Voevodsky's Nordfjordeid Lectures: Motivic Homotopy Theory

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1 Introduction

Motivic homotopy theory is a new and in vogue blend of algebra and topology. Its primary object is to study algebraic varieties from a homotopy theoretic viewpoint. Many of the basic ideas and techniques in this subject originate in algebraic topology.

This text is a report from Voevodsky's summer school lectures on motivic homotopy in Nordfjordeid. Its first part consists of a leisurely introduction to motivic stable homotopy theory, cohomology theories for algebraic varieties, and some examples of current research problems. As background material, we recommend the lectures of Dundas [Dun] and Levine [Lev] in this volume. An introductory reference to motivic homotopy theory is Voevodsky's ICM address [Voe98]. The appendix includes more in depth background material required in the main body of the text. Our discussion of model structures for motivic spectra follows Jardine's paper [Jar00].

In the first part, we introduce the motivic stable homotopy category. The examples of motivic cohomology, algebraic K-theory, and algebraic cobordism illustrate the general theory of motivic spectra. In March 2000, Voevodsky [Voe02b] posted a list of open problems concerning motivic homotopy theory. There has been so much work done in the interim that our update of the status of these conjectures may be useful to practitioners of motivic homotopy theory.

The second and third author would like to thank Vladimir Voevodsky for helpful discussions concerning the content of this text, and his kind permission to include a sketch proof of Theorem 4.3. The actual wording here, and the responsibility for any misinterpretations, are our own.

2 Motivic Stable Homotopy Theory

In this section, we introduce the motivic stable homotopy category. Although the construction of this category can be carried out for more general base schemes, we shall only consider Zariski spectra of fields.

A final word about precursors: In what follows, we use techniques which are basic in the study of both model categories and triangulated categories. Introductory textbooks on these subjects include [Hov99] and [Nee01].

2.1 Spaces

Let k be a field and consider the category Sm/k of smooth separated schemes of finite type over $\mathrm{Spec}(k)$. From a homotopical point of view, the category Sm/k is intractable because it is not closed under colimits.

The spaces we consider are the objects in the category

$$\mathrm{Spc}(k) := \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}/k)$$

of Nisnevich sheaves on Sm/k [Lev] with values in simplicial sets [Dun].

We mention two typical types of examples of such sheaves. First, any scheme in Sm/k determines a representable space via the Yoneda embedding. This holds since the Nisnevich topology is sub-canonical [Lev]. Second, any simplicial set can be viewed as a constant Nisnevich sheaf on Sm/k, and also as a constant sheaf in any other Grothendieck topology.

A pointed space consists of a space X together with a map

$$x\colon \mathrm{Spec}(k) \longrightarrow X .$$

Here, we consider $\mathrm{Spec}(k)$ as a representable sheaf with constant simplicial structure. Let $\mathrm{Spc}_\bullet(k)$ denote the category of pointed spaces. If X is a space, let X_+ denote the canonically pointed space $X \amalg \mathrm{Spec}(k)$. By adding disjoint base-points, it follows that the forgetful functor from $\mathrm{Spc}_\bullet(k)$ to $\mathrm{Spc}(k)$ has a left adjoint.

It is important to note that the category of pointed spaces has a symmetric monoidal structure: Suppose X and Y are pointed spaces. Then their smash product $X \wedge Y$ is the space associated to the presheaf

$$U \longmapsto X(U) \wedge Y(U) .$$

The sheaf represented by the Zariski spectrum $\mathrm{Spec}(k)$ is the terminal presheaf with value the one-point set

$$U \longmapsto * .$$

Clearly, this shows that $\mathrm{Spec}(k)_+$ is a unit for the smash product.

Recall that in classical stable homotopy theory, when constructing spectra of pointed simplicial sets one inverts only one suspension coordinate, namely the simplicial circle. This part works slightly differently in the motivic context. An exotic aspect, which turns out to play a pivotal role in the motivic stable homotopy theory, is the use of two radically different suspension coordinates. In order to define the motivic stable homotopy category, we shall make use of bispectra of pointed spaces.

The first of the motivic circles is well-known to topologists: Let $\Delta[n]$ denote the standard simplicial n-simplex [Dun]. Recall that the simplicial circle S^1 is the coequalizer of the diagram

$$\Delta[0] \rightrightarrows \Delta[1] .$$

We denote by S^1_s the corresponding pointed space.

The second motivic circle is well-known to algebraic geometers: Denote by $\mathbb{A}^1 \in \mathrm{Sm}/k$ the affine line. Then the Tate circle S^1_t is the space $\mathbb{A}^1 \smallsetminus 0$, pointed by the global section given by the identity; this is the underlying scheme of the multiplicative group.

Since pointed spaces $\mathrm{Spc}_\bullet(k)$ acquires a smash product, we may form the n-fold smash products S^n_s and S^n_t of the simplicial circle and the Tate circle. A mixed sphere refers to a smash product of S^m_s and S^n_t.

2.2 The Motivic s-Stable Homotopy Category $\mathbf{SH}_s^{\mathbb{A}^1}(k)$

To invert S_s^1 we shall consider spectra of pointed spaces. This is analogous to the situation with ordinary spectra and the simplicial circle.

Definition 2.1 *An s-spectrum E is a sequence of pointed spaces $\{E_n\}_{n\geq 0}$ together with structure maps*

$$S_s^1 \wedge E_n \longrightarrow E_{n+1} .$$

A map of s-spectra

$$E \longrightarrow E'$$

consists of degree-wise maps of pointed spaces

$$E_n \longrightarrow E'_n$$

which are compatible with the structure maps.

Let $\mathrm{Spt}_s(k)$ *denote the category of s-spectra.*

Pointed spaces give examples of s-spectra:

Example 2.2 *The s-suspension spectrum of a pointed space X is the s-spectrum $\Sigma_s^\infty X$ with n-th term $S_s^n \wedge X$ and identity structure maps.*

The next step is to define weak equivalences of s-spectra. If $n \geq 1$ and (X, x) is a pointed space, let $\pi_n(X, x)$ denote the sheaf of homotopy groups associated to the presheaf

$$U \longmapsto \pi_n(X(U), x|U) ,$$

where $x|U$ is the image of x in $X(U)$. If $n \geq 2$, this is a Nisnevich sheaf of abelian groups.

The suspension homomorphism for ordinary pointed simplicial sets yields suspension homomorphisms of sheaves of homotopy groups

$$\pi_n(X) \longrightarrow \pi_{n+1}(S_s^1 \wedge X) .$$

If E is an s-spectrum and $m > n$ are integers, consider the sequence

$$\pi_{n+m}(E_m) \longrightarrow \pi_{n+m+1}(S_s^1 \wedge E_m) \longrightarrow \pi_{n+m+1}(E_{m+1}) \longrightarrow \cdots .$$

The sheaves of stable homotopy groups of E are the sheaves of abelian groups

$$\pi_n(E) \colon = \operatorname*{colim}_{m>n} \pi_{n+m}(E_m) .$$

A map between s-spectra E and E' is called an s-stable weak equivalence if for every integer $n \in \mathbb{Z}$, there is an induced isomorphism of sheaves

$$\pi_n(E) \longrightarrow \pi_n(E') .$$

Definition 2.3 *Let* $\mathrm{SH}_s(k)$ *be the category obtained from* $\mathrm{Spt}_s(k)$ *by inverting the* s*-stable weak equivalences.*

Remark 2.4 *One can show that* $\mathrm{Spt}_s(k)$ *has the structure of a proper simplicial stable model category. The associated homotopy category is* $\mathrm{SH}_s(k)$*. In this model structure, the weak equivalences are the* s*-stable weak equivalences.*

There is an obvious way to smash an s-spectrum with any pointed space. If we smash with the simplicial circle, the induced simplicial suspension functor Σ_s^1 becomes an equivalence of $\mathrm{SH}_s(k)$: Indeed, an s-spectrum is a Nisnevich sheaf on Sm/k with values in the category Spt of ordinary spectra. Since the simplicial suspension functor is an equivalence of the stable homotopy category SH, it is also an equivalence of $\mathrm{SH}_s(k)$. The term 's-stable' refers to this observation.

Remark 2.5 *There is a canonical functor*

$$\mathrm{Spt} \longrightarrow \mathrm{Spt}_s(k)$$

obtained by considering pointed simplicial sets as pointed spaces. A stable weak equivalence of spectra induces an s*-stable weak equivalence of* s*-spectra, and there is an induced functor*

$$\mathrm{SH} \longrightarrow \mathrm{SH}_s(k)$$

between the corresponding homotopy categories.

The basic organizing principle in motivic homotopy theory is to make the affine line contractible. One way to obtain this is as follows: An s-spectrum F is $\mathbb{A}^1$-local if for all $U \in \mathrm{Sm}/k$ and $n \in \mathbb{Z}$, there is a bijection

$$\mathrm{Hom}_{SH_s(k)}(\Sigma_s^\infty U_+, \Sigma_s^n F) \longrightarrow \mathrm{Hom}_{SH_s(k)}(\Sigma_s^\infty (U \times \mathbb{A}^1)_+, \Sigma_s^n F)$$

defined by the projection

$$U \times \mathbb{A}^1 \longrightarrow U\ .$$

We say that a map

$$E \longrightarrow E'$$

of s-spectra is an $\mathbb{A}^1$-stable weak equivalence if for any $\mathbb{A}^1$-local s-spectrum F, there is a canonically induced bijection

$$\mathrm{Hom}_{\mathrm{SH}_s(k)}(E', F) \longrightarrow \mathrm{Hom}_{\mathrm{SH}_s(k)}(E, F)\ .$$

Definition 2.6 *Let the motivic* s*-stable homotopy category* $\mathrm{SH}_s^{\mathbb{A}^1}(k)$ *be the category obtained from* $\mathrm{Spt}_s(k)$ *by inverting the* $\mathbb{A}^1$*-stable weak equivalences.*

The following Lemma is now evident.

Lemma 2.7 *Assume $X \in \mathrm{Sm}/k$. Then the canonical map*

$$X \times \mathbb{A}^1 \longrightarrow X$$

induces an $\mathbb{A}^1$-stable weak equivalence of s-spectra.

In particular, the s-suspension spectrum $\Sigma_s^\infty(\mathbb{A}^1, 0)$ is contractible.

We note that the simplicial suspension functor is an equivalence of $\mathrm{SH}_s^{\mathbb{A}^1}(k)$. The last step in the construction of the motivic stable homotopy category is to invert the Tate circle so that smashing with S_t^1 becomes an equivalence. But first we discuss an unsatisfactory facet of the motivic s-stable homotopy category; part of this is motivation for work in the next section.

