mathbf{Z} \cong \mu_k^{\otimes n}$, which is equivalent by the definition of $\mathbf{Z}$(n) to the isomorphism $\mathbf{G}_m^{otimesn} \otimes \mathbf{Z}/k\mathbf{Z} \cong \mu_k^{\otimes n}[n]$ (note that tensor product on the left side is tensor product in the category DM which corresponds to the L-tensor product on the level of the derive category of sheaves).

Note first that μ_k is, by the definition, a kernel of the morphism of the sheaves $\mathbf{G}_m \longrightarrow \mathbf{G}_m$ which corresponds to the morphism of schemes $\mathbf{A}^1 - 0 \longrightarrow \mathbf{A}^1 - 0$ which takes z to z^k. In h-topology it is a surjection. (It is not true in the case of etale topology, say, where we should restrict our considerations to the case of schemes over $Spec\mathbf{Z}[1/k]$.) Therefore one has $\mathbf{G}_m \overset{L}{\otimes} \mathbf{Z}/k\mathbf{Z} \cong \mu_k[1]$. To finish the proof of the proposition one should show that $\mu_k^{\otimes n} \overset{L}{\otimes} \mathbf{G}_m \cong \mu_k^{\otimes(n+1)}[1]$, which is easy.

For any scheme X we define its motivic cohomologies to be the groups

$$H^p(X, \mathbf{Z}(\mathrm{q})) = \mathrm{DM(M(X)}, \mathbf{Z}(\mathrm{q})$$

When it is necessary I shall use the notations $H^p_{qfh}(X, \mathbf{Z}(\mathrm{q}))$ and $H^p_h(X, \mathbf{Z}(\mathrm{q}))$ for these groups defined with respect to qfh- and h-topology respectively.

There is defined an obvious multiplication of the form

$$H^p(X, \mathbf{Z}(\mathrm{q})) \otimes \mathrm{H^{p'}(X}, \mathbf{Z}(\mathrm{q'})) \longrightarrow \mathrm{H^{p+p'}(X}, \mathbf{Z}(\mathrm{q}+\mathrm{q'})$$

which satisfies all standard properties. In particular a direct sum $\oplus_{p,q} H^p(X, \mathbf{Z}(\mathrm{q}))$ has a natural structure of bigraded ring, which is commutative as a bigraded ring by the axioms of the tensor triangle categories (see Appendix A).

Proposition 4.23 *Let X be a scheme. For any q for any k prime to characteristic of X one has a long exact sequence*

$$\ldots \longrightarrow H^p(X, \mathbf{Z}(\mathrm{q})) \xrightarrow{\mathrm{k}} \mathrm{H^p(X}, \mathbf{Z}(\mathrm{q})) \longrightarrow \mathrm{H^p_{et}(X}, \mu_{\mathrm{k}}^{\otimes \mathrm{n}}) \longrightarrow \mathrm{H^(p+1)(X}, \mathbf{Z}(\mathrm{q})) \longrightarrow \ldots$$

Proof: It follows from the proposition ?? that the only thing we have to prove is that under our assumptions one has an isomorphism

$$DM(M(X), \mu_k^{\otimes n}[p]) \cong H^p_{et}(X, \mu_k^{\otimes n}).$$

It follows from the proposition ?? and a remark that $\mu_k^{\otimes n}$ is a locally free in etale topology sheaf over $Spec(\mathbf{Z}[1/\mathrm{k}])$.

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Proposition 4.24 *Let X be a scheme of the characteristic l then all the groups $H^p(X, \mathbf{Z}(q))$ for $q > 0$ are $\mathbf{Z}[1/l]$ modules.*

Proof: It is a particular case of ??.

Theorem 4.25 *A natural section of the sheaf $\mathbf{G}_m$ over $\mathbf{A}^1 - \{0\}$ defines isomorphism in DM*

$$\tilde{M}(\mathbf{A}^1 - \{0\}) \cong \mathbf{Z}(1)[1].$$

Proof: Consider a covering of $\mathbf{P}^1$ of the form

$$\mathbf{P}^1 = (\mathbf{P}^1 - \{0\}) \cup (\mathbf{P}^1 - \{\infty\})$$

It defines an exact sequence of the sheaves

$$0 \longrightarrow \tilde{\mathbf{Z}}_{qfh}(\mathbf{A}^1 - \{0\}) \longrightarrow \tilde{\mathbf{Z}}_{qfh}(\mathbf{A}^1) \oplus \tilde{\mathbf{Z}}_{qfh}(\mathbf{A}^1) \longrightarrow \tilde{\mathbf{Z}}_{qfh}(\mathbf{P}^1) \longrightarrow 0$$

Since $\tilde{\mathbf{Z}}_{qfh}(\mathbf{A}^1)$ is acyclic a morphism

$$\tilde{\mathbf{Z}}_{qfh}(\mathbf{P}^1) \longrightarrow \tilde{\mathbf{Z}}_{qfh}(\mathbf{A}^1 - \{0\})[1]$$

defined by this exact sequence is isomorphism in DM and our result follows from ??

Theorem 4.26 *Let X be a scheme and E be a vector bundle on X. Denote by $P(E) \longrightarrow X$ a projectivization of E. One has a natural isomorphism in DM*

$$M(P(E)) \cong \bigoplus_{i=0}^{dimE-1} M(X)(i)[2i].$$

Proof: We may suppose X to be our base scheme. Let $\mathcal{O}(-1)$ be a tautological line bundle on $P(E)$ and a be a morphism $M(P(E)) \longrightarrow \mathbf{Z}(1)[2]$ in the category $DM(X)$ which corresponds to the class of this bundle in $H^1(P(E), \mathbf{G}_m)$. Using a morphism $M(P(E)) \longrightarrow M(P(E)) \otimes M(P(E))$ induced by the diagonal we can define elements $a^i \in DM(M(P(E)), \mathbf{Z}(i)[2i])$ as tensor powers of $a = a^1$. I claim that a direct sum

$$\phi : \bigoplus_{i=0}^{dimE-1} a^i : M(P(E)) \longrightarrow \bigoplus_{i=0}^{dimE-1} \mathbf{Z}(i)[2i]$$

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is an isomorphism in $DM(X)$.

Consider a trivializing open covering $X = \cup U_i$ of X. Let me suppose for a simplicity of the notations that this covering consists only of two open subsets. By the proposition ?? we have an exact sequence of sheaves

$$0 \longrightarrow \mathbf{Z}(\mathrm{U} \cap \mathrm{V}) \longrightarrow \mathbf{Z}(\mathrm{U}) \oplus \mathbf{Z}(\mathrm{V}) \longrightarrow \mathbf{Z} = \mathbf{Z}(\mathrm{X}) \longrightarrow 0.$$

Since our construction of the map ϕ is natural with respect to the restrictions to the open subsets it is easy to see that the existence of this exact sequence let us restrict our considerations to the case of a trivial bundle E.

In other words we should consider a scheme $\mathbf{P}^n$ over S and prove, that a morphism in $DM(S)$ which is defined as

$$\phi = \bigoplus_{i=0}^{n} a_n^i$$

where a corresponds to the line bundle $\mathcal{O}(-1)$ is an isomorphism. We use an induction on n. For $n = 0$ our statement is trivial. Consider a covering of $\mathbf{P}^n$ of the form

$$\mathbf{P}^n = \mathbf{P}^n - \{0\} \cup \mathbf{A}^n$$

where $\{0\}$ is a point with coordinates $[1, 0, ..., 0]$. We have a following exact triangle in DM

$$M(\mathbf{A}^n - \{0\}) \longrightarrow M(\mathbf{P}^n - \{0\}) \oplus M(\mathbf{A}^n) \longrightarrow M(\mathbf{P}^n) \longrightarrow M(\mathbf{A}^n - \{0\})[1].$$

I am going to construct a morhpism from this exact triangle to the exact triangle of the form

$$\mathbf{Z}(\mathrm{n})[2\mathrm{n}-1] \oplus \mathbf{Z} \longrightarrow \oplus_{\mathrm{i}=0}^{\mathrm{n}-1}\mathbf{Z}(\mathrm{i})[2\mathrm{i}] \oplus \mathbf{Z} \longrightarrow \oplus_{\mathrm{i}=0}^{\mathrm{n}}\mathbf{Z}(\mathrm{i})[2\mathrm{i}] \longrightarrow \mathbf{Z}(\mathrm{n})[2\mathrm{n}] \oplus \mathbf{Z},$$

and to show that it is an isomorphism on the first two terms, which would imly that it is an isomorphism of exact triangles. Define a cohomological class $\psi \in H^{n-1}(\mathbf{A}^n - \{0\}, \mathbf{G}_m^{\otimes n})$ as follows. Consider a covering of the scheme $\mathbf{A}^n - \{0\}$ of the form

$$\mathbf{A}^n - \{0\} = \bigcup_{i=1}^{n} \mathbf{A}^n - H_i$$

where H_i is a hyperplane $x_i = 0$. A Čech cocycle in $Z^{n-1}(\mathbf{A}^n - \{0\}, \mathbf{G}_m^{\otimes n})$ with respect to this covering is nothing but a section of the sheaf $\mathbf{G}_m^{\otimes n}$ over

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$\cap_{i=1}^{n} \mathbf{A}^n - H_i$. So we set ψ to be a cohomological class which corresponds to a tautological section of the form

$$(x_1, ..., x_n) \longrightarrow x_1 \otimes ... \otimes x_n.$$

Define a morphism $f : M(\mathbf{A}^n - \{0\}) \longrightarrow \mathbf{Z}(\mathrm{n})[2\mathrm{n}-1] \oplus \mathbf{Z}$ as a direct sum of the morphism which corresponds to ψ and a structural morphism.