In topology, a finite unramified covering map of topological spaces

$$Y \longrightarrow Z ,$$

induces a transfer map

$$\Sigma^\infty Z_+ \longrightarrow \Sigma^\infty Y_+$$

in the ordinary stable homotopy category. The first algebraic analogue of a covering map is a finite Galois extension of fields, say k'/k. However, in the s-stable homotopy category $\mathrm{SH}_s^{\mathbb{A}^1}(k)$ there is no non-trivial transfer map

$$\Sigma_s^\infty \mathrm{Spec}(k)_+ \longrightarrow \Sigma_s^\infty \mathrm{Spec}(k')_+ .$$

This follows by explicit calculations: If E is an s-spectrum and π^s denotes ordinary stable homotopy groups, then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SH}_s(k)}(\Sigma_s^\infty S_s^n, E) &= \mathrm{Hom}_{\mathrm{SH}}(\Sigma^\infty S^n, E(\mathrm{Spec}(k))) \\ &= \pi_n^s(E(\mathrm{Spec}(k))) .\end{aligned}$$

Now, let $X \in \mathrm{Sm}/k$ and suppose that

$$\mathrm{Hom}_{\mathrm{Sm}/k}(\mathbb{A}^1, X) = \mathrm{Hom}_{\mathrm{Sm}/k}(Spec(k), X) .$$

This is satisfied if $X = \mathrm{Spec}(k')$, where k' is a field extension of k. Then the above implies an isomorphism

$$\mathrm{Hom}_{\mathrm{SH}_s^{\mathbb{A}^1}(k)}(\Sigma_s^\infty S_s^n, \Sigma_s^\infty X_+) = \mathrm{Hom}_{\mathrm{SH}}(\Sigma^\infty S^n, \Sigma^\infty X(\mathrm{Spec}(k))_+) .$$

When $X = \mathrm{Spec}(k')$, note that

$$\mathrm{Hom}_{Sm/k}(Spec(k), Spec(k')) = \emptyset .$$

By letting $n = 0$, we find

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{SH}_s^{\mathbb{A}^1}(k)}(\Sigma_s^\infty \mathrm{Spec}(k)_+, \Sigma_s^\infty \mathrm{Spec}(k')_+)\\
&= \mathrm{Hom}_{\mathrm{SH}}(\Sigma^\infty S^0, \Sigma^\infty \mathrm{Spec}(k')(\mathrm{Spec}(k))_+)\\
&= \mathrm{Hom}_{\mathrm{SH}}(\Sigma^\infty S^0, \Sigma^\infty *)\\
&= 0\,.
\end{aligned}$$

Remark 2.8 *It turns out that the existence of transfer maps for finite Galois extensions and the Tate circle being invertible in the homotopy category are closely related issues. Transfer maps are incorporated in Voevodsky's derived category* $\mathrm{DM}_-^{\mathrm{eff}}(k)$ *of effective motivic complexes over k [Voe00b]. There is a canonical Hurewicz map relating the motivic stable homotopy category with* $\mathrm{DM}_-^{\mathrm{eff}}(k)$*. If k is a perfect field, the cancellation theorem [Voe02a] shows that tensoring with the Tate object* $\mathbb{Z}(1)$ *in* $\mathrm{DM}_-^{\mathrm{eff}}(k)$ *induces an isomorphism*

$$\mathrm{Hom}_{\mathrm{DM}_-^{\mathrm{eff}}(k)}(C,D) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{DM}_-^{\mathrm{eff}}(k)}\big(C\otimes\mathbb{Z}(1), D\otimes\mathbb{Z}(1)\big)\,.$$

2.3 The Motivic Stable Homotopy Category SH(k)

The definition of SH(k) combines the category of s-spectra and the Tate circle. We use bispectra in order to make this precise.

Let $m, n \geq 0$ be integers. An (s,t)-bispectrum E consists of pointed spaces $E_{m,n}$ together with structure maps

$$\sigma_s \colon S_s^1 \wedge E_{m,n} \longrightarrow E_{m+1,n}\ ,$$

$$\sigma_t \colon S_t^1 \wedge E_{m,n} \longrightarrow E_{m,n+1}\ .$$

In addition, the structure maps are required to be compatible in the sense that the following diagram commutes.

$$\begin{array}{ccccc}
S_s^1\wedge S_t^1\wedge E_{m,n} & & \xrightarrow{\tau\wedge E_{m,n}} & & S_t^1\wedge S_s^1\wedge E_{m,n}\\
\downarrow{\scriptstyle S_s^1\wedge\sigma_t} & & & & \downarrow{\scriptstyle S_t^1\wedge\sigma_s}\\
S_s^1\wedge E_{m,n+1} & \xrightarrow{\sigma_s} & E_{m+1,n+1} & \xleftarrow{\sigma_t} & S_t^1\wedge E_{m+1,n}
\end{array}$$

Here, τ flips the copies of S_s^1 and S_t^1. There is an obvious notion of maps between such bispectra. An (s,t)-bispectrum can and will be interpreted as a t-spectrum object in the category of s-spectra, that is, a collection of s-spectra

$$E_n \colon = E_{*,n}$$

together with maps of s-spectra induced by the structure maps

$$S_t^1 \wedge E_n \longrightarrow E_{n+1}\ .$$

We write $\mathrm{Spt}_{s,t}(k)$ for the category of (s,t)-bispectra. If X is a pointed space, let $\Sigma^\infty_{s,t}X$ denote the corresponding suspension (s,t)-bispectrum.

If E is an (s,t)-bispectrum and p,q are integers, denote by $\pi_{p,q}(E)$ the sheaf of bigraded stable homotopy groups associated to the presheaf

$$U \longmapsto \operatorname*{colim}_m \mathrm{Hom}_{\mathrm{SH}^{\mathbb{A}^1}_s(k)}(S^{p-q}_s \wedge S^{q+m}_t \wedge \Sigma^\infty_s U_+, E_m) .$$

This expression makes sense for $p < q$, since smashing with S^1_s yields an equivalence of categories

$$\mathrm{SH}^{\mathbb{A}^1}_s(k) \longrightarrow \mathrm{SH}^{\mathbb{A}^1}_s(k), \quad E \longmapsto S^1_s \wedge E .$$

Moreover, in the above we assume $q+m \geq 0$.

Definition 2.9 *A map $E \to E'$ of (s,t)-bispectra is a stable weak equivalence if for all $p,q \in \mathbb{Z}$, there is an induced isomorphism of sheaves of bigraded stable homotopy groups*

$$\pi_{p,q}(E) \longrightarrow \pi_{p,q}(E') .$$

We are ready to define our main object of study:

Definition 2.10 *The motivic stable homotopy category* $\mathrm{SH}(k)$ *of* k *is obtained from* $\mathrm{Spt}_{s,t}(k)$ *by inverting the stable weak equivalences.*

Remark 2.11 *There is an underlying model category structure on* $\mathrm{Spt}_{s,t}(k)$*; the weak equivalences are the stable weak equivalences defined in 2.9. Moreover, this model structure is stable, proper, and simplicial.*

By construction, the suspension functors Σ^1_s and Σ^1_t induce equivalences of the motivic stable homotopy category. Hence, analogous to 2.7, we have:

Lemma 2.12 *Assume $X \in \mathrm{Sm}/k$. Then the canonical map*

$$X \times \mathbb{A}^1 \longrightarrow X$$

induces a stable weak equivalence of suspension (s,t)-bispectra.

In 2.13, we note that maps between suspension spectra in $\mathrm{SH}(k)$ can be expressed in terms of maps in $\mathrm{SH}^{\mathbb{A}^1}_s(k)$. As the proof in the Nisnevich topology hinges on finite cohomological dimension, it is not clear whether there is a similarly general result in the etale topology.

Proposition 2.13 *If $X \in \mathrm{Sm}/k$ and $E \in \mathrm{Spt}_{s,t}(k)$, there is an isomorphism*

$$\mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_{s,t} X_+, E) = \operatorname*{colim}_n \mathrm{Hom}_{\mathrm{SH}^{\mathbb{A}^1}_s(k)}(S^n_t \wedge \Sigma^\infty_s X_+, E_n) .$$

Proof. If $E = (E_{m,n})$ is some (s,t)-bispectrum, recall that E_n is the s-spectrum defined by the sequence $(E_{0,n}, E_{1,n}, \ldots)$. By model category theory, there exists a fibrant replacement E^f of E in $\mathrm{Spt}_{s,t}(k)$ and isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_{s,t} X_+, E) &= \mathrm{Hom}_{\mathrm{Spt}_{s,t}(k)}(\Sigma^\infty_{s,t} X_+, E^f)/_\simeq \\ &= \mathrm{Hom}_{\mathrm{Spt}_s(k)}(\Sigma^\infty_s X_+, E^f_0)/_\simeq \,.\end{aligned}$$

The relation $\simeq$ is the homotopy relation on maps; in our setting, homotopies are parametrized by the affine line $\mathbb{A}^1$. By using properties of the Nisnevich topology – every scheme has finite cohomological dimension, coverings are generated by so-called upper distinguished squares which will be introduced in Definition 2.18 – one can choose a fibrant replacement E^f so that

$$E^f_0 = \operatorname*{colim}_n \left((E_0)^f \longrightarrow \Omega_t((E_1)^f) \longrightarrow \Omega_t(\Omega_t((E_2)^f)) \longrightarrow \cdots\right).$$

Here, $(E_n)^f$ is a fibrant replacement of E_n in $\mathrm{Spt}_s(k)$, and Ω_t is the right adjoint of the functor

$$S^1_t \wedge -\colon \mathrm{Spt}_s(k) \longrightarrow \mathrm{Spt}_s(k)\,.$$

Since the s-suspension spectra $\Sigma^\infty_s X_+$ and $\Sigma^\infty_s X_+ \wedge \mathbb{A}^1_+$ are both finitely presentable objects in $\mathrm{Spt}_s(k)$, we have

$$\mathrm{Hom}_{\mathrm{Spt}_s(k)}(\Sigma^\infty_s X_+, E^f_0)/_\simeq = \operatorname*{colim}_n \mathrm{Hom}_{\mathrm{Spt}_s(k)}(\Sigma^\infty_s X_+, \Omega^n_t((E_n)^f))/_\simeq \,.$$

The latter and the isomorphism

$$\mathrm{Hom}_{\mathrm{SH}_s(k)}(\Sigma^\infty_s X_+, \Omega^n_t(E_n)) = \mathrm{Hom}_{\mathrm{SH}_s(k)}(S^n_t \wedge \Sigma^\infty_s X_+, E_n)\,,$$

obtained from the adjunction, imply the claimed group isomorphism. □

Suppose $X, Y \in \mathrm{Sm}/k$. As a special case of 2.13, we obtain an isomorphism between $\mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_{s,t} X_+, \Sigma^\infty_{s,t} Y_+)$ and

$$\operatorname*{colim}_m \mathrm{Hom}_{\mathrm{SH}^{\mathbb{A}^1}_s(k)}(S^m_t \wedge \Sigma^\infty_s X_+, S^m_t \wedge \Sigma^\infty_s Y_+)\,.$$

Remark 2.14 *We will not discuss details concerning the notoriously difficult notion of an adequate smash product for (bi)spectra. Rather than using spectra, one solution is to consider Jardine's category of motivic symmetric spectra [Jar00]. An alternate solution using motivic functors is discussed by Dundas in this volume [Dun].*

For our purposes, it suffices to know that a handicrafted smash product of bispectra induces a symmetric monoidal structure on $\mathrm{SH}(k)$*. The proof of this fact is tedious, but straight-forward. The unit for the monoidal structure is the 'sphere spectrum' or the suspension (s,t)-bispectrum* $\Sigma^\infty_{s,t}\,\mathrm{Spec}(k)_+$ *of the base scheme k. Moreover, for all* $X, Y \in \mathrm{Sm}/k$*, there is an isomorphism between the smash product* $\Sigma^\infty_{s,t}X_+ \wedge \Sigma^\infty_{s,t}Y_+$ *and* $\Sigma^\infty_{s,t}(X \times Y)_+$.

Next we summarize the constructions

$$\mathrm{Spc}_\bullet(k) \xrightarrow{\Sigma_s^\infty} \mathrm{SH}_s^{\mathbb{A}^1}(k) \xrightarrow{\Sigma_t^\infty} \mathrm{SH}(k)\ .$$

The spaces in the motivic setting are the pointed simplicial sheaves on the Nisnevich site of Sm/k. By using the circles S_s^1 and S_t^1, one defines spectra of pointed spaces as for spectra of simplicial pointed sets. The notion of $\mathbb{A}^1$-stable weak equivalences of s-spectra forces the s-suspension spectrum of $(\mathbb{A}^1, 0)$, the affine line pointed by zero, to be contractible. We use the same class of maps to define the motivic s-stable homotopy category $\mathrm{SH}_s^{\mathbb{A}^1}(k)$.

Our main object of interest, the motivic stable homotopy category $\mathrm{SH}(k)$ is obtained by considering (s,t)-bispectra and formally inverting the class of stable weak equivalences; such maps are defined in terms of Nisnevich sheaves of bigraded homotopy groups.

If

$$\Sigma_{s,t}^\infty \colon \mathrm{Sm}/k \longrightarrow \mathrm{SH}(k)$$

denotes the suspension (s,t)-bispectrum functor, there is a natural equivalence of functors

$$\Sigma_{s,t}^\infty = \Sigma_t^\infty \circ \Sigma_s^\infty\ .$$

Remark 2.15 *Work in progress by Voevodsky suggests yet another construction of* $\mathrm{SH}(k)$*. This uses a theory of framed correspondences; a distant algebraic relative of framed cobordisms, which may have computational advantages.*

Next, we shall specify a triangulated structure on $\mathrm{SH}(k)$. As for a wide range of other examples, this additional structure provides a convenient tool to construct long exact sequences.[4]

The category $\mathrm{Spt}_{s,t}(k)$ is obviously complete and cocomplete: Limits and colimits are formed degree-wise in $\mathrm{Spc}_\bullet(k)$. In particular, there is an induced coproduct $\vee$ in $\mathrm{SH}(k)$. We also claim that the latter is an additive category: Since the simplicial circle S_s^1 is a cogroup object in the ordinary unstable homotopy category, we conclude that every spectrum in $\mathrm{SH}(k)$ is a two-fold simplicial suspension. Thus all objects in $\mathrm{SH}(k)$ are abelian cogroup objects, and the set of maps out of any object is an abelian group.

To define a triangulated category structure on $\mathrm{SH}(k)$, we need to specify a class of distinguished triangles and also a shift functor $[-]$. Suppose that E is an (s,t)-bispectrum. Its shift $E[1]$ is the s-suspension of E. Any map

$$f\colon E \longrightarrow E'$$

of (s,t)-bispectra has an associated cofibration sequence

[4] The homotopy categories $\mathrm{SH}_s(k)$ and $\mathrm{SH}_s^{\mathbb{A}^1}(k)$ are also triangulated.

$$E \longrightarrow E' \longrightarrow \mathrm{Cone}(f) \longrightarrow \Sigma^1_s E\ .$$

The cone of f is defined in terms of a push-out square in $\mathrm{Spt}_{s,t}(k)$, where $\Delta[1]$ is pointed by zero:

$$\begin{array}{ccc} E & \xrightarrow{E\wedge 0} & E \wedge \Delta[1] \\ {\scriptstyle f}\downarrow & & \downarrow \\ E' & \longrightarrow & \mathrm{Cone}(f) \end{array}$$

The map

$$\mathrm{Cone}(f) \longrightarrow \Sigma^1_s E$$

collapses E' to a point.

A distinguished triangle in $\mathrm{SH}(k)$ is a sequence that is isomorphic to the image of a cofibration sequence in $\mathrm{Spt}_{s,t}(k)$. It follows that any distinguished triangle in $\mathrm{SH}(k)$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]\ ,$$

induces long exact sequences of abelian groups

$$\cdots [E, X[n]] \longrightarrow [E, Y[n]] \longrightarrow [E, Z[n]] \longrightarrow [E, X[n+1]] \cdots\ ,$$

$$\cdots [Z[n], E] \longrightarrow [Y[n], E] \longrightarrow [X[n], E] \longrightarrow [Z[n-1], E] \cdots\ .$$

Here, $[-,-]$ denotes $\mathrm{Hom}_{\mathrm{SH}(k)}(-,-)$.

The next lemma points out an important class of distinguished triangles.

Lemma 2.16 *If*

$$X \rightarrowtail Y$$

is a monomorphism of pointed spaces, then

$$\Sigma^\infty_{s,t}\ X \longrightarrow \Sigma^\infty_{s,t}\ Y \longrightarrow \Sigma^\infty_{s,t}\ Y/X \longrightarrow \Sigma^\infty_{s,t}\ X[1]$$

is a distinguished triangle in $\mathrm{SH}(k)$*.*

Proof. The model structure in Remark 2.11 shows that the canonically induced map

$$\mathrm{Cone}(\Sigma^\infty_{s,t}(X \rightarrowtail Y)) \longrightarrow \Sigma^\infty_{s,t}\ Y/X$$

is a stable weak equivalence. □

Remark 2.17 *Note that 2.16 applies to open and closed embeddings in* Sm/k*.*

Definition 2.18 *An upper distinguished square is a pullback square in* Sm/k

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

where i is an open embedding, p is an etale map, and

$$p|_{p^{-1}(X \smallsetminus U)} \colon p^{-1}(X \smallsetminus U) \longrightarrow (X \smallsetminus U)$$

induces an isomorphism of reduced schemes.

Any Zariski open covering gives an example of an upper distinguished square: If $X = U \cup V$, let $W = U \cap V$.

The Nisnevich topology is discussed in detail in [Lev]. The generating coverings are of the form

$$\{i \colon U \longrightarrow X, p \colon V \longrightarrow X\} \ .$$

where i, p and $W = p^{-1}(U)$ form an upper distinguished square.

Since spaces are sheaves in the Nisnevich topology, we get:

Lemma 2.19 *A square of representable spaces which is obtained from an upper distinguished square is a pushout square.*

In the following, we show that upper distinguished squares give examples of distinguished triangles in SH(k). Part (a) of the next result can be thought of as a generalized Mayer-Vietoris property:

Corollary 2.20 *For an upper distinguished square, the following holds.*

(a) There is a distinguished triangle in SH(k)*:*

$$\Sigma^\infty_{s,t}\ W_+ \longrightarrow \Sigma^\infty_{s,t}\ U_+ \vee \Sigma^\infty_{s,t}\ V_+ \longrightarrow \Sigma^\infty_{s,t}\ X_+ \longrightarrow \Sigma^\infty_{s,t}\ W_+[1] \ .$$

(b) There is a naturally induced stable weak equivalence:

$$\Sigma^\infty_{s,t}\ V/W \longrightarrow \Sigma^\infty_{s,t}\ X/U \ .$$

Proof. By Remark 2.17 and Lemma 2.19, the pushout square of representable spaces associated to an upper distinguished square is a homotopy pushout square. This implies item (a) because the suspension functor $\Sigma^\infty_{s,t}$ preserves homotopy pushout squares.

To prove (b), we use 2.19 to conclude there is even an isomorphism between the underlying pointed spaces. □

In the next result, the projective line $\mathbb{P}^1$ is pointed by the rational point at infinity. Another piece of useful information about mixed spheres is:

Lemma 2.21 *In* SH(k)*, there are canonical isomorphisms:*

(a) $\Sigma^\infty_{s,t}\ (\mathbb{A}^1/\mathbb{A}^1 \smallsetminus 0) = \Sigma^\infty_{s,t}\ (S^1_s \wedge S^1_t)$.

(b) $\Sigma^\infty_{s,t}\ (\mathbb{P}^1, \infty) = \Sigma^\infty_{s,t}\ (S^1_s \wedge S^1_t)$.

Proof. To prove (a), we use 2.12, and 2.16 to conclude there is a distinguished triangle:

$$\Sigma^\infty_{s,t}\,(\mathbb{A}^1 \smallsetminus 0, 1) \longrightarrow \Sigma^\infty_{s,t}\,(\mathbb{A}^1, 1) \longrightarrow \Sigma^\infty_{s,t}\,(\mathbb{A}^1/\mathbb{A}^1 \smallsetminus 0) \longrightarrow \Sigma^\infty_{s,t}\,(\mathbb{A}^1 \smallsetminus 0, 1)[1]\,.$$

To prove (b), cover the projective line by two affine lines; the choice of a point on $\mathbb{P}^1$ is not important, since all such points are $\mathbb{A}^1$-homotopic. Moreover, by 2.20(a), there is a distinguished triangle

$$\Sigma^\infty_{s,t}\ (\mathbb{A}^1 \smallsetminus 0)_+ \longrightarrow \Sigma^\infty_{s,t}\ \mathbb{A}^1_+ \vee \Sigma^\infty_{s,t}\ \mathbb{A}^1_+ \longrightarrow \Sigma^\infty_{s,t}\ \mathbb{P}^1_+ \longrightarrow \Sigma^\infty_{s,t}\ (\mathbb{A}^1 \smallsetminus 0)_+[1]\ .$$

To remove the disjoint base-points, we point all spaces by $1\colon \mathrm{Spec}(k) \to \mathbb{A}^1 \smallsetminus 0$ and consider the quotients. For instance, we have

$$\Sigma^\infty_{s,t}(\mathbb{A}^1 \smallsetminus 0)_+/\Sigma^\infty_{s,t}\mathrm{Spec}(k)_+ = \Sigma^\infty_{s,t}(\mathbb{A}^1 \smallsetminus 0, 1)\ .$$

By considering the resulting distinguished triangle, the claim follows from homotopy invariance 2.12. □

Remark 2.22 *The pointed space* $T\colon = \mathbb{A}^1/\mathbb{A}^1 \smallsetminus 0$ *is called the Tate sphere. There is also a 'motivic unstable homotopy category' for spaces, and 2.21 holds unstably. We refer the reader to Sect. 5.2 for the unstable theory.*

We place emphasis on a typical instance of 2.20(b):

Corollary 2.23 *Suppose there is an etale morphism*

$$p\colon V \longrightarrow X\ ,$$

in Sm$/k$*, and* Z *is a closed sub-scheme of* X *such that there is an isomorphism*

$$p^{-1}(Z) \longrightarrow Z\ .$$

Then there is a canonical isomorphism in SH(k)*:*

$$\Sigma^\infty_{s,t}\ (V/V \smallsetminus p^{-1}(Z)) = \Sigma^\infty_{s,t}\ (X/X \smallsetminus Z)\ .$$

As in classical homotopy theory, there is also a notion of Thom spaces in motivic homotopy theory.

Definition 2.24 *Suppose there is a vector bundle $\mathcal{E} \to X$ in Sm/k, with zero section $i\colon X \to \mathcal{E}$. Its Thom space is defined by setting*

$$\mathrm{Th}(\mathcal{E}/X) := \mathcal{E}/\mathcal{E} \smallsetminus i(X) \; ,$$

pointed by the image of $\mathcal{E} \smallsetminus i(X)$.

Example 2.25 *We give some examples of Thom spaces.*

(a) Suppose we have given vector bundles

$$\mathcal{E}_1 \longrightarrow X_1, \mathcal{E}_2 \longrightarrow X_2$$

in Sm/k. Then, as pointed spaces

$$\mathrm{Th}(\mathcal{E}_1 \times \mathcal{E}_2/X_1 \times X_2) = \mathrm{Th}(\mathcal{E}_1/X_1) \wedge \mathrm{Th}(\mathcal{E}_2/X_2) \; .$$

(b) The Thom space of the trivial 1-bundle

$$\mathbb{A}^1 \times X \longrightarrow X$$

is the smash product $T \wedge X_+$.

Theorem 2.26 (Homotopy Purity) *Suppose there is a closed embedding in Sm/k*

$$i\colon Z \hookrightarrow X \; .$$

Denote the corresponding normal vector bundle of Z in X by $\mathbb{N}_{X,Z}$.