Lemma 4.27 *f is an isomorphism.*

Proof: Easy by the induction on n starting with a theorem ??

Let $p : \mathbf{P}^n - \{0\} \longrightarrow \mathbf{P}^{n-1}$ be a natural projection which fibers are affine lines. It is obviously an isomorphism in DM. Define now a morphism $g : M(\mathbf{P}^n - \{0\}) \oplus M(\mathbf{A}^n) \longrightarrow \oplus_{i=0}^{n-1} \mathbf{Z}(\mathrm{i})[2\mathrm{i}] \oplus \mathbf{Z}$ as a direct sum of the morphism

$$\oplus_{i=0}^{n-1} M(p) a_{n-1}^i$$

and a structural morphism of $\mathbf{A}^n$. Note, that g is an isomorphism according to our inductive assumption. I claim now that a family of morphisms $f, g, \psi, f[1]$ is indeed a morphism of the exact triangles. To prove it one has to verify a commutativity of three squares. It is almost tautology.

Theorem is proven.

4.4 Characteristic classes

In this section I construct for any scheme X a family of maps

$$c_j^i : K_i(X) \longrightarrow H^{2j-i}(X, \mathbf{Z}(\mathrm{i}))$$

from Quillen K-groups (of locally free sheaves) to our motivic cohomologies, which satisfies all the usual properties of the characteristic classes.

All through this section I am working with qfh-topology. To obtain characteristic classes which take value in the motivic cohomology groups defined with respect to h-topology one should just consider a composition of C_j^i with a natural map from one cohomologies to another.

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We shall construct firs our classes for K_0. Let X be a (which I suppose to be connected for a simplicity) scheme and E be a vector bundle on X. We use an induction on $dim(E)$. If $dim(E) = 1$ then it defines a class in $H^1(X, \mathbf{G}_m)$ and, therefore a class c in the group $H^2(X, \mathbf{Z}(1))$. We set $c_0^0(E) = 1, c_1^0(E) = c$ and all the higher classes are equal to zero.

Suppose now that $dim(E) = n$. Denote by $p : P(E) \longrightarrow X$ a projectivization of X considering as a scheme over X. By the theorem ?? above we have a decomposition:

$$M(P(E)) \cong \bigoplus_{i=0}^{dimE-1} M(X)(i)[2i].$$

Consider an inverse image $p^*(E)$ of E with respect to the projection p. It contains one dimensional subbundle $F \subset E$. By the induction there is defined an element

$$(\oplus c_i^0(F)) \otimes (\oplus c_i^0(E/F)) \in \oplus_j H^{2j}(P(E), \mathbf{Z}(\mathrm{j}))$$

where $\otimes$ here means a multiplication in the ring of motivic cohomologies of $P(E)$. A decomposition above defines a morphism $M(X) \longrightarrow M(P(E))$ and we set $c_j^0(E)$ to be the components of an inverse image of this class with respect to this morphism.

It is very easy to see that this construction defines indeed a family of maps from $K_0(X)$ to the corresponding motivic cohomologies, which is natural with respect to the morphisms of scheme (but not in general with respect to morphisms in DM).

If one consider a λ-ring $\mathcal{H}(X)$ associated with a graded ring $\oplus_j H^{2j}(P(E), \mathbf{Z}(\mathrm{j}))$, then one can define a Chern character

$$ch : K_0(X) \longrightarrow \mathcal{H}(X)$$

which will be a morphism of λ-rings, which means that our classes satisfy all the usual properties. The proof of this result is similar to its proof for usual characteristic classes.

To extend our construction to the higher K-groups it is sufficient to use a following remark. For any scheme X there is a natural homomorphism

$$K_i(X) \longrightarrow K_0(X \times \partial\Delta^{i+1}).$$

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From the other hand almost by the definition we have

$$H^{2j}(X \times \partial\Delta^{i+1}, \mathbf{Z}(j)) = H^{2j-i}(X, \mathbf{Z}(j)) \oplus H^{2j}(X, \mathbf{Z}(j)),$$

which leads immediately to the definition of the higher classes c_j^i.

In the composition with natural morphisms $H^{2j-i}(X, \mathbf{Z}(j)) \longrightarrow H_{et}^{2j-i}(X, \mu_n^{\otimes j})$ our classes give us a family of maps from K-theory to etale cohomologies which also satisfies all the standard properties and coincide with the usual characteristic classes for line bundles. It implies, by the well known uniqueness theorem that they coincide with usual classes everywhere.

4.5 Monoidal transformations

All through this section I am working with qfh-topology. In particular a notation $DM(S)$ is used for a category $DM_{fqh}(S)$.

Let us recall some notations. Fos a scheme X and its closed subscheme Z we denote by X_Z a blow up of X with a center in Z and by $p_Z : X_Z \longrightarrow X$ a corresponding projection.

By PN_Z we denote a projectivization of a normal cone to Z in X and by $p : PN_Z \longrightarrow Z$ a natural morphism, which is a restriction of p_Z. Let $O_X(Z)$ be a kernel of the corresponding morphism of qfh-sheaves

$$\mathbf{Z}_{qfh}(p) : \mathbf{Z}_{qfh}(PN_Z) \longrightarrow \mathbf{Z}_{qfh}(Z).$$

By the proposition ?? it is naturally isomorphic to the kernel of the morphism $\mathbf{Z}_{qfh}(p_Z)$.

Theorem 4.28 *Let $Z \subset X$ be a smooth pair over S, then a sequence of sheaves*

$$O_X(Z) \longrightarrow \mathbf{Z}_{qfh}(X_Z) \longrightarrow \mathbf{Z}_{qfh}(X)$$

defines an exact triangle in $DM(S)$ of the form

$$O_X(Z) \longrightarrow M(X_Z) \longrightarrow M(X) \longrightarrow O_X(Z)[1].$$

In other words a cokernel of the morphism $\mathbf{Z}_{qfh}(p_Z)$ represents zero object in $DM(S)$.

Proof: Let us prove first the following lemma.

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Lemma 4.29 *Let $X \cup U_i$ be an open covering of X and $X_Z = \cup V_i$ be a corresponding covering of X_Z. Consider long exact sequences of sheaves which are defined by this coverings and a natural morphism between them*

$$\begin{array}{ccccccccccc} 0 & \to & \mathbf{Z}_{qfh}(\cap V_i) & \to & \ldots & \to & \oplus \mathbf{Z}_{qfh}(V_i) & \to & \mathbf{Z}_{qfh}(X_Z) & \to & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z}_{qfh}(\cap U_i) & \to & \ldots & \to & \oplus \mathbf{Z}_{qfh}(U_i) & \to & \mathbf{Z}_{qfh}(X) & \to & 0 \end{array}$$

Then a complex which is a cokernel of this morphism is exact.

Proof: An exactness of the cokernel of this morphism is equivalent to the exactness of the kernel of this morphism. By the proposition **??** this kernel is isomorphic to the kernel of the morphism of complexes

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z}_{qfh}(\cap V_i \cap PN_Z) & \to \ldots \to & \oplus \mathbf{Z}_{qfh}(V_i \cap PN_Z) & \to & \mathbf{Z}_{qfh}(PN_Z) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z}_{qfh}(\cap U_i \cap Z) & \to \ldots \to & \oplus \mathbf{Z}_{qfh}(U_i \cap Z) & \to & \mathbf{Z}_{qfh}(Z) & \to & 0 \end{array}$$

This two complexes are obviously exact, since they correspond to the covering of PN_Z and Z respectively which are induced by $\{U_i\}$. From the other hand in our case normal cone to Z is a vector bundle and, therefore, a morphism $PN_Z \longrightarrow Z$ is flat. In particular it admits a splitting over some qfh-covering which implies that the vertical arrows in the diagram above are surjections. It is well known, that a kernel of a surjection of exact complexes is exact which proves our lemma.