Then there is a canonical isomorphism in $\mathrm{SH}(k)$:

$$\Sigma^\infty_{s,t}\, \mathrm{Th}(\mathbb{N}_{X,Z}) = \Sigma^\infty_{s,t}\, (X/X \smallsetminus i(Z)) \; .$$

To prove the general case of the homotopy purity theorem, one employs the well-known deformation to the normal cone construction based on the blow-up of $X \times \mathbb{A}^1$ with center in $Z \times \{0\}$. This is an algebraic analog of the notion of tubular neighborhood in topology. In what follows, by 'isomorphism' we mean a canonical isomorphism in $\mathrm{SH}(k)$.

Next we construct the isomorphism in the homotopy purity theorem for any finite separable field extension k'/k. By the Primitive Element Theorem, there exists an element α such that $k' = k(\alpha)$. Consider the surjective map

$$\phi\colon k[X] \longrightarrow k', X \longmapsto \alpha \; .$$

If f is the minimal polynomial of α, then the induced closed embedding

$$i\colon \mathrm{Spec}(k') \hookrightarrow \mathbb{A}^1_k$$

sends the closed point of $\mathrm{Spec}(k')$ to the closed point x of $\mathbb{A}^1_k$, corresponding to the prime ideal (f). We claim there is an isomorphism

$$\Sigma^\infty_{s,t}\ \mathrm{Th}(\mathbb{N}_{\mathbb{A}^1_k,k'}) = \Sigma^\infty_{s,t}\ (\mathbb{A}^1_k/\mathbb{A}^1_k \smallsetminus \{x\})\ .$$

With respect to the identification of the normal bundle of i with $\mathbb{A}^1_{k'}$, the Thom space $\mathrm{Th}(\mathbb{N}_{\mathbb{A}^1_k,k'})$ is isomorphic to $\mathbb{A}^1_{k'}/\mathbb{A}^1_{k'} \smallsetminus \{0\}$.

In $k'[X]$, f factors into irreducible polynomials, say $f_1,\ldots,f_q$. Denote by $x_1,\ldots,x_q$ the corresponding closed points. Since the extension k'/k is separable, we may assume $f_1 = X-\alpha$ and $f_i(\alpha)\neq 0$ if $i\neq 1$. We have an automorphism

$$k'[X] \longrightarrow k'[X], X \longmapsto X-\alpha\ .$$

By the above, there is an isomorphism

$$\Sigma^\infty_{s,t}\ \mathrm{Th}(\mathbb{N}_{\mathbb{A}^1_k,k'}) = \Sigma^\infty_{s,t}\ (\mathbb{A}^1_{k'}/\mathbb{A}^1_{k'} \smallsetminus \{x_1\})\ .$$

We note that

$$\mathbb{A}^1_{k'} \smallsetminus \{x_1\} \hookrightarrow \mathbb{A}^1_{k'}$$

fits into the upper distinguished square:

$$\begin{array}{ccc}
\mathbb{A}^1_{k'} \smallsetminus \{x_1,x_2,\ldots,x_q\} & \longrightarrow & \mathbb{A}^1_{k'} \smallsetminus \{x_2,\ldots,x_q\} \\
\downarrow & & \downarrow \\
\mathbb{A}^1_{k'} \smallsetminus \{x_1\} & \longrightarrow & \mathbb{A}^1_{k'}
\end{array}$$

By applying 2.20(b), we find

$$\Sigma^\infty_{s,t}\ \mathrm{Th}(\mathbb{N}_{\mathbb{A}^1_k,k'}) = \Sigma^\infty_{s,t}\ (\mathbb{A}^1_{k'} \smallsetminus \{x_2,\ldots,x_q\}/\mathbb{A}^1_{k'} \smallsetminus \{x_1,x_2,\ldots,x_q\})\ .$$

The right hand side of this isomorphism is related to $\mathbb{A}^1_k/\mathbb{A}^1_k \smallsetminus \{x\}$ via the upper distinguished square

$$\begin{array}{ccc}
\mathbb{A}^1_{k'} \smallsetminus \{x_1,x_2,\ldots,x_q\} & \longrightarrow & \mathbb{A}^1_{k'} \smallsetminus \{x_2,\ldots,x_q\} \\
\downarrow & & \downarrow \\
\mathbb{A}^1_{k} \smallsetminus \{x\} & \longrightarrow & \mathbb{A}^1_{k}
\end{array}$$

which defines an isomorphism

$$\Sigma^\infty_{s,t}\ (\mathbb{A}^1_{k'} \smallsetminus \{x_2,\ldots,x_q\}/\mathbb{A}^1_{k'} \smallsetminus \{x_1,x_2,\ldots,x_q\}) = \Sigma^\infty_{s,t}\ (\mathbb{A}^1_k/\mathbb{A}^1_k \smallsetminus \{x\})\ .$$

By using the fact that the isomorphism of 2.26 is compatible in the obvious sense with etale morphisms, one shows that the isomorphism we constructed above coincides with the isomorphism in the homotopy purity theorem.

Remark 2.27 *Given a finite etale map*

$$Y \longrightarrow X\ ,$$

the proof of 2.26 implies there exists, in $\mathrm{SH}(k)$*, a transfer map*

$$\Sigma^\infty_{s,t}\ X_+ \longrightarrow \Sigma^\infty_{s,t}\ Y_+\ .$$

3 Cohomology Theories

The introduction of the motivic stable homotopy category via spectra provides a convenient framework for defining cohomology theories of algebraic varieties. Such theories encode important data about the input, often in a form which allows to make algebraic manipulations. We shall consider the examples of motivic cohomology, algebraic K-theory and algebraic cobordism.

3.1 The Motivic Eilenberg-MacLane Spectrum $\mathbf{H}\mathbb{Z}$

In stable homotopy theory, the key to understand cohomology theories is to study their representing spectra. Singular cohomology is a prime example. This motivates the construction of the motivic Eilenberg-MacLane spectrum, which we denote by $\mathbf{H}\mathbb{Z}$.

Let $K(\mathbb{Z},n)$ denote the Eilenberg-MacLane simplicial set with homotopy groups

$$\pi_i K(\mathbb{Z},n) = \begin{cases} \mathbb{Z} & i = n, \\ 0 & i \neq n. \end{cases} \tag{1}$$

The simplicial set $K(\mathbb{Z},n)$ is uniquely determined up to weak equivalence by (1). A model for $K(\mathbb{Z},n)$ can be constructed as follows: Denote by $\mathbb{Z}(-)$ the functor which associates to a pointed simplicial set (X,x) the simplicial abelian group $\mathbb{Z}[X]/\mathbb{Z}[x]$. That is, the simplicial free abelian group generated by X modulo the copy of the integers generated by the base-point. Then $K(\mathbb{Z},n)$ is the underlying simplicial set of the simplicial abelian group $\mathbb{Z}(\Delta[n]/\partial\Delta[n])$. Moreover, the $K(\mathbb{Z},n)$'s assemble to define a spectrum $H\mathbb{Z}$ which represents singular cohomology: The integral singular cohomology group $H^n(X,\mathbb{Z})$ coincides with $\mathrm{Hom}_{\mathrm{SH}}(\Sigma^\infty X_+, H\mathbb{Z})$.

In motivic homotopy theory, there is a closely related algebro-geometric analog of the spectrum $H\mathbb{Z}$. But as it turns out, it is impossible to construct an (s,t)-bispectrum whose constituent terms satisfy a direct motivic analog of (1); however, by using the theory of algebraic cycles one can define an (s,t)-spectrum $\mathbf{H}\mathbb{Z}$ that represents motivic cohomology. Next, we will indicate the construction of $\mathbf{H}\mathbb{Z}$.

If $X \in \mathrm{Sm}/k$, let $L(X)$ be the following functor: Its value on $U \in \mathrm{Sm}/k$ is the free abelian group generated by closed irreducible subsets of $U \times X$ which are finite over U and surjective over a connected component of U. For simplicity, we refer to elements in this group as cycles. The graph $\Gamma(f)$ of a morphism

$$f\colon U \longrightarrow X$$

is an example of a cycle in $U \times X$.

It turns out that $L(X)$ is a Nisnevich sheaf, hence a pointed space by forgetting the abelian group structure. Moreover, there is a map

$$\Gamma(X)\colon X \longrightarrow L(X)\ .$$

One can extend $L(-)$ to a functor from pointed spaces to Nisnevich sheaves with values in simplicial abelian groups.[5] For example, we have that

$$L(S^0_s \wedge S^1_t) = L((\mathbb{A}^1 \smallsetminus 0, 1))$$

is the quotient sheaf of abelian groups $L(\mathbb{A}^1 \smallsetminus 0)/L(\mathrm{Spec}(k))$, considered as a pointed space. The pointed space $L(S^1_s \wedge S^1_t)$ turns out to be equivalent to

$$L(\mathbb{P}^1, \infty) = L(\mathbb{P}^1)/L(\mathrm{Spec}(k)) .$$

Remark 3.1 *One can show that $L(\mathbb{P}^1, \infty)$ is weakly equivalent to the infinite projective space $\mathbb{P}^\infty$. Hence, if k admits a complex embedding, taking complex points yields an equivalence*

$$L(\mathbb{P}^1, \infty)(\mathbb{C}) \sim \mathbb{CP}^\infty = K(\mathbb{Z}, 2).$$

The exterior product of cycles induces a pairing

$$L(S^p_s \wedge S^q_t) \wedge L(S^m_s \wedge S^n_t) \longrightarrow L(S^{p+m}_s \wedge S^{q+n}_t) .$$

In particular, we obtain the composite maps

$$\sigma_s \colon S^1_s \wedge L(S^m_s \wedge S^n_t) \xrightarrow{\Gamma(S^1_s)\wedge \mathrm{id}} L(S^1_s) \wedge L(S^m_s \wedge S^n_t) \longrightarrow L(S^{m+1}_s, S^n_t) ,$$
$$\sigma_t \colon S^1_t \wedge L(S^m_s \wedge S^n_t) \xrightarrow{\Gamma(S^1_t)\wedge \mathrm{id}} L(S^1_t) \wedge L(S^m_s \wedge S^n_t) \longrightarrow L(S^m_s, S^{n+1}_t) . \tag{2}$$

Remark 3.2 *A topologically inclined reader might find it amusing to compare the above with Bökstedt's notion of functors with smash products.*

Definition 3.3 *The Eilenberg-MacLane spectrum $\mathbf{H}\mathbb{Z}$ is the (s,t)-bispectrum with constituent pointed spaces $\mathbf{H}\mathbb{Z}_{m,n} \colon = L(S^m_s \wedge S^n_t)$ and structure maps given by (2).*

Remark 3.4 *Let ℓ be a prime number. The Eilenberg-MacLane spectrum with mod-ℓ coefficients is defined as above by taking the reduction of $L(X)$ modulo ℓ in the category of abelian sheaves.*

We can now define the motivic cohomology groups of (s,t)-bispectra.

Definition 3.5 *Let E be an (s,t)-bispectrum and let p, q be integers.*
The integral motivic cohomology groups of E are defined by

$$\mathbf{H}\mathbb{Z}^{p,q}(E) \colon = \mathrm{Hom}_{\mathrm{SH}(k)}(E, S^{p-q}_s \wedge S^q_t \wedge \mathbf{H}\mathbb{Z}) .$$

[5] Since every pointed space is a colimit of representable functors and L preserves colimits, it suffices to describe the values of L on Sm/k.