It follows from this lemma, that it is sufficient to prove our proposition locally. More precisely, it is sufficient to construct an open covering $X = \cup U_i$ of X such that all the cokernels of the maps $\mathbf{Z}_{qfh}(p_{Z \cap U_i})$ represent zero object in $DM(S)$.

Since $Z \subset X$ is a smooth pair there exists a covering $X = \cup U_i$ such that for any i there is an etale morphism $f_i : U_i \longrightarrow \mathbf{A}^N$ such that $Z \cap U_i = f^{-1}(\mathbf{A}^k)$, where $N = dim_S X$ and $k = dim_S Z$ (see [?, 2.4.9]). Let U be one of these open subschemes. We are going to prove that $coker(\mathbf{Z}_{qfh}(p_{Z \cap U}))$ represents zero object in $DM(S)$. Denote $U \cap Z$ by Y. Consider a diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z}_{qfh}(U - Y) & \to & \mathbf{Z}_{qfh}(U_Y) & \to & \mathbf{Z}_{qfh}(U_Y)/\mathbf{Z}_{qfh}(U - Y) & \to & 0 \\ & & \| & & \downarrow a & & \downarrow b & & \\ 0 & \to & \mathbf{Z}_{qfh}(U - Y) & \to & \mathbf{Z}_{qfh}(U) & \to & \mathbf{Z}_{qfh}(U)/\mathbf{Z}_{qfh}(U - Y) & \to & 0 \end{array}$$

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It is easy to see that a natural morphism $coker(a) \longrightarrow coker(b)$ is an isomorphism. It is sufficient, therefore, to prove, that $coker(b)$ represents zero in $DM(S)$. We shall need a following lemma.

Lemma 4.30 *Let $Z \longrightarrow X$ be a closed embedding and $f : U \longrightarrow X$ be an etale surjective morphism such that $U \times_X Z \longrightarrow Z$ is an isomorphism. Then one has a natural isomorphism of sheaves* $\mathbf{Z}(\mathrm{U})/\mathbf{Z}(\mathrm{U}-\mathrm{f}^{-1}(\mathrm{Z})) = \mathbf{Z}(\mathrm{X})/\mathbf{Z}(\mathrm{X}-\mathrm{Z})$.

Proof: Consider the diagram of sheaves:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}(\mathrm{U}-\mathrm{f}^{-1}(\mathrm{Z})) & \xrightarrow{i} & \mathbf{Z}(\mathrm{U}) & \longrightarrow & \mathbf{Z}(\mathrm{U})/\mathbf{Z}(\mathrm{U}-\mathrm{f}^{-1}(\mathrm{Z})) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}(\mathrm{X}-\mathrm{Z}) & \longrightarrow & \mathbf{Z}(\mathrm{X}) & \longrightarrow & \mathbf{Z}(\mathrm{X})/\mathbf{Z}(\mathrm{X}-\mathrm{Z}) & \longrightarrow & 0 \end{array}$$

We are to prove that the right vertical arrow is an isomorphism. It is obviously epimorphism, so, it is sufficient to prove that $ker\mathbf{Z}(\mathrm{f})$ lies in $Im(i)$. Note that we may prove it not for the morphisms of sheaves by for the morphisms of the presheaves of the form $\mathbf{Z}_0(X)(Y) = \oplus\mathbf{Z}(\mathrm{Hom}(\mathrm{Y_i},\mathrm{X}))$, where Y_i are the connected components of the scheme Y. Let Y be a connected scheme. Then $ker(\mathbf{Z}_0(f))$ is a group of expressions of the form $\sum_{i\in I} n_i g_i$, where $g_i : Y \longrightarrow U$ such that there exists a decomposition $I = \coprod I_k$ such that $f \circ g_i = f \circ g_j$ for $i, j \in I_k$ and $\sum_{i\in I_k} n_i = 0$ for any k. Therefore, we are to prove only that if $f \circ g = f \circ h$ for some $g, h : Y \longrightarrow U$ then either $g = h$ or g and h can be factorized through $U - f^{-1}(Z)$. Let g, h be such morphisms. Then there exists a morphism $g \times h : Y \longrightarrow U \times_X U$, which projections are g and h resp. To finish the proof one should notice that under the assumptions of our lemma there is the decomposition $U \times_X U = \Delta(U) \coprod U_0$ where Δ is the diagonal embedding and the projections $pr_1, pr_2 : U_0 \longrightarrow U$ can be factorized through $U - f^{-1}(Z)$.

Let $W = \mathbf{A}^{N-k} \times (\mathbf{A}^k \cap f(Y))$. We may replace U by $f^{-1}(W)$ and suppose, that $f(U) \subset W$. Denote by V a product $\mathbf{A}^{N-k} \times Y$. There is an etale morphism of the form

$$Id_{\mathbf{A}^{N-k}} \times f_{|_Y} : V \longrightarrow W.$$

Consider a fiber product $V \times_W U$ and let $U' = (V \times_W U) - (pr_1^{-1}(Z) - \Delta(Z))$, where $\Delta(Z) \longrightarrow V \times_W U$ is a diagonal. One can easily see that both

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projections $pr_1 : U' \longrightarrow V$ and $pr_2 : U' \longrightarrow W$ satisfy the conditions of the lemma above.

Note now, that since our construction is based on the etale morphisms it is natural with respect to blow up. It implies that $coker(b)$ is isomorphic to a cokernel of th morphism

$$\mathbf{Z}_{qfh}(Y \times (\mathbf{A}^{N-k}_{\{0\}}/(\mathbf{A}^{N-k} - \{0\}))) \longrightarrow \mathbf{Z}_{qfh}(Y \times (\mathbf{A}^{N-k}/(\mathbf{A}^{N-k} - \{0\})).$$

We reduced our problem, therefore to the case of a blow up of the point on the affine space.

It is sufficient to show, therefore that a cokernel of the morphism $\mathbf{Z}_{qfh}(\mathbf{A}^n_{\{0\}}) \longrightarrow \mathbf{Z}_{qfh}(\mathbf{A}^n)$ represents zero in $DM(S)$, or, equivalentely, that a kernel of this morphism is isomorphic to its cone in $DM(S)$. It follows from the proposition ?? and a remark that $\mathbf{A}^n_{\{0\}}$ is a total space of the vector bundle $\mathcal{O}[-1]$ on $\mathbf{P}^{n-1}$ and, therefore, $M(\mathbf{A}^n_{\{0\}})$ is isomorphic to $M(\mathbf{P}^{n-1})$. Theorem is proven.

Theorem 4.31 *Let $Z \subset X$ is a smooth pair over S. Then one has a natural isomorphism in $DM(S)$:*

$$M(X_Z) = M(X) \oplus (\oplus_{i=1}^{codim Z-1} Z(i)[2i]).$$

Proof: By the theorem ?? we have an exact triangle

$$O_X(Z) \longrightarrow M(X_Z) \longrightarrow M(X) \longrightarrow O_X(Z)[1].$$

By the definition $O_X(Z)[1]$ is a cone of the natural morphism $M(PN_Z) \longrightarrow M(Z)$. Since $PN(Z)$ is a projectivization of a normal bundle to Z in X it follows from the theorem ??, that

$$O_X(Z) \cong \oplus_{i=1}^{codim Z-1} Z(i)[2i].$$

To prove our theorem it is sufficient to construct a splitting of the exact triangle above. Let $i_0 : X \longrightarrow X \times \mathbf{A}^1$ be an embedding of the form $i_0 = Id_X \times \{0\}$. Consider a diagram

$$\begin{array}{ccc} O_X(Z) & \longrightarrow & O_{X\times\mathbf{A}^1}(Z \times \{0\}) \\ \downarrow & & \downarrow \\ M(X_Z) & \xrightarrow{\tilde{i}_0} & M(X \times \mathbf{A}^1_{Z\times\{0\}}) \\ \downarrow & & \downarrow \\ M(X) & \xrightarrow{i_0} & M(X \times \mathbf{A}^1) \end{array} \qquad (1)$$

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There is a canonical splitting of the morphism $M(p_{Z\times\{0\}})$ by the morphism $M(X \times \mathbf{A}^1) \cong M(X) \longrightarrow M(X \times \mathbf{A}^1_{Z\times\{0\}})$ where the last map is induced by an obvious lifting of the embedding $Id_X \times \{1\} : X \longrightarrow X \times \mathbf{A}^1$. To define a splitting of the projection $M(X_Z) \longrightarrow M(X)$ (or, equivalently, of an embedding $O_X(Z) \longrightarrow M(X_Z)$) it is sufficient to define a splitting of the morphism $O_X(Z) \longrightarrow O_{X\times\mathbf{A}^1}(Z \times \{0\})$. It existence (and, moreover a canonical choice) follows from the theorem ??. Theorem is proven.

4.6 Gysin exact triangle.

The goal of this section is to prove the following theorem. Similarly to the previous section I denote by DM a category DM_{qfh}.

Theorem 4.32 *Let $Z \subset X$ be a smooth pair over S and $U = X - Z$. Then there is defined a natural exact triangle in $DM(S)$ of the form*

$$M(U) \longrightarrow M(X) \longrightarrow M(Z)(d)[2d] \longrightarrow M(U)[1]$$

where d is a codimension of Z. In other words we have a natural isomorphism $M(X/U) \cong M(Z)(d)[2d]$ in $DM(S)$.

Proof: Let us first construct a morphism $M(X/U) \longrightarrow M(Z)(d)[2d]$ in $DM(S)$. Consider once more a diagram ??. A morphism $Id \times 1 : X \longrightarrow X \times \mathbf{A}^1$ has a naturallifting to $X \times \mathbf{A}^1_{Z\times\{0\}}$, which in the composition with $M(p_Z) : M(X_Z) \longrightarrow M(X)$ defines a morphism $\tilde{i}_1 : M(X_Z) \longrightarrow M(X \times \mathbf{A}^1_{Z\times\{0\}})$. One obviously has

$$M(p_{Z\times\{0\}})\tilde{i}_1 = M(p_Z)\tilde{i}_0,$$

which implies that there exists a lifting of $\tilde{i}_0 - \tilde{i}_1$ to a morphism $M(X_Z) \longrightarrow O_{X\times\mathbf{A}^1}(Z \times \{0\})$. It follows from the theorem ??, that this lifting is well defined. It composition with a natural morphism

$$O_{X\times\mathbf{A}^1}(Z \times \{0\}) \longrightarrow O_{X\times\mathbf{A}^1}(Z \times \{0\})/O_X(Z)$$

can be descended to the morphism $M(X) \longrightarrow O_{X\times\mathbf{A}^1}(Z\times\{0\})/O_X(Z)$ which is also well defined by the theorem ??. We have by ??

$$O_X(Z) \cong \oplus_{i=1}^{d-1} M(Z)(i)[2i]$$

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$$O_{X\times\mathbf{A}^1}(Z\times\{0\})\cong\oplus_{i=1}dM(Z)(i)[2i]$$

and, therefore

$$O_{X\times\mathbf{A}^1}(Z\times\{0\})/O_X(Z)\cong M(Z)(d)[2d].$$

This construction provides us with a morphism $M(X)\longrightarrow M(Z)(d)[2d]$. Considering it more carefully one can easily see, that this morphism can, in fact, be factorized through $M(X/U)$. Denote this last morphism $M(X/U)\longrightarrow M(Z)(d)[2d]$ by $G_{(X,Z)}$. To finish the proof of our theorem it is sufficient to show that it is an isomorphism in DM.

Consider first a special case $X=\mathbf{P}^n$, $Z=\{x\}$ where x is an S-point of $\mathbf{P}^N$. In this special case our diagram 1 has a following form

$$\begin{array}{ccc}
\tilde{M}(\mathbf{P}^{n-1}) & \longrightarrow & \tilde{M}(\mathbf{P}^n)\\
\downarrow & & \downarrow\\
M(\mathbf{P}^n_{\{x\}}) & \stackrel{\tilde{i}_0,\tilde{i}_1}{\longrightarrow} & M((\mathbf{P}^n\times\mathbf{A}^1)_{\{x\}\times\{0\}})\\
\downarrow & & \downarrow\\
M(\mathbf{P}^n) & \stackrel{M(i_0),M(i_1)}{\longrightarrow} & M(\mathbf{P}^n\times\mathbf{A}^1)
\end{array}$$

By the theorem 4.31 we have

$$M(\mathbf{P}^n_{\{x\}})\cong(\oplus_{i=0}^n\mathbf{Z}(i)[2i])\oplus(\oplus_{j=1}^{n-1}\mathbf{Z}(j)[2j]) \quad (2)$$

$$M((\mathbf{P}^n\times\mathbf{A}^1)_{\{x\}\times\{0\}})\cong(\oplus_{i=0}^n\mathbf{Z}(i)[2i])\oplus(\oplus_{j=1}^n\mathbf{Z}(j)[2j]) \quad (3)$$

Let us describe these isomorphisms explicitly. Denote by $a,b\in H^1((\mathbf{P}^n\times\mathbf{A}^1)_{\{x\}\times\{0\}},\mathbf{G}_m)$ classes which correspond to the divisor $p^{-1}_{\{x\}\times\{0\}}(\mathbf{P}^{n-1}\times\mathbf{A}^1)$ and a special divisor respectively. It is easy to see that the isomorphism (3) has a form $\oplus_{i=0}^n a^i\oplus\oplus_{j=1}^n b^j$. Similarly, if we denote by $a_0,b_0\in H^1(\mathbf{P}^n_{\{x\}},\mathbf{G}_m)$ elements which correspond to $p^{-1}_{\{x\}}(\mathbf{P}^{n-1})$ and a special divisor respectively an isomorphism (2) can be written as $\oplus_{i=0}^n a_0^i\oplus\oplus_{j=1}^{n-1}b_0^j$.

One obviously has

$$\tilde{i}_0 a=\tilde{i}_1 a=a_0$$

$$\tilde{i}_1 b=0,\tilde{i}_0 b=b_0.$$

which implies that the morphism $h_{\mathbf{P}^n,\{x\}}$ has with respect to the isomorphisms above a form $h_{\mathbf{P}^n,\{x\}}=b_0^n$. To prove that it is an isomorphism it is

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sufficient to show, that $b_0^n = a_0^n$. since $a_0 b_0 = 0$ it is equivalent to the equality $(a_0 - b_0)^n = 0$.

Consider a projection $q : \mathbf{P}^n_{\{x\}} \longrightarrow \mathbf{P}^{n-1}$ which corresponds to the rational map $\mathbf{P}^n \longrightarrow \mathbf{P}^{n-1}$ which is defined as a projection from the point x to $\mathbf{P}^{n-1}$. Let $c \in H^1(\mathbf{P}^{n-1}, \mathbf{G}_m)$ be a class of the hyperplane. One can easily se, that

$$q^*(c) = a_0 - b_0$$

which implies our result, since c^n is obviously zero.

To prove our theorem in general case one should use exactly the same localization technique as in the proof of the theorem 4.28.

Theorem is proven.

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5 Categories DM over a field of characteristic zero.

5.1 One comparison result for the categories DM_{qfh} and DM_h over a field of characteristic zero.

Let k be a field of characteristic zero. Denote by $\phi : Sch/Spec(k)_h \longrightarrow Sch/Spec(k)_{qfh}$ a natural morphism of sites. The goal of this section is to prove the following theorem.

Theorem 5.1 *Let k be a field of characteristic zero and X be a smooth variety over k. Then for any object F of $DM_{qfh}(Speck(k))$ a natural morphism*

$$DM_{qfh}(M(X), F) \otimes \mathbf{Q} \longrightarrow DM_h(M(X), H(\phi)(F)) \otimes \mathbf{Q}$$

is bijective.

To prove this theorem we need one technical generalization of the theorem 4.28. More precisely we are going to define a class of successive blowups $(X_i, f_i : X_{i+1} \longrightarrow X_i)$ such, that cokernel of the composition $f_0 ... f_n$ represents zero object in DM_{qfh}. Note, that it is not true in general, that a cokernel of the composition of two morphisms $\mathbf{Z}_{qfh}(f), \mathbf{Z}_{qfh}(g)$ such, that $coker(\mathbf{Z}_{qfh}(f)), coker(\mathbf{Z}_{qfh}(g))$ represent zero in DM_{qfh} represents zero.

Let $(X_i, D_i)_{0 \leq i \leq n}$ be a sequence of smooth pairs such that X_{i+1} is a blow up of X_i with center in D_i. Denote by $f_i : X_{i+1} \longrightarrow X_i$ corresponding birational morphisms. Let W_i be a closed subspace in X_i which is the image of the exceptional divisor of the composition $f_i ... f_{n-1}$. We say, that $(X_i, D_i)_{0 \leq i \leq n}$ satisfyies a condition (*) if for any $i \leq n-1$ a morphism

$$f_i^{-1}(D_i) - f_i^{-1}(D_i) \cap W_{i+1} \longrightarrow D_i$$

is surjective.