Suppose $X \in \mathrm{Sm}/k$. The integral motivic cohomology group $\mathbf{H}^{p,q}(X,\mathbb{Z})$ of X in degree p and weight q is by definition $\mathbf{H}\mathbb{Z}^{p,q}(\Sigma^\infty_{s,t}X_+)$. One can show that $\mathbf{H}^{p,q}(X,\mathbb{Z})$ is isomorphic to the higher Chow group $\mathrm{CH}^q(X,2q-p)$ introduced by Bloch in [Blo86].

In Sect. 4, we shall outline an approach to construct a spectral sequence whose E_2-terms are the integral motivic cohomology groups of X. Its target groups are the algebraic K-groups of X. Since there is a spectral sequence for topological K-theory whose input terms are singular cohomology groups, this would allow to make more precise the analogy between motivic and singular cohomology.

3.2 The Algebraic K-Theory Spectrum KGL

In 2.21(b), we noted that the suspension (s,t)-bispectra of $(\mathbb{P}^1,\infty)$ and $S^1_s \wedge S^1_t$ are canonically isomorphic in $\mathrm{SH}(k)$. In fact, we may replace the suspension coordinates S^1_s and S^1_t by $\mathbb{P}^1$ without introducing changes in the motivic stable homotopy category. In order to represent cohomology theories on Sm/k, it turns out to be convenient to consider $\mathbb{P}^1$-spectra. To define the category of such spectra, one replaces in 2.1 every occurrence of the simplicial circle by the projective line. If E is a $\mathbb{P}^1$-spectrum, we may associate a bigraded cohomology theory by setting

$$E^{p,q}(X){:}\;= \mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}X_+, E \wedge S_s^{p-2q} \wedge (\mathbb{P}^1)^{\wedge q})\;. \tag{3}$$

An important example is the spectrum representing algebraic K-theory, which we describe next.

If m is a non-negative integer, we denote the Grassmannian of vector spaces of dimension n in the $n+m$-dimensional vector space over k by $\mathrm{Gr}_n(\mathbb{A}^{n+m})$. By letting m tend to infinity, it results a directed system of spaces; denote the colimit by BGL_n. There are canonical monomorphisms

$$\cdots \hookrightarrow \mathrm{BGL}_n \hookrightarrow \mathrm{BGL}_{n+1} \hookrightarrow \cdots\;.$$

Denote by BGL the sequential colimit of this diagram.

Let KGL be a fibrant replacement of

$$\mathbb{Z} \times \mathrm{BGL}$$

in the unstable motivic homotopy theory of k. Then there exists a map

$$\beta{:}\; \mathbb{P}^1 \wedge \mathrm{KGL} \longrightarrow \mathrm{KGL}\;, \tag{4}$$

which represents the canonical Bott element of

$$K_0(\mathbb{P}^1 \wedge (\mathbb{Z} \times \mathrm{BGL}))\;.$$

More precisely, the map (4) is adjoint to a lift of the isomorphism

$$\mathbb{Z} \times \mathrm{BGL} \longrightarrow \Omega_{\mathbb{P}^1}(\mathbb{Z} \times \mathrm{BGL})$$

in the unstable motivic homotopy category which induces Bott periodicity in algebraic K-theory. Details are recorded in [Voe98].

Definition 3.6 *The algebraic K-theory spectrum* **KGL** *is the $\mathbb{P}^1$-spectrum*

$$(\mathrm{KGL}, \mathrm{KGL}, \cdots, \mathrm{KGL}, \dots)\ ,$$

together with the structure maps in (4).

As in topological K-theory, there is also Bott periodicity in algebraic K-theory: The structure maps of **KGL** are defined by lifting an isomorphism in the unstable motivic homotopy category, so there is an isomorphism

$$\mathbb{P}^1 \wedge \mathbf{KGL} = \mathbf{KGL}\ . \tag{5}$$

If k admits a complex embedding, taking $\mathbb{C}$-points defines a realization functor

$$t_{\mathbb{C}} \colon \mathrm{SH}(k) \longrightarrow \mathrm{SH}\ .$$

This functor sends $\Sigma^{\infty}_{s,t} S^{p,q}$ to the suspension spectrum of the p-sphere and **KGL** to the ordinary complex topological K-theory spectrum.

3.3 The Algebraic Cobordism Spectrum MGL

In what follows, we use the notation in 3.2. Denote the tautological vector bundle over the Grassmannian by

$$\gamma_{n,m} \longrightarrow \mathrm{Gr}_n(\mathbb{A}^{n+m})$$

The canonical morphism

$$\mathrm{Gr}_n(\mathbb{A}^{n+m}) \longrightarrow \mathrm{Gr}_n(\mathbb{A}^{n+m+1})$$

is covered by a bundle map $\gamma_{n,m} \longrightarrow \gamma_{n,m+1}$. Taking the colimit over m yields the universal n-dimensional vector bundle

$$\gamma_n \longrightarrow \mathrm{BGL}_n\ .$$

The product

$$\mathbb{A}^1 \times \gamma_n \longrightarrow \mathrm{BGL}_n$$

with the trivial one-dimensional bundle

$$\mathbb{A}^1 \longrightarrow \mathrm{Spec}(k)$$

is classified by the canonical map

$$\mathrm{BGL}_n \longrightarrow \mathrm{BGL}_{n+1} \ .$$

In particular, there exists a bundle map

$$\mathbb{A}^1 \times \gamma_n \longrightarrow \gamma_{n+1} \ .$$

On the level of Thom spaces, we obtain the map

$$\mathrm{Th}(\mathbb{A}^1) \wedge \mathrm{Th}(\gamma_n) = (\mathbb{A}^1/\mathbb{A}^1 \smallsetminus 0) \wedge \mathrm{Th}(\gamma_n) \longrightarrow \mathrm{Th}(\gamma_{n+1}) \ . \qquad (6)$$

Here, note that we may apply 2.25(a) since the map between γ_n and BGL_n is a colimit of vector bundles of smooth schemes. From (6) and Remark 2.22, we get

$$c_n \colon \mathbb{P}^1 \wedge \mathrm{Th}(\gamma_n) \longrightarrow \mathrm{Th}(\gamma_{n+1}) \ . \qquad (7)$$

Definition 3.7 *The algebraic cobordism spectrum* **MGL** *is the* $\mathbb{P}^1$*-spectrum*

$$(\mathrm{Th}(\gamma_0), \mathrm{Th}(\gamma_1), \ldots, \mathrm{Th}(\gamma_n), \ldots) \ ,$$

together with the structure maps in (7).

The algebraic cobordism spectrum is the motivic analog of the ordinary complex cobordism spectrum MU. One can check that

$$t_{\mathbb{C}}(\mathbf{MGL}) = \mathrm{MU} \ .$$

The notions of orientation, and formal group laws in ordinary stable homotopy theory have direct analogs for $\mathbb{P}^1$-spectra.

4 The Slice Filtration

In classical homotopy theory, the Eilenberg-MacLane space $K(\mathbb{Z}, n)$ has a unique non-trivial homotopy group. And up to homotopy equivalence there is a unique such space for each n (1). The situation in motivic homotopy theory is quite different. For example, the homotopy groups $\pi_{p,q}(\mathbf{H}\mathbb{Z})$ are often non-zero, as one may deduce from the isomorphism between $\pi_{q,q}(\mathbf{H}\mathbb{Z})$ and the Milnor K-theory of k. To give an internal description of $\mathbf{H}\mathbb{Z}$ within the stable motivic homotopy category, we employ the so-called slice filtration. In what follows, we recall and discuss the status of Voevodsky's conjectures about the slices of the sphere spectrum $\mathbf{1} = \Sigma^\infty_{s,t}\mathrm{Spec}(k)_+$, $\mathbf{H}\mathbb{Z}$, and **KGL**. For more details, we refer to the original papers [Voe02b] and [Voe02c].

Let $\mathrm{SH}^{\mathrm{eff}}(k)$ denote the smallest triangulated sub-category of $\mathrm{SH}(k)$ which is closed under direct sums and contains all (s,t)-bispectra of the form $\Sigma^\infty_{s,t}X_+$. If $n \geq 1$, the desuspension spectrum $\Sigma^{-n}_t\Sigma^\infty_{s,t}X_+$ is not contained in $\mathrm{SH}^{\mathrm{eff}}(k)$.

The 'effective' s-stable homotopy category $\mathrm{SH}^{\mathrm{eff}}_s(k)$ is defined similarly, by replacing $\Sigma^\infty_{s,t}$ with the s-suspension Σ^∞_s.

In this section, we shall study the sequence of full embeddings of categories

$$\cdots \hookrightarrow \Sigma^1_{s,t}\mathrm{SH}^{\mathrm{eff}}(k) \hookrightarrow \mathrm{SH}^{\mathrm{eff}}(k) \hookrightarrow \Sigma^{-1}_{s,t}\mathrm{SH}^{\mathrm{eff}}(k) \hookrightarrow \cdots . \tag{8}$$

The sequence (8) is called the slice filtration. For an alternative formulation of the slice filtration, consider [Lev03]. The above is a filtration in the sense that $\mathrm{SH}(k)$ is the smallest triangulated category which contains $\Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)$ for all n and is closed under arbitrary direct sums in $\mathrm{SH}(k)$. For each n, the category $\Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)$ is triangulated. Moreover, this category has arbitrary direct sums and a set of compact generators.

Neeman's work on triangulated categories in [Nee96] shows that the full inclusion functor

$$i_n\colon \Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k) \hookrightarrow \mathrm{SH}(k)$$

has a right adjoint

$$r_n\colon \mathrm{SH}(k) \longrightarrow \Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)\ .$$

such that the unit of the adjunction is an isomorphism

$$\mathrm{Id} \longrightarrow r_n \circ i_n\ .$$

Consider now the reverse composition

$$f_n\colon \mathrm{SH}(k) \xrightarrow{r_n} \Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k) \xrightarrow{i_n} \mathrm{SH}(k)\ .$$

The counit

$$f_{n+1} \longrightarrow \mathrm{Id}$$

applied to the functor f_n determines a natural transformation

$$f_{n+1} = f_{n+1} \circ f_n \longrightarrow f_n\ .$$

If E is an (s,t)-bispectrum and $n \in \mathbb{Z}$, then the slice tower of E consists of the distinguished triangles in $\mathrm{SH}(k)$

$$f_{n+1}E \longrightarrow f_nE \longrightarrow s_nE \longrightarrow f_{n+1}E[1]\ . \tag{9}$$

Definition 4.1 *The n-th slice of E is s_nE.*

Remark 4.2 *Since $f_{n+1}E, f_nE \in \Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)$, we get that $s_nE \in \Sigma^n_{s,t}$ $\mathrm{SH}^{\mathrm{eff}}(k)$. The adjunction above implies that s_nE receives only the trivial map from $\Sigma^{n+1}_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)$. Standard arguments show that these properties characterize, up to canonical isomorphism, the triangulated functors*

$$s_n \colon \mathrm{SH}(k) \longrightarrow \mathrm{SH}(k) .$$

Suppose that $E \in \Sigma^n_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)$ *and let* $k \leq n$*. Then* $f_k E = E$ *and the* k*-th slice* $s_k E$ *of* E *is trivial for all* $k < n$*.*

Slice towers are analogous to Postnikov towers in algebraic topology; the slices corresponding to the cofibers or quotients. If E is an ordinary spectrum, recall that its Postnikov tower expresses E as the sequential colimit of a diagram of cofibrations

$$\cdots \longrightarrow P_{-1}E \xrightarrow{p_0} P_0 E \xrightarrow{p_1} P_1 E \longrightarrow \cdots \xrightarrow{p_n} P_n E \xrightarrow{p_{n+1}} \cdots .$$

The canonical map

$$P_n E \longrightarrow E$$

induces isomorphisms on all stable homotopy groups π_i and $i \leq n$, whereas for $i > n$ we have

$$\pi_i(P_n E) = 0 .$$

It follows that the cofiber of p_n is an Eilenberg-MacLane spectrum $\Sigma^n H\pi_n(E)$; hence, in particular an $H\mathbb{Z}$-module. Now recall that the zeroth stage of the Postnikov tower of the ordinary sphere spectrum is the Eilenberg-MacLane spectrum $H\mathbb{Z}$.