Let $Z \subset X$ be a smooth subscheme in X_0. We say that Z is transversal to the sequence $(X_i, D_i)_{0 \leq i \leq n}$ if $Z \cap D_0$ is smooth and subvarieties $f_0^{-1}(Z \cap D_0), Z_{Z \cap D_0}, Z_{Z \cap D_0} \cap f_0^{-1}(Z \cap D_0)$ of X_1 are transversal to the sequence $(X_i, D_i)_{0 \leq i \leq n-1}$. We say that our sequence $(X_i, D_i)_{0 \leq i \leq n}$ satisfies a condition (**) if X_0 is transversal to it in this sense.

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Proposition 5.2 *Let $(X_i, D_i)_{0\leq i\leq n}$ be a sequence of blowups satisfying (*) and $Z \subset X_0$ is a smooth subscheme transversal to $(X_i, D_i)_{0\leq i\leq n}$. Then cokernel of the natural morphism*

$$\mathbf{Z}_{qfh}(Z \times_{X_0} X_n) \longrightarrow \mathbf{Z}_{qfh}(Z)$$

represents zero in DM_{qfh}.

Proof: We use an induction by n. Let $n = 1$. We have

$$X_1 \times_{X_0} Z = f_0^{-1}(Z) = Z_{D_0 \cap Z} \cup f_0^{-1}(Z \cap D_0).$$

Restriction of the morphism $p_{Z\cap D_0} : Z_{Z\cap D_0} \longrightarrow Z$ to $Z \cap D_0{}^{-1}(D_0)$ is flat and surjective over D_0 which implies that it induces a surjection of the corresponding qfh-sheaves. Therefore a cokernel of the morphism $\mathbf{Z}_{qfh}(X_1 \times X_0 Z) \longrightarrow \mathbf{Z}_{qfh}(Z)$ is naturally isomorphic to the cokernel of the morphism $\mathbf{Z}_{qfh}(Z_{Z\cap D_0}) \longrightarrow \mathbf{Z}_{qfh}(Z)$ which represents zero in DM_{qfh} by the theorem 4.28.

Suppose now, that $n > 1$. denote by g a composition $f_{n-1}...f_1$. Consider a following diagram:

$$\begin{array}{ccc} Z \times_{X_0} X_n & \longrightarrow & X_n \\ \downarrow g' & & \downarrow g \\ Z \times_{X_0} X_1 & \longrightarrow & X_1 \\ \downarrow f_0' & & \downarrow f_0 \\ Z & \longrightarrow & X_0 \end{array}$$

One can easily see, that there is a following exact sequence of sheaves of abelien groups

$$0 \to ker(\mathbf{Z}_{qfh}(g')) \to ker(\mathbf{Z}_{qfh}(f_0'g')) \to ker(\mathbf{Z}_{qfh}(f_0')) \xrightarrow{d} coker(\mathbf{Z}_{qfh}(g'))$$

$$\to coker(\mathbf{Z}_{qfh}(f_0'g')) \to coker(\mathbf{Z}_{qfh}(f_0')) \to 0$$

It is sufficient to prove, that $coker(\mathbf{Z}_{qfh}(g'))$ and $Im(d)$ represent zero objects in DM_{qfh}.

Since

$$X_1 \times_{X_0} Z = f_0^{-1}(Z) = Z_{D_0 \cap Z} \cup f_0^{-1}(Z \cap D_0)$$

is a qfh-covering it is sufficient (by the proposition 2.5) to show that $coker(\mathbf{Z}_{qfh}(g'))$ represents zero in DM_{qfh} to show that the following sheaves represent zero:

$$coker(\mathbf{Z}_{qfh}((Z_{D_0 \cap Z} \cap f_0^{-1}(Z \cap D_0)) \times_{X_1} X_n) \longrightarrow \mathbf{Z}_{qfh}(Z_{D_0 \cap Z} \cap f_0^{-1}(Z \cap D_0)))$$

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$$coker(\mathbf{Z}_{qfh}(Z_{D_0 \cap Z} \times_{X_1} X_n) \longrightarrow \mathbf{Z}_{qfh}(Z_{D_0 \cap Z}))$$
$$coker(\mathbf{Z}_{qfh}(f_0^{-1}(Z \cap D_0) \times_{X_1} X_n) \longrightarrow \mathbf{Z}_{qfh}(f_0)^{-1}(Z \cap D_0))$$

It follows from the transversality of Z to our sequence of blowups and inductive assumption.

Let us consider now a sheaf $Im(d)$. It is a cokernel of the morphism $ker(\mathbf{Z}_{qfh}(f_0'g')) \rightarrow ker(\mathbf{Z}_{qfh}(f_0'))$. Let me show, that it is naturally isomorphic to the cokernel of the morphism

$$\mathbf{Z}_{qfh}(f_0^{-1}(Z \cap D_0) \times_{X_1} X_n) \longrightarrow \mathbf{Z}_{qfh}(f_0^{-1}(Z \cap D_0)).$$

which represents zero object by the inductive assumption.

One has a natural isomorphism (by theorem 3.28)

$$ker(\mathbf{Z}_{qfh}(f_0)) = ker(\mathbf{Z}_{qfh}(f_0^{-1}(D_0)) \longrightarrow \mathbf{Z}_{qfh}(D_0))$$

It implies (by proposition 2.7) that one has a natural isomorphism

$$ker(\mathbf{Z}_{qfh}(f_0')) = ker(\mathbf{Z}_{qfh}(f_0^{-1}(Z \cap D_0)) \longrightarrow \mathbf{Z}_{qfh}(Z \cap D_0)).$$

Let $B = ker(\mathbf{Z}_{qfh}(f_0^{-1}(Z \cap D_0) \times_{X_1} X_n) \longrightarrow \mathbf{Z}_{qfh}(Z \cap D_0))$. There is a natural morphism $B \longrightarrow ker(\mathbf{Z}_{qfh}(f_0'))$ and one can easily see from the proposition 2.7 that its image coincide with the image of $ker(\mathbf{Z}_{qfh}(g'f_0'))$. An isomorphism between two cokernels we want to prove follows now easily from the surjectivity of the morphism

$$\mathbf{Z}_{qfh}(f_0^{-1}(Z \cap D_0) \times_{X_1} X_n) \longrightarrow \mathbf{Z}_{qfh}(Z \cap D_0)$$

which is a corollary of our condition (*) on the sequence (X_i, D_i). Proposition is proven.

The proof of our theorem is based on the following proposition, which is a corollary of the Hironaka's theorem on the simplification of the coherent sheaf of ideals ([7,]).

Proposition 5.3 *Let X be a smooth variety over a field k of characteristic zero and $f : Y \longrightarrow X$ be a proper surjective morphism such, that $f_*(\mathcal{O}_Y) \cong \mathcal{O}_X$. then there exists a sequence (X_i, D_i) of blowups which satisfies a condition (**) such, that $X_0 = X$ and composition $f_0 ... f_n : X_n \longrightarrow X_0$ can be factorized through f.*

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Proof: Note first of all, that our condition on the morphism f implies, that there exists a coherent sheaf of ideals $\mathcal{J}$ on X such, that if $p_{\mathcal{J}} : X_{\mathcal{J}} \longrightarrow X$ is a blowup of $\mathcal{J}$, then $p_{\mathcal{J}}$ can be factorized through f.

Let me recall some notations from [7]. Let $\mathcal{J}$ be a coherent sheaf of ideals on X. For any point $x \in X$ we denote by $\nu_x(\mathcal{J})$ maximal n such that a fiber $\mathcal{J}_x$ of $\mathcal{J}$ in x lies in the n-th power of the maximal ideal of the local ring $\mathcal{O}_{X,x}$ of x on X. Let $p_Z : X_Z \longrightarrow X$ be a blowup with center in smooth connected subvariety Z of X and $z \in Z$ be a general point of Z. Let I_Z be an invertible sheaf which corresponds to the exceptional divisor of X_Z. A weak transform of $\mathcal{J}$ with respect to Z is defined as a coherent sheaf of ideals of the form $p_Z^{-1}(\mathcal{J})I_Z^{-\nu_z(\mathcal{J})}$. Let (X_i, D_i) be a sequence of blowups with smooth centers $D_i \subset X_i$ and $X_0 = X$. Let $\mathcal{J}_0 = \mathcal{J}$ and $\mathcal{J}_{n+1}$ be a weak transform of $\mathcal{J}_i$ with respect to D_i. Let $E_i \subset X_i$ be an exceptional divisor of the composition $f_0 ... f_{i-1} : X_i \longrightarrow X_0$. By the theorem [7,] there exists a sequence $(X_i, D_i)_{0 \le i \le n}$ of blowups with smooth connected centers such, that

1. For any $i > 0$ a divisor E_i has only normal crossings with D_i.
2. For any $i < n$ one has $\nu_z(\mathcal{J}_i) = const > 0$ for $z \in D_i$ and $\mathcal{J}_n \cong \mathcal{O}_{X_n}$.