This gives some topological motivation for the following conjecture, which in turn implies a characterization of motivic cohomology entirely in terms of the motivic stable homotopy category.

Conjecture 1. $s_0\mathbf{1} = \mathbf{H}\mathbb{Z}$.

The collection of the functors s_n is compatible with the smash product, meaning that if E and F are objects of $\mathrm{SH}(k)$, there is a natural map

$$s_n(E) \wedge s_m(F) \longrightarrow s_{n+m}(E \wedge F) .$$

In particular, we get a map

$$s_0(\mathbf{1}) \wedge s_n(E) \longrightarrow s_n(E) .$$

If Conjecture 1 holds, then the above allows us to conclude that each slice of an (s,t)-bispectrum has a natural and unique structure of a module over the motivic cohomology spectrum.

Theorem 4.3 *Conjecture 1 holds for all fields of characteristic zero.*

Remark 4.4 *According to [Lev03, Theorem 8.4.1], Theorem 4.5 holds for every perfect field.*

In our sketch proof of 4.3, we start with the zeroth slice of $\mathbf{H}\mathbb{Z}$.

Lemma 4.5 *Let k be a field of characteristic zero. Then $s_0\mathbf{H}\mathbb{Z} = \mathbf{H}\mathbb{Z}$.*

To prove 4.5, we will make use of the following facts: First, the motivic Eilenberg-MacLane spectrum is an effective spectrum. Thus

$$f_0\mathbf{H}\mathbb{Z} = \mathbf{H}\mathbb{Z} .$$

Second, if X is a scheme in Sm/k, then every map in $\mathrm{SH}(k)$ from $\Sigma^1_{s,t}\Sigma^\infty_{s,t} X_+$ to $\mathbf{H}\mathbb{Z}$ is trivial by Voevodsky's cancellation theorem [Voe02a].[6]

Hence, for every $E \in \Sigma^1_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)$, we get isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\Sigma^1_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)}(E, r_1\mathbf{H}\mathbb{Z}) &= \mathrm{Hom}_{\mathrm{SH}(k)}(i_1E, \mathbf{H}\mathbb{Z}) \\ &= 0 . \end{aligned}$$

This proves

$$r_1\mathbf{H}\mathbb{Z} = 0 .$$

In particular, $f_1\mathbf{H}\mathbb{Z} = 0$ and hence

$$\begin{aligned} s_0\mathbf{H}\mathbb{Z} &= f_0\mathbf{H}\mathbb{Z} \\ &= \mathbf{H}\mathbb{Z} . \end{aligned}$$

Denote by $\mathcal{C}$ the cone of the unit map

$$\mathbf{1} \longrightarrow \mathbf{H}\mathbb{Z} .$$

To finish the proof of 4.3, note that by 4.5 it remains to show

$$s_0\mathcal{C} = 0 .$$

Remark 4.2 shows it suffices to prove that $\mathcal{C}$ is contained in $\Sigma^1_{s,t}\mathrm{SH}^{\mathrm{eff}}(k)$.

If k is a field of characteristic zero, then $\mathbf{H}\mathbb{Z}$ has an explicit description in terms of infinite symmetric products of $\mathbb{A}^n$ and $\mathbb{A}^n \smallsetminus 0$. This allows to conclude the statement about $\mathcal{C}$. We shall sketch a proof.

Recall that

$$\mathbf{H}\mathbb{Z}_{n,n} = L(S^n_s \wedge S^n_t)$$

is weakly equivalent to the quotient sheaf $L(\mathbb{A}^n)/L(\mathbb{A}^n \smallsetminus 0)$.

Let $L^{\mathrm{eff}}(\mathbb{A}^n)$ denote the sheaf which maps $U \in \mathrm{Sm}/k$ to the free abelian monoid generated by closed irreducible subsets of $U \times \mathbb{A}^n$ which are finite over U and also surjective over a connected component of U.

The sheaf $L^{\mathrm{eff}}(\mathbb{A}^n \smallsetminus 0)$ is defined similarly. In particular, L^{eff} consists of cycles with nonnegative coefficients. Denote the quotient $L^{\mathrm{eff}}(\mathbb{A}^n)/L^{\mathrm{eff}}(\mathbb{A}^n \smallsetminus 0)$

[6] In other words, motivic cohomology of X in weight -1 is zero.

by $\mathbf{H}\mathbb{Z}^{\text{eff}}_{n,n}$. It is straightforward to define $\mathbf{H}\mathbb{Z}^{\text{eff}}_{m,n}$ and moreover note that, along the same lines as above, these pointed spaces form an (s,t)-bispectrum $\mathbf{H}\mathbb{Z}^{\text{eff}}$.

If $n \geq 1$, the canonical map

$$\mathbf{H}\mathbb{Z}^{\text{eff}}_{n,n} \longrightarrow \mathbf{H}\mathbb{Z}_{n,n}$$

turns out to be a weak equivalence of pointed spaces.

Furthermore, the canonical inclusion

$$\mathbf{1} \longrightarrow \mathbf{H}\mathbb{Z}$$

factors through

$$\mathbf{1} \longrightarrow \mathbf{H}\mathbb{Z}^{\text{eff}}.$$

Hence the quotient of the latter map is equivalent to $\mathcal{C}$.

Next, we consider $T^n = \mathbb{A}^n/\mathbb{A}^n \smallsetminus 0$. Let $\mathbf{H}\mathbb{Z}^{\leq d}_{n,n}$ be the subsheaf of $\mathbf{H}\mathbb{Z}^{\text{eff}}_{n,n}$ mapping $U \in \mathrm{Sm}/k$ to the cycles of degree $\leq d$ over U. Then the natural inclusion

$$T^n \hookrightarrow \mathbf{H}\mathbb{Z}^{\text{eff}}_{n,n}$$

has a filtration

$$T^n = \mathbf{H}\mathbb{Z}^{\leq 1}_{n,n} \hookrightarrow \mathbf{H}\mathbb{Z}^{\leq 2}_{n,n} \hookrightarrow \cdots \hookrightarrow \mathbf{H}\mathbb{Z}^{\leq d}_{n,n} \hookrightarrow \cdots \hookrightarrow \mathbf{H}\mathbb{Z}^{\text{eff}}_{n,n}.$$

Taking quotients $\mathbf{H}\mathbb{Z}^{\leq d}_{n,n}/\mathbf{H}\mathbb{Z}^{\leq d-1}_{n,n}$ induces a filtration of the quotient sheaf $\mathbf{H}\mathbb{Z}^{\text{eff}}_{n,n}/T^n$. Results of Suslin-Voevodsky [SV96] imply that $\mathbf{H}\mathbb{Z}^{\leq d}_{n,n}/\mathbf{H}\mathbb{Z}^{\leq d-1}_{n,n}$ is isomorphic to the d-th symmetric power sheaf $(T^n)^{\wedge d}/S_d$ of T^n, where S_d denotes the symmetric group on d letters. This uses the characteristic zero assumption.

One can show that $(T^n)^{\wedge d}/S^d$ is contained in the smallest class of pointed spaces which is closed under homotopy colimits and contains all $X/(X-Z)$, where Z is a closed subscheme of $X \in \mathrm{Sm}/S$ of codimension $\geq n+1$. After resolving all singularities in Z, homotopy purity 2.26 implies that the (s,t)-suspension spectrum of any space in this class is contained in $\Sigma^{n+1}_{s,t}\mathrm{SH}^{\text{eff}}(k)$. This ends our sketch proof of 4.3.

Analogous to (8), there exists a slice filtration in $\mathrm{SH}^{\mathbb{A}^1}_s(k)$. For $n \geq 0$, let $\Sigma^n_t \mathrm{SH}^{\mathbb{A}^1}_s(k)$ be the smallest compactly generated triangulated sub-category of $\mathrm{SH}^{\mathbb{A}^1}_s(k)$ which is closed under arbitrary direct sums, and generated by the objects $\Sigma^n_t \Sigma^\infty_s X_+$. We obtain the filtration

$$\cdots \hookrightarrow \Sigma^n_t \mathrm{SH}^{\mathbb{A}^1}_s(k) \hookrightarrow \Sigma^{n-1}_t \mathrm{SH}^{\mathbb{A}^1}_s(k) \cdots \hookrightarrow \Sigma^0_t \mathrm{SH}^{\mathbb{A}^1}_s(k) = \mathrm{SH}^{\mathbb{A}^1}_s(k)\ .$$

The t-suspension functor Σ^∞_t preserves the slice filtrations in the sense that

$$\Sigma^\infty_t(\Sigma^n_t \mathrm{SH}^{\mathbb{A}^1}_s(k)) \subseteq \Sigma^n_{s,t}\mathrm{SH}^{\text{eff}}(k)\ .$$

Denote the right adjoint of this inclusion by

$$\Omega_t^\infty \colon \mathrm{SH}(k) \longrightarrow \mathrm{SH}_s^{\mathbb{A}^1}(k) .$$

Conjecture 2. There is an inclusion $\Omega_t^\infty(\Sigma_{s,t}^n \mathrm{SH}^{\mathrm{eff}}(k)) \subseteq \Sigma_t^n \mathrm{SH}_s^{\mathbb{A}^1}(k)$.

In topology, recall that first applying the suspension functor, and second the loop space functor preserve connectivity. An inductive argument shows that the following implies Conjecture 2.

Conjecture 3. If $X \in \mathrm{Sm}/k$ and $n \geq 0$, then $\Omega_t^1 \Sigma_t^1 \Sigma_s^\infty(S_t^n \wedge X_+) \in \Sigma_t^n \mathrm{SH}_s^{\mathbb{A}^1}(k)$.

This conjecture is not known at present, even if k is a field of characteristic zero. A proof seems to require a fair amount of work. A possible approach to prove Conjecture 3 is to develop an analog of the theory of operads, at least for A_∞-operads, in order to have an explicit model for $\Omega_{\mathbb{P}^1}\Sigma_{\mathbb{P}^1}(X)$. Framed correspondences might be a useful tool in working out such a theory.

At last in the section, we relate the above machinery to a possibly new approach to some recent advances in algebraic K-theory.

In [Bei87], Beilinson conjectured the existence of an Atiyah-Hirzebruch type spectral sequence for the algebraic K-groups of nice schemes. In [BL95], Bloch and Lichtenbaum constructed such a spectral sequence. Their work has been expanded by Friedlander and Suslin [FS02], by Levine [Lev01], and by Suslin [Sus03].

The slice tower (9) acquires an associated spectral sequence. In the example of **KGL**, one obtains

$$\mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma_{s,t}^\infty X_+, S_s^{p-q} \wedge S_t^q \wedge s_0\mathbf{KGL}) \Rightarrow K_{-p-q}(X) . \tag{10}$$

The main problem with this spectral sequence is to identify the input terms in (10) with motivic cohomology.

Conjecture 4. $s_0\mathbf{KGL} = \mathbf{H}\mathbb{Z}$.

Because of Bott periodicity (5), the above describe all the slices of **KGL**. More precisely, one expects

$$s_n\mathbf{KGL} = \Sigma_{s,t}^n \mathbf{H}\mathbb{Z} .$$

As a consequence, Conjecture 4 and (10) would imply the Atiyah-Hirzebruch type spectral sequence

$$\mathbf{H}^{p-q,q}(X,\mathbb{Z}) \Rightarrow K_{-p-q}(X) . \tag{11}$$

Strong convergence of (11) is shown in [Voe02c]. In the same paper, it was noted that Conjectures 1 and 2 imply Conjecture 4.

5 Appendix

In this appendix, written by the second and third author, we shall discuss in some details the homotopy theoretic underpinnings of the theory presented in the main body of this note.

Some results in motivic homotopy theory depend on a characterization of Nisnevich sheaves in terms of upper distinguished squares. For completeness, in the following section we review the connection between upper distinguished squares and the Nisnevich topology, as described by Morel and Voevodsky in [MV99].

This brings us to the topic of model structures for simplicial presheaves on the smooth Nisnevich site of k. Nowadays there exist several such model structures. At a first encounter, the choice of a model structure might be a quite confusing foundational aspect of the theory. However, the flexibility this choice offers clearly outweighs the drudgery involved in learning about the various models; in fact, this development is a result of various quests to improve the machinery leading to the construction of $\mathrm{SH}(k)$. Our exposition follows the paper of Goerss and Jardine [GJ98].

The final section deals with the motivic stable model structure as presented in [Jar00]. In the 1990's topologists discovered model structures with compatible monoidal structures, and such that the associated homotopy categories are all equivalent as monoidal categories to the ordinary stable homotopy category. Motivic stable homotopy theory have in a few years time acquired the same level of technical sophistication as found in ordinary stable homotopy theory. An example of a highly structured model for $\mathrm{SH}(k)$ is the model of motivic functors described in [Dun].

In this appendix, using the Zariski spectrum $\mathrm{Spec}(k)$ of a field k as the base-scheme is mainly for notational convenience. Although smoothness is essential in the proof of the homotopy purity Theorem 2.26, it is not a foundational requirement for setting up motivic model structures. The theory we shall discuss works well for categories of schemes of (locally) finite type over a finite dimensional Noetherian base-scheme.

5.1 The Nisnevich Topology

In Sect. 2, we constructed several examples of distinguished triangles by means of upper distinguished squares:

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{i} & X \end{array} \tag{12}$$

Recall the conditions we impose on (12): i is an open embedding, p is an etale map, and $p^{-1}(X \setminus U) \to (X \setminus U)$ induces an isomorphism of reduced schemes.

Exercise 5.1 *Show that the real affine line* $\mathbb{A}^1_{\mathbb{R}}$, $\mathbb{A}^1_{\mathbb{R}} \smallsetminus \{(x^2+1)\}$, $\mathbb{A}^1_{\mathbb{C}} \smallsetminus \{(x-i)\}$, *and* $\mathbb{A}^1_{\mathbb{C}} \smallsetminus \{(x+i)(x-i)\}$ *define an upper distinguished square. Here, irreducible polynomials are identified with the corresponding closed points on affine lines.*

The Grothendieck topology obtained from the collection of all upper distinguished squares is by definition the smallest topology on Sm/k such that, for each upper distinguished square (12), the sieve obtained from the morphisms i and p is a covering sieve of X, and the empty sieve is a covering sieve of the empty scheme $\emptyset$. Note that for a sheaf F in this topology we have $F(\emptyset) = *$. A sieve of X is a subfunctor of the space represented by X under the Yoneda embedding. A sieve is a covering sieve if and only if it contains a covering arising from an upper distinguished square.

To tie in with the Nisnevich topology, we record the following result due to Morel-Voevodsky [MV99].

Proposition 5.2 *The coverings associated to upper distinguished squares form a basis for the Nisnevich topology on* Sm/k.

In the proof of 5.2, we use the notion of a splitting sequence for coverings: Suppose $X \in \mathrm{Sm}/k$ and the following is a Nisnevich covering

$$\{f_\alpha \colon X_\alpha \longrightarrow X\}_{\alpha \in I} \ . \tag{13}$$

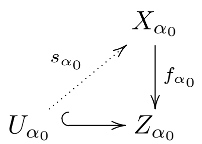
We claim there exists a finite sequence of closed embeddings

$$\emptyset = Z_{\alpha_{n+1}} \longrightarrow Z_{\alpha_n} \longrightarrow \cdots \longrightarrow Z_{\alpha_0} = X \ , \tag{14}$$

and for $0 \leq i \leq n$, $\mathrm{Spec}(k)$-sections s_{α_i} of the natural projections

$$f_{\alpha_i} \times_X (Z_{\alpha_i} \smallsetminus Z_{\alpha_{i+1}}) \colon X_{\alpha_i} \times_X (Z_{\alpha_i} \smallsetminus Z_{\alpha_{i+1}}) \longrightarrow (Z_{\alpha_i} \smallsetminus Z_{\alpha_{i+1}}) \ .$$

To construct the sequence $(Z_{\alpha_i})_{i=0}^{n+1}$ we set $Z_{\alpha_0} := X$. For each generic point x of X, the Nisnevich covering condition requires that there exists an index $\alpha_0 \in I$ and a generic point x_{α_0} of X_{α_0} such that f_{α_0} induces an isomorphism of residue fields $k(x) \to k(x_{\alpha_0})$. The induced morphism of closed integral subschemes corresponding to the generic points is an isomorphism over x, hence an isomorphism over an open neighborhood U_{α_0} of x. It follows that f_{α_0} has a section s_{α_0} over U_{α_0} as shown in the diagram:



Next, set $Z_{\alpha_1} := Z_{\alpha_0} \smallsetminus U_{\alpha_0}$. With this definition, there exists a Nisnevich covering $\{X_\alpha \times_X Z_{\alpha_1} \to Z_{\alpha_1}\}_{\alpha \in I}$. The next step is to run the same argument for Z_{α_1}. Iterating this procedure, we obtain a strictly decreasing sequence of closed subsets of X with the ascribed property. Since X is Noetherian, the sequence will terminate.

We note that the existence of splitting sequences for coverings implies that the Nisnevich topology on Sm/k is Noetherian in the sense that every covering allows a finite refinement. This follows because there is only a finite number of the pairs $(Z_{\alpha_i} \smallsetminus Z_{\alpha_{i+1}})$ and $\{f_{\alpha_i} \colon X_\alpha \to X\}_{i=0}^n$ is a Nisnevich covering.

Next, we sketch a proof of 5.2: First, it is clear that every covering obtained from an upper distinguished square is indeed a Nisnevich covering. Conversely, consider the covering sieve R generated by a Nisnevich covering of X (13) and the corresponding splitting sequence (14). Since X is Noetherian, we may assume that I is finite and the sieve R is obtained from the morphism

$$f \colon \coprod_{\alpha \in I} X_\alpha \longrightarrow X \ .$$

The idea is now to construct an upper distinguished square where the scheme subject to the open embedding i has a splitting sequence of length less than that of (14).

Denote by s the $\mathrm{Spec}(k)$-section of f over $X \smallsetminus Z_{\alpha_n}$. Note that we have obtained the upper distinguished square:

$$\begin{array}{ccc} W & \longrightarrow & V = (\coprod_{\alpha \in I} X_\alpha) \smallsetminus (f^{-1}(X \smallsetminus Z_{\alpha_n}) \smallsetminus \mathrm{Im}(s)) \\ \downarrow & & \downarrow p \\ U = (X \smallsetminus Z_{\alpha_n}) & \xrightarrow{\ i\ } & X \end{array}$$

Here, the splitting sequence associated to the Nisnevich covering

$$\coprod_{\alpha \in I} X_\alpha \times_X (X \smallsetminus Z_{\alpha_n}) \longrightarrow (X \smallsetminus Z_{\alpha_n}) \qquad (15)$$

has length one less than (14).

The covering sieve R is obtained by composing the Nisnevich coverings

$$\{\coprod_{\alpha \in I} X_\alpha \times_X (X \smallsetminus Z_{\alpha_n}) \longrightarrow (X \smallsetminus Z_{\alpha_n}), V = V\} \ , \qquad (16)$$

$$\{U \longrightarrow X, V \longrightarrow X\} \ . \qquad (17)$$

By considering the length of the splitting sequence for (15), we may assume that (16) is a covering in the topology generated by upper distinguished squares. The same holds trivially for (17).

Corollary 5.3 *A presheaf on the smooth Nisnevich site of k is a sheaf if and only if it maps every upper distinguished square to a cartesian diagram of sets and the empty scheme to $*$.*

The empty scheme $\emptyset$ represents a simplicial presheaf on Sm/k. Its value on the empty scheme is $*$, not the empty set, which distinguishes it from the initial presheaf. Recall that a Nisnevich neighborhood of $x \in X$ consists of an etale morphism $f\colon V \to X$ together with a point $v \in f^{-1}(x)$ such that the induced map $k(x) \to k(v)$ is an isomorphism. The Nisnevich neighborhoods of $x \in X$ yield a cofiltering system. Let $\mathcal{O}^h_{X,x}$ denote the henselization of the local ring of X at x. Its Zariski spectrum equals the limit of all Nisnevich neighborhoods of $x \in X$.

When F is a sheaf on the smooth Nisnevich site of k, denote by $F(\mathcal{O}^h_{X,x})$ the filtered colimit of $F(V)$ indexed over all the Nisnevich neighborhoods of x. By restricting to a small skeleton of Sm/k, we obtain a family of conservative points for the Nisnevich topos $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}/k)$, namely

$$F \longmapsto F(\mathcal{O}^h_{X,x}) \ .$$

In other terms, a morphism of sheaves $F \to G$ on the smooth Nisnevich site of k is an isomorphism if and only if $F(\mathcal{O}^h_{X,x}) \to G(\mathcal{O}^h_{X,x})$ is an isomorphism for all $x \in X$.

Denote by $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ the category of simplicial presheaves on the smooth Nisnevich site of k. Recall from [Dun] the notion of weak equivalence between simplicial sets. A morphism $\mathcal{X} \to \mathcal{Y}$ in $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ is called a schemewise weak equivalence if for all $X \in \mathrm{Sm}/k$ there is an induced weak equivalence $\mathcal{X}(X) \to \mathcal{Y}(Y)$. In particular, a morphism of discrete simplicial presheaves is a schemewise weak equivalence if and only if it is an isomorphism. There is the much coarser notion of a stalkwise weak equivalence.

Definition 5.4 *A morphism $\mathcal{X} \to \mathcal{Y}$ in $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ is called a stalkwise weak equivalence if for all $X \in \mathrm{Sm}/k$ and $x \in X$ there is an induced weak equivalence of simplicial sets $\mathcal{X}(\mathcal{O}^h_{X,x}) \to \mathcal{Y}(\mathcal{O}^h_{X,x})$.*

An important observation is that the simplicial presheaves are evaluated at Hensel local rings; this is particular to the Nisnevich topology. If we instead considered the etale topology, we would have evaluated at strict Hensel local rings. In this way, the stalkwise weak equivalences in $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ depend on some of the finest properties of the Nisnevich topology.

Exercise 5.5 *Give an example of a stalkwise weak equivalence which is not a weak equivalence on all local rings. (Hint: An example can be obtained by considering the pushout of the upper distinguished square in Exercise 5.1.)*

So far we have encountered two important properties of the Nisnevich topology: First, the collection of all upper distinguished squares generates the Nisnevich topology. This implies a useful characterization of Nisnevich sheaves. Second, the stalkwise weak equivalences are completely determined by Henselizations of Zariski local rings. These two facts are among the chief reasons why developing motivic homotopy theory in the Nisnevich topology turns out to give a whole host of interesting results.