Obviously, the second condition implies, that the composition $f_0 ... f_n : X_n \longrightarrow X_0$ can be factorized through p_Z. To finish the proof of our proposition it is sufficient to show, that this sequence (X_i, D_i) satisfies our condition (**). Let us show first that it satisfies the condition (*). it is sufficient to show, that a morphism $f_0^{-1} - f_0^{-1} \cap f_1 ... f_{n-1}(E_n) \longrightarrow D_0$ is surjective. It means, that for any $z \in D_0$ a general point $\tilde{z}$ of the fiber of f_0 over z is not contained in $f_1 ... f_{n-1}(E_n)$. Condition 2 above implies, that for any $x \in f_1 ... f_{n-1}(E_n)$ one has $\nu_x(\mathcal{J}_1) > 0$. From the other hand $\nu_{\tilde{z}}(\mathcal{J}_1) = 0$ by the definition of weak transform and the part of condition 2, which states that $\nu_z(\mathcal{J}_0)$ is constant for $z \in D_0$.

To finish the proof it is sufficient to notice, that condition 1 above obviously implies a transversality of X_0 to this sequence of blowups in our sense.

Now we are ready to prove theorem 5.1.

Proof of the theorem 5.1: Denote by DM_h^0 (resp. by DM_{qfh}^0) category obtained by means of the same construction as DM_h (resp. DM_{qfh}) but using usual localization instead of strong one. By 3.35,3.30,?? and 2.18 we

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have natural bijections

$$DM_{qfh}(M(X),F)\otimes\mathbf{Q}=DM^0_{qfh}(M(X),F)\otimes\mathbf{Q}$$

$$DM_h(M(X),F)\otimes\mathbf{Q}=DM^0_h(M(X),F)\otimes\mathbf{Q}.$$

It is sufficient (by proposition 2.20) to show, therefore, that for any qfh-sheaf F of $\mathbf{Q}$-vector spaces such, that associated with F h-sheaf is isomorphic to zero one has

$$DM_{qfh}(M(X),F[n])=0$$

for any $n\in\mathbf{Z}$. It follows easily from proposition 2.14 that it is sufficient to show that for any such F and any $n>0,k\geq 0$ image of the natural map

$$H^k_{qfh}(X\times\partial\Delta^n,F)\longrightarrow DM(M(X),F[k+1-n])$$

is zero.

Lemma 5.4 *Let X be a scheme and $U\longrightarrow X$ be an etale morphism. then for any closed subscheme Z in U there exists a closed subscheme Z' in X such that projection $X_{Z'}\times_X U\longrightarrow U$ can be factorized through natural morphism $U_Z\longrightarrow U$.*

Proof: Obviously.

Lemma 5.5 *Let X be a smooth variety and F be a qfh-sheaf of $\mathbf{Q}$-vector spaces such, that associated with F h-sheaf is isomorphic to zero. Then for any $a\in H^k_{qfh}(X,F)$ there exists a closed subscheme Z of X such, that a as a morphism $\mathbf{Z}_{qfh}(X)\longrightarrow F[k]$ in derived category can be factorized through the natural morphism $\mathbf{Z}_{qfh}(X)\longrightarrow coker(\mathbf{Z}_{qfh}(p_Z):\mathbf{Z}_{qfh}(X_Z)\longrightarrow\mathbf{Z}_{qfh}(X))$.*

Proof: By the theorem 3.30 we have a natural isomorphism

$$H^k_{qfh}(X,F)=H^k_{et}(X,F).$$

It is well known (see [8,]) that in etale topology usual cohomologies coincide with Čech cohomologies. There exists therefore an etale covering $\mathcal{U}=\{U_i\longrightarrow X\}$ of X and a section $\tilde{a}\in F(\mathcal{U}^{k+1}_X)$ of F over $\mathcal{U}^{k+1}_X$ which

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represents our cohomology class a. since h-sheaf associated with F is isomorphic to zero,. it follows from theorem 3.9 that there exists a blow up $p: Y \longrightarrow \mathcal{U}_X^{k+1}$ of $\mathcal{U}_X^{k+1}$ such, that a restriction of $\tilde{a}$ to Y is equal to zero. It follows from the lemma 5.4 that there exists a blowup $Y' \longrightarrow X$ such, that a projection $Y' \times_X \mathcal{U}_X^{k+1} \longrightarrow \mathcal{U}_X^{k+1}$ can be factorized through Y. Consider complexes of sheaves of abelien groups

$$K = (\ldots \longrightarrow \mathbf{Z}_{qfh}(\mathcal{U}_X^{k+1}) \longrightarrow \ldots \longrightarrow \mathbf{Z}_{qfh}(\mathcal{U}))$$

and

$$K' = (\ldots \longrightarrow \mathbf{Z}_{qfh}(Y' \times_X \mathcal{U}_X^{k+1}) \longrightarrow \ldots \longrightarrow \mathbf{Z}_{qfh}(Y' \times_X \mathcal{U})).$$

There are resolvents of $\mathbf{Z}_{qfh}(X)$ and $\mathbf{Z}_{qfh}(Y')$ respectively (by proposition 2.4). It follows from our construction, that the morphism $K \longrightarrow F[k]$ which corresponds to $\tilde{a}$ can be factorized through the cokernel of the natural projection $K \longrightarrow K'$. By the proposition 2.5 this cokernel is a resolvent of the cokernel of the morphism $\mathbf{Z}_{qfh}(Y') \longrightarrow \mathbf{Z}_{qfh}(X)$ which finish the prove of our lemma.

let now a be a class in $H^k_{qfh}(X \times \partial\Delta^n, F)$. Consider a covering

$$\coprod_{i=0}^{n} X \times \mathbf{A}^{n-1} \longrightarrow X \times \partial\Delta^{n-1}.$$

It follows from lemma 5.5 that there exists a closed subscheme Z in $\coprod_{i=0}^n X \times \mathbf{A}^{n-1}$ such that a can be factorized through the natural morphism

$$\mathbf{Z}_{qfh}(X \times \partial\Delta^n) \longrightarrow coker(\mathbf{Z}_{qfh}((\coprod_{i=0}^{n} X \times \mathbf{A}^{n-1})_Z) \longrightarrow \mathbf{Z}_{qfh}(X \times \partial\Delta^n)).$$

it is sufficient to prove our theorem to show that this morphism is zero in DM_{qfh}.

Consider a semi-simplicial scheme of the form

$$\coprod X \; \begin{matrix}\rightarrow\\ \vdots \\ \rightarrow\end{matrix} \; \coprod X \times \mathbf{A}^1 \; \begin{matrix}\rightarrow\\ \vdots \\ \rightarrow\end{matrix} \; \ldots \coprod X \times \mathbf{A}^{n-2} \; \begin{matrix}\rightarrow\\ \\ \rightarrow\end{matrix} \; \coprod X \times \mathbf{A}^{n-1}$$

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A normalization of the corresponding freely generated semi-simplicial sheaf of abelien groups is a resolvent for $\mathbf{Z}_{qfh}(X \times \partial\Delta^n)$. Denote $(\coprod_{i=0}^{n} X \times \mathbf{A}^{n-1})_Z$ by Y and this semi-simplicial scheme by $\mathcal{S}$. To prove our theorem it is sufficient to construct a semi-simplicial scheme $\mathcal{S}'$ and a morphism $F : \mathcal{S}' \longrightarrow \mathcal{S}$ such, that $F_0 : \mathcal{S}'_0 \longrightarrow \mathcal{S}_0$ can be factorized through $Y \longrightarrow \mathcal{S}_0$ and all the cokernels $coker(\mathbf{Z}_{qfh}(F_i))$ represent zero object in DM_{qfh}. The existence of such morphism follows from propositions 5.3 and 5.2 by successive application of the following technical result.

Lemma 5.6 *Let C be a category with fiber products and $\mathcal{X} = (X_i, \partial_i^j : X_{i+1} \longrightarrow X_i)$ be a semi-simplicial object in C. Then for any $k \geq 0$ and any morphism $f : Y \longrightarrow X_k$ there exists a semi-simplicial object $\mathcal{Y}$ and a morphism $F : \mathcal{Y} \longrightarrow calX$ such, that*

1. *$\mathcal{Y}_k = Y$ and F_k coincide with f.*
2. *All the morphisms F_i are compositions of the morphisms obtained by some base change of f.*

Proof: Let $g_k^{ij} : X_i \longrightarrow X_j, k = 1, ..., C_i^j$ be different compositions of face maps of $\mathcal{X}$. We define terms of $\mathcal{Y} = (Y_i, \partial_i^j)$ as follows

$$Y_i = X_i for i < k$$

$$Y_k = Y$$

$$Y_i = (X_i \times_{X_k, f_1^{ik}} Y) \times_{X_i} ... \times_{X_i} (X_i \times_{X_k, f_{C_i^k}^{ik}} Y) for i > k.$$

The definition of morphisms $\partial_i^j : Y_i \longrightarrow Y_j$ is obvious.

Theorem is proven.

Theorem 5.7 *Let X be a smooth variety over a field of characteristic zero, then one has natural isomorphisms*

$$H^n(X, \mathbf{Z}(1)) = \mathrm{H}_{et}^{n-1}(\mathrm{X}, \mathbf{G}_m).$$

In particular $H^2(X, \mathbf{Z}(1)) = \mathrm{Pic(X)}$ and $H^n(X, \mathbf{Z}(1)) \otimes \mathbf{Q} = 0$ for all $n > 2$.

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Proof: Denote by μ an inductive limit of sheaves of roots of unit. It follows from the proposition ??, that the family of natural morphisms $H^i_{et}(X, \mathbf{G}_m) \longrightarrow H^{i-1}(X, \mathbf{Z}(1))$ can be included in the following diagram with exact strings:

$$\begin{array}{ccccccc} \dots \to & H^i_{et}(X,\mathbf{G}_m) & \to & H^i_{et}(X,\mathbf{G}_m)\otimes\mathbf{Q} & \to & H^{i-1}_{et}(X,\mu) & \to \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ \dots \to & H^{i-1}(X,\mathbf{Z}(1)) & \to & H^{i-1}(X,\mathbf{Z}(1))\otimes\mathbf{Q} & \to & H^{i-1}_{et}(X,\mu) & \to \dots \end{array}$$

Where morphisms $H^{i-1}_{et}(X, \mu) \longrightarrow H^{i-1}_{et}(X, \mu)$ are identities. To prove our proposition it is sufficient, therefore to show, that the morphisms

$$H^i_{et}(X, \mathbf{G}_m) \otimes \mathbf{Q} \longrightarrow H^{i-1}(X, \mathbf{Z}(1)) \otimes \mathbf{Q}$$

are isomorphisms. Our theorem follows from the theorems 5.1,??, proposition 2.17 and the following lemma.

Lemma 5.8 *Let X be a regular scheme, then for any $n \geq 2$ one has*

$$H^n_{et}(X, \mathbf{G_m}) \otimes \mathbf{Q} = 0$$

and for any $n \leq 1$ a natural morphism

$$H^n_{et}(X, \mathbf{G_m}) \otimes \mathbf{Q} \longrightarrow H^n_{et}(X \times \mathbf{A}^1, \mathbf{G}_m) \otimes \mathbf{Q}$$

is an isomorphism.

Proof: We may suppose, that X is connected. Let $i : Spec(K) \longrightarrow X$ be a general point of X. There is defined a following exact sequence of sheaves (see [8, 2.3.9]):

$$0 \longrightarrow \mathbf{G}_m \longrightarrow i_*\mathbf{G}_m \longrightarrow D \longrightarrow 0$$

where

$$D = \bigoplus_{codim(x)=1} (i_x)_*(\mathbf{Z})$$

is a direct sum of the direct images of constant sheaves on points of codimension 1 on X. One can easily see (using Leray spectral sequence) that

$$H^n_{et}(X, i_*\mathbf{G}_m) \otimes \mathbf{Q} = H^n_{et}(X, D) \otimes \mathbf{Q} = 0$$

for $n > 0$. It implies the first part of our lemma. The second part follows now from the well known homotopy invariance of Picard group over regular

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schemes.

Our theorem is proven.

5.2 Categories DM_{ft}.

Let k be afield of characteristic zero. Denote by $DM_{ft}(k)$ a full tensor triangle subcategory in $DM(Spec(k))$ generated by motives of the schemes of finite type over $Spec(k)$. In this section I shall prove some elementary properties of these categories.

Proposition 5.9 *Let X,X' be a pair of birationally equivalent schemes of finite type over k. Denote by C (resp. C') a full tensor triangle subcategory of DM_{ft} which is generated by the objects $M(Y)$ for Y such, that $dim(Y) < dim(X)$ and $M(X)$ (resp. $M(X')$). Then one has $C = C'$.*

Proof: It is direct corollary of the theorem of Hironaka about resolution of singularities and our theorems 4.28 and 4.32.

Theorem 5.10 *Category $DM_{ft}(k)$ is generated as tensor triangle category by the motives of smooth projective varieties.*

Proof: It is a direct corollary of the proposition 5.9 and resolution of singularities.

Proposition 5.11 *Let X be an object of $DM_{ft}(k)$, then for any $n > 0$ an object $X \times \mathbf{Z}/n\mathbf{Z}$ is isomorphic in $DM_{ft}(k)$ to an object which corresponds to a finite complex of locally free (in etale topology) sheaves of finite groups over k.*

Proof:

——————to be continued——————

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A Tensor triangle categories.

I want to give here several definitions and examples concerning tensor triangle categories, because as I know there are no any paper where such structures would be considering.

Tensor triangle category is by the definition the category equipped by both tensor and triangle structure together with some additional data describing there concordance.

Definition A.1 *The tensor triangle category is the collection of the following data:*

1. Triangle category C.

2. Tensor structure on C in the sense of []. We shall denote by $\otimes$ the tensor product on C, by $a_{X,Y,Z} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z$ - the isomorphisms of the associativity and by $\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ - the isomorphisms of the commutativity in C. We assume that C is equipped with the strict unit object $\mathbb{Z}$ and that $(\sigma_{X,X})^2 = 1$.

3. For any two objects X, Y of C - the isomorphisms:

$$\alpha_{X,Y} : (X \otimes Y)[1] \longrightarrow X \otimes (Y[1])$$

$$\beta_{X,Y} : (X \otimes Y)[1] \longrightarrow (X[1]) \otimes Y$$

This set of data should satisfy the following conditions:

1. The functors $? \otimes X : C \longrightarrow C$ and $X \otimes ? : C \longrightarrow C$ are exact for any $X \in obC$.

2. For any $X, Y \in obC$ the following diagram is commutative:

$$\begin{array}{ccc} (X \otimes Y)[1] & \xrightarrow{\alpha_{X,Y}} & X \otimes (Y[1]) \\ \sigma_{X,Y}[1] \downarrow & & \downarrow \sigma_{X,Y[1]}[1] \\ (Y \otimes X)[1] & \xrightarrow{\beta_{Y,X}} & (Y[1]) \otimes X \end{array}$$

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3. For any $X, Y \in obC$ the following diagram is commutative:

$$\begin{array}{ccc}
(X \otimes Y)[2] & \xrightarrow{\beta_{X,Y}[1]} & (X[1] \otimes Y)[1] \\
\Big\downarrow{\scriptstyle \alpha_{X,Y}[1]} & & \Big\downarrow{\scriptstyle -\alpha_{X[1],Y}} \\
(X \otimes Y[1])[1] & \xrightarrow{\beta_{X,Y[1]}} & X[1] \otimes Y[1]
\end{array}$$

The morphism $\alpha_{X,Y}, \beta_{X,Y}$ in this definition are nothing but the components of the natural isomorphisms of functors $(? \otimes X)[?] \longrightarrow [?](? \otimes X)$ and $(X \otimes ?)[?] \longrightarrow [?](X \otimes ?)$ the existence of which is the part of the definition of the exact functor.

Note that we should not include in our definition any concordance conditions for $\alpha_{X,Y}$ or $\beta_{X,Y}$ and the isomorphisms of the associativity, because they can be obtained from the standard diagrams for the tensor categories using the isomorphisms $\alpha_{X,\mathbf{Z}} : X[1] \longrightarrow X \otimes \mathbf{Z}[1]$. In fact, the only nontrivial point in this definition is the negative sign in the condition 3.

Proposition A.2 *Let A be a tensor abelien category of the finite Tor-dimension, then the derived category $D(A)$ of A equipped with the $\overset{\otimes}{L}$ is the tensor triangle category in the sense of the above definition.*

Proof: Direct computation.

Definition A.3 *The tensor exact functor from the tensor triangle category D to the tensor triangle category D' is the functor $F : D \longrightarrow D'$ which is tensor functor with respect to the tensor structures on D, D' and the exact functor with respect to the corresponding triangle structures together with the fixed isomorphism $\phi F \circ [?] \longrightarrow [?] \circ F$ such that the following diagrams are commutative for any $X, Y \in obD$:*

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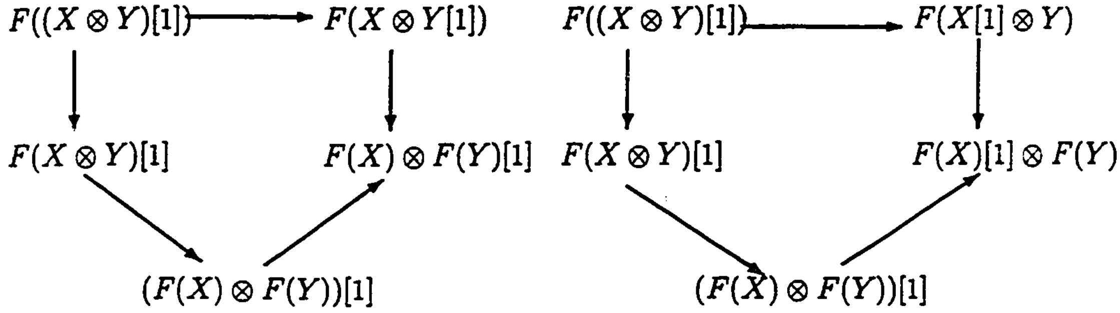

Now I want to prove one result which shows what kind of effects one can expect working with the tensor triangle categories.

For the pair of objects $X, Y \in obD$ denote by $\gamma_{X,Y} : (X \otimes Y)[2] \longrightarrow X[1] \otimes Y[1]$ the composition

$$\gamma_{X,Y} = \alpha_{X,Y}[1] \circ \beta_{X,Y[1]}$$

I shall need the following lemma:

Lemma A.4 *Let $X, Y \in obD$ then one has $\gamma_{Y,X} \circ \sigma_{X,Y}[2] = -\sigma_{X[1],Y[1]} \circ \gamma_{X,Y}$.*

Proof: Consider the diagram:

$$\begin{array}{ccccc} (X\otimes Y)[2] & \longrightarrow & (X\otimes Y[1])[1] & \longrightarrow & X[1]\otimes Y[1] \\ \downarrow & & \downarrow & & \downarrow \\ (Y\otimes X)[2] & \longrightarrow & (Y[1]\otimes X)[1] & \longrightarrow & Y[1]\otimes x[1] \end{array}$$

It is commutative according to the condition 2 of the definition A.1. The upper string represents by the definition $\gamma_{X,Y}$ and according to the condition 3 of A.1 the lower string represents $-\gamma_{Y,X}$ which proves the lemma.

Proposition A.5 *Let A be a tensor k-linear abelien category such that $chark \neq 2$, D be a tensor triangle category and $F : A \longrightarrow D$ be a tensor exact functor.*

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Suppose that $X, Y \in obA$ are flat objects such, that there exists an isomorphism $\phi : F(A) \longrightarrow F(B)[1]$ then one has:

$$F(S^2X) \cong F(\bigwedge^2 Y)[2]$$

$$F(\bigwedge^2 X) \cong F(S^2Y)$$

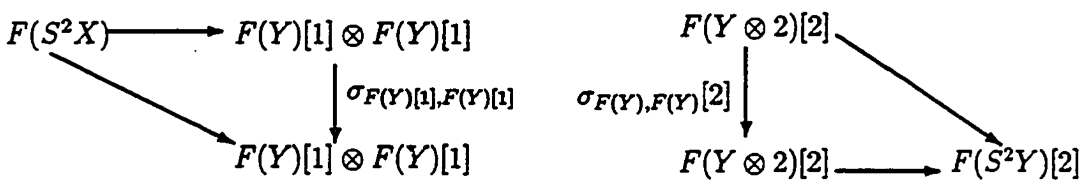
where S^2Z and $\bigwedge^2 Z$ denote the simmetric and exterior squares of an object Z respectively.

Proof: To avoid a waste of paper I shall assume that $F(Z_1 \otimes Z_2) = F(Z_1) \otimes F(Z_2)$. Consider the decompositions $X^{\otimes 2} = S^2X \oplus \bigwedge^2 X$ and $Y^{\otimes 2} = S^2Y \oplus \bigwedge^2 Y$ (we can do it because *chark* $\neq 2$). Any morphism $F(X^{\otimes 2}) \longrightarrow F(Y^{\otimes 2})[2]$ can be represented by 2×2 matrix, which "elements" are the morphisms of the form $F(S^2X) \longrightarrow F(S^2Y)[2], F(S^2X) \longrightarrow F(\bigwedge^2 Y)[2]$ etc. It is sufficient to prove that the diagonal ones are zero, then the other two will give us the isomorphisms we need. Let us prove, say, that the composition

$$F(S^2X) \longrightarrow F(X) \otimes F(X) \longrightarrow F(Y)[1] \otimes F(Y)[1] \longrightarrow F(Y^{\otimes 2})[2] \longrightarrow F(S^2Y)[2]$$

is equial to zero.

Considering the commutative diagrams



we see that to prove it is sufficient to show that the diagram

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$$\begin{array}{ccc}
F(Y)[1] \otimes F(Y)[1] & \longrightarrow & F(Y \otimes 2)[2] \\
{\scriptstyle \sigma_{F(Y)[1],F(Y)[1]}} \big\downarrow & & \big\downarrow {\scriptstyle \sigma_{F(Y),F(Y)}[2]} \\
F(Y)[1] \otimes F(Y)[1] & \longrightarrow & F(Y \otimes 2)[2]
\end{array}$$

is commutative up to the multiplication by -1, which is nothing but the statment of the lemma A.4.

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B Strong localization of derived categories

This appendix is devoted to the construction which we call strong localization for the derived categories of abelien categories. It is a modification of the construction of the localization for triangle categories due to Verdier ([12]), which seems to be more convenient than the original one in the case of possibly infinite Ext-dimension of the abelien category.

Let me briefly recall first an original construction by Verdier.

Definition B.1 *Let D be a triangle category. A full subcategory C of D is called thick if it satisfies the following three conditions.*

1. *If $X \in ob(C)$, then for any n one has $X[n] \in ob(C)$.*

2. *Let*
$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$
be an exact triangle in D such that $X, Y \in ob(C)$, then $Z \in ob(C)$.

3. *Let*
$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow x[1]$$
be an exact triangle in D such, that $Z \in ob(C)$ and f can be factorized through an object from C, then $X, Y \in ob(C)$.

For any class P of objects of D there is a smallest thick subcategory of D which contains all the objects from P. We denote it by $< P >$.

The main result of the Verdier's theory of the localizations of triangle categories can be formulated as follows.

Theorem B.2 *Let D be a triangle category and P be a class of objects of D. Then a localization D/A of D with respect to the class of morphisms such that their cones lies in $< P >$ has a natural structure of triangle category and a functor $D \longrightarrow D/P$ is universal with respect to exact functors which take objects of P to zero object.*

Proof: See [12].

One can easily see that if D is a derived category of an abelien category A

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constructed by means of bounded complexes, then any object of A which has a finite resolvent consisting of the objects from P is contained in $< P >$, and, therefore, represents zero in the localization D/P. If A has a finite Ext-dimension, then one can easily see, that this remark admit a generalization. Namely, if an object X of A has an infinite to the left resolvent consisting of the objects from P, then it also represents zero in the localized category. In the case of infinite Ext-dimension it is not in general true, i.e. there might exist objects which have an infinite to the left resolvent consisting of objects of P which do not represent zero object in the localized category.

The following definition of the strong localization let us to eliminate such effects. Denote by D_- a full subcategory of D consisting of complexes which are acyclic in positive dimension.

Definition B.3 *a. An object $X \in ob(D)$ is called unbounded with respect to P if there exists N such that for any $n > N$ there exist an object $Y_n \in ob(D_-)$ and a morphism $X \longrightarrow Y_n[n]$ in D, which represents isomorphism in the localized category D/P.*

b. A strong localization $D/\bar{P}$ of D with respect to P is defined as a localization of D/P with respect to thick subcategory generated by unbounded objects.

Proposition B.4 *Let X be an object of A such, that there exists a resolvent*

$$\dots \xrightarrow{d_n} X_n \xrightarrow{d_{n-1}} X_{n-1} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_1} X_1$$

of X consisting of the objects from $< P >$, then X represents zero in $D/\bar{P}$.

Proof: Consider an exact sequence

$$ker(d_n) \longrightarrow X_n \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} X_1 \longrightarrow X$$

It defines a morphism $X \longrightarrow ker(d_n)[n]$ in D. It follows immediately from the condition $X_k \in ob(< P >)$, that this morphism represents isomorphism in the localized category. Therefore X is unbounded, i.e. represents zero in $D/\bar{P}$.

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Proposition B.5 *Let X be an object of D such, that for any $Y \in P$ and any n one has*

$$Hom_D(Y, X) = 0,$$

then

$$Hom_{D/P}(Z, X) = Hom_D(Z, X)$$

for any $Z \in ob(D)$.

Proof: Obviously.

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