There exist two other characterizations of stalkwise weak equivalences. To review these, we generalize combinatorial and topological homotopy groups for simplicial sets to the setting of Nisnevich sheaves of homotopy groups of simplicial presheaves. Recall that Kan employed the subdivision functor Ex^{∞} to define combinatorial homotopy groups of simplicial sets without reference to topological spaces [Dun]. We recall Jardine's generalization to simplicial presheaves [Jar87, §1].

First, we require an extension of Ex^{∞} to the simplicial presheaf category $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$. If $\mathcal{X}$ is a simplicial presheaf on the smooth Nisnevich site of k, let $\mathrm{Ex}^m\mathcal{X}$ denote the simplicial presheaf with n-simplices

$$[n] \longmapsto \Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)(\mathrm{sd}^m \Delta[n], \mathcal{X}) \ .$$

In the above expression, sd^m denotes the subdivision functor iterated m times. Its simplicial structure is obtained by precomposition with the simplicial sets maps involving the subdivision sd^m.

Using the natural last vertex maps $\mathrm{sd}\Delta[n] \to \Delta[n]$, for $n \geq 0$, and iterating, we get the diagram

$$\mathcal{X} \longrightarrow \mathrm{Ex}^1\mathcal{X} \longrightarrow \mathrm{Ex}^2\mathcal{X} \longrightarrow \cdots \ . \tag{18}$$

Denote by $\mathrm{Ex}^{\infty}\mathcal{X}$ the colimit of (18) in the presheaf category. There is, by construction, a canonically induced schemewise weak equivalence, and hence stalkwise weak equivalence

$$\mathcal{X} \longrightarrow \mathrm{Ex}^{\infty}\mathcal{X} \ .$$

A morphism $\mathcal{X} \to \mathcal{Y}$ of simplicial presheaves is a local fibration if for every commutative diagram of simplicial set maps

$$\begin{array}{ccc} \Lambda_k[n] & \longrightarrow & \mathcal{X}(X) \\ \big\downarrow & & \big\downarrow \\ \Delta[n] & \longrightarrow & \mathcal{Y}(X) \end{array}$$

there exists a covering sieve $R \subseteq \mathrm{Sm}/k(-, X)$ such that for every $\phi\colon Y \to X$ in R and in every commutative diagram as below, there exists a lift:

$$\begin{array}{ccccc} \Lambda_k[n] & \longrightarrow & \mathcal{X}(X) & \longrightarrow & \mathcal{X}(Y) \\ \big\downarrow & & & \nearrow & \big\downarrow \\ \Delta[n] & \longrightarrow & \mathcal{Y}(X) & \longrightarrow & \mathcal{Y}(Y) \end{array}$$

Local fibrations are the morphisms with the local right lifting property with respect to the inclusions $\Lambda_k[n] \subseteq \Delta[n]$, $n > 0$, of the boundary $\partial\Delta[n]$ having the k-th face deleted from its list of generators. Simplicial presheaf morphisms having the analogously defined local right lifting property with respect to all inclusions $\partial\Delta[n] \subseteq \Delta[n]$, $n > 0$, are local fibrations. We say that $\mathcal{X}$ is locally fibrant if the morphism $\mathcal{X} \to *$ to the simplicial presheaf represented by $\mathrm{Spec}(k)$ is a local fibration. Schemewise Kan fibrations (i.e. morphisms $\mathcal{X} \to \mathcal{Y}$ which for every member X of Sm/k gives a Kan fibration of simplicial sets $\mathcal{X}(X) \to \mathcal{Y}(X)$) are local fibrations. The simplicial presheaf $\mathrm{Ex}^\infty\mathcal{X}$ is a typical example of a locally fibrant object.

Exercise 5.6 *Show that $\mathcal{X} \to \mathcal{Y}$ is a local fibration if and only if for all $X \in \mathrm{Sm}/k$ and $x \in X$, the map $\mathcal{X}(\mathcal{O}^h_{X,x}) \to \mathcal{Y}(\mathcal{O}^h_{X,x})$ is a Kan fibration of simplicial sets.*

Conclude that X is locally fibrant if and only if $\mathcal{X}(\mathcal{O}^h_{X,x})$ is a Kan complex for all $X \in \mathrm{Sm}/k$ and $x \in X$.

When comparing model structures for simplicial presheaves and simplicial sheaves on Sm/k, we shall employ the Nisnevich sheafification functor for presheaves [Lev]. Recall that the functor a_{Nis} is left adjoint to the inclusion

$$\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}/k) \subseteq \mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k).$$

A degreewise application extends it to simplicial presheaves.

Exercise 5.7 *Show that $\mathcal{X} \to a_{\mathrm{Nis}}\mathcal{X}$ is a local fibration by proving that it has the local right lifting property with respect to the inclusions $\partial\Delta[n] \subseteq \Delta[n]$.*

Consider a locally fibrant simplicial presheaf $\mathcal{X}$ on Sm/k and a pair of simplicial set maps

$$\Delta[n] \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{X}(X).$$

Then f is locally homotopic to g if there exists a covering sieve $R \subseteq \mathrm{Sm}/k(-, X)$ such that, for each $\phi\colon Y \to X$ in R, there is a commutative diagram:

$$\begin{array}{ccccc}
\Delta[n] & \xrightarrow{d^0} & \Delta[n] \times \Delta[1] & \xleftarrow{d^1} & \Delta[n] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h_\phi} & & \downarrow{\scriptstyle g} \\
\mathcal{X}(X) & \xrightarrow{\phi^*} & \mathcal{X}(Y) & \xleftarrow{\phi^*} & \mathcal{X}(X)
\end{array}$$

In addition, f and g are locally homotopic relative to $\partial\Delta[n]$ provided each homotopy h_ϕ is constant on $\partial\Delta[n] \subseteq \Delta[n] \times \Delta[1]$ and

$$f|_{\partial\Delta[n]} = g|_{\partial\Delta[n]} .$$

One shows easily that local homotopy relative to $\partial\Delta[n]$ is an equivalence relation for locally fibrant simplicial presheaves [Jar87, Lemma 1.9].

If x is a zero-simplex of $\mathcal{X}(k)$, let $x|X$ be the image of x in $\mathcal{X}(X)_0$ under the canonically induced morphism $\mathcal{X}(\mathrm{Spec}(k)) \to \mathcal{X}(X)$. Consider the following set of all equivalence classes of maps of pairs

$$\big(\Delta[n], \partial\Delta[n]\big) \longrightarrow \big(\mathcal{X}(X), x|X\big)$$

where the equivalence relation is generated by relative local homotopies. For $n \geq 1$, the associated Nisnevich sheaves $\pi_n^{\mathrm{loc}}(\mathcal{X}, x)$ of combinatorial homotopy groups are formed by letting X vary over the Nisnevich site on Sm/k. When $n = 0$, we take the sheaf associated with local homotopy classes of vertices.

As for simplicial sets, a tedious check reveals that $\pi_n^{\mathrm{loc}}(\mathcal{X}, x)$ is a sheaf of groups for $n \geq 1$, which is abelian for $n \geq 2$.

The Nisnevich site $\mathrm{Sm}/k \downarrow X$ has the terminal object Id_X, with topology induced from the Nisnevich topology on the big site Sm/k. So for a locally fibrant simplicial presheaf $\mathcal{X}$, the zero-simplex $x \in \mathcal{X}|X(\mathrm{Id}_X)_0$ determines a sheaf of homotopy groups $\pi_n^{\mathrm{loc}}(\mathcal{X}|X, x)$.

Definition 5.8 *A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of simplicial presheaves on the smooth Nisnevich site of k is a combinatorial weak equivalence if for all $n \geq 1$, $X \in \mathrm{Sm}/k$, and zero-simplices $x \in \mathcal{X}(X)_0$ there are induced isomorphisms of Nisnevich sheaves*

$$\pi_0^{\mathrm{loc}}(\mathcal{X}) \longrightarrow \pi_0^{\mathrm{loc}}(\mathcal{Y}) ,$$
$$\pi_n^{\mathrm{loc}}(\mathrm{Ex}^\infty \mathcal{X}|X, x) \longrightarrow \pi_n^{\mathrm{loc}}(\mathrm{Ex}^\infty \mathcal{Y}|X, f(x)).$$

Exercise 5.9 *Show the following assertions.*

(i) There is a combinatorial weak equivalence of simplicial presheaves $\mathcal{X} \to \mathcal{Y}$ if and only if for each $X \in \mathrm{Sm}/k$ and $x \in X$ there is a naturally induced weak equivalence of simplicial sets $\mathcal{X}(\mathcal{O}^h_{X,x}) \to \mathcal{Y}(\mathcal{O}^h_{X,x})$.
(ii) If $\mathcal{X}$ is a locally fibrant simplicial presheaf, then $\pi_n^{\mathrm{loc}}(\mathcal{X}) \to \pi_n^{\mathrm{loc}}(\mathrm{Ex}^\infty \mathcal{X})$ is an isomorphism for every $n \geq 0$.

If $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ and $x \in \mathcal{X}(X)_0$ is a zero-simplex, let $\pi_n(\mathcal{X}, x)$ denote the sheaf on $\mathrm{Sm}/k \downarrow X$ associated to the presheaf

$$(U \longrightarrow X) \longmapsto \pi_n(\mathcal{X}(U), x|U) .$$

Note that this definition uses homotopy groups obtained by passing to the geometrical realization of the simplicial set $\mathcal{X}(U)$.

The sheaf of path components $\pi_0(\mathcal{X})$ of a simplicial presheaf $\mathcal{X}$ is the Nisnevich sheafification of the coequalizer of the presheaf diagram

$$\mathcal{X}_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathcal{X}_0 .$$

The definition of topological weak equivalence is strictly parallel to that of combinatorial weak equivalences:

Definition 5.10 *A morphism $\mathcal{X} \to \mathcal{Y}$ in $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ is a topological weak equivalence if for all $n \geq 1$, $X \in \mathrm{Sm}/k$, and $x \in \mathcal{X}(X)_0$ there are naturally induced isomorphisms of Nisnevich sheaves*

$$\pi_0(\mathcal{X}) \longrightarrow \pi_0(\mathcal{Y}) \ ,$$
$$\pi_n(\mathcal{X}|X,x) \longrightarrow \pi_n(\mathcal{Y}|X,f(x)) \ .$$

For proofs of the following result, see [DI04, 6.7] and [Jar87, 1.18].

Lemma 5.11 *For any simplicial presheaf $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$, $X \in \mathrm{Sm}/k$, and $x \in \mathcal{X}(X)_0$ there are naturally induced isomorphisms of Nisnevich sheaves*

$$\pi_n(\mathcal{X}|X,x) \longrightarrow \pi_n^{\mathrm{loc}}(\mathcal{X}|X,x) \ .$$

Exercise 5.12 *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of simplicial presheaves. If $\mathcal{X}$ and $\mathcal{Y}$ are locally fibrant, show that $\mathcal{X} \to \mathcal{Y}$ is a topological weak equivalence if and only if there are naturally induced isomorphisms of Nisnevich sheaves*

$$\pi_0^{\mathrm{loc}}(\mathcal{X}) \longrightarrow \pi_0^{\mathrm{loc}}(\mathcal{Y}) \ ,$$
$$\pi_n^{\mathrm{loc}}(\mathcal{X}|X,x) \longrightarrow \pi_n^{\mathrm{loc}}(\mathcal{Y}|X,f(x)).$$

Exercises 5.9 and 5.12 show that the classes of combinatorial, stalkwise, and topological weak equivalences coincide. To emphasize the local structure, we refer to them as local weak equivalences. We also use the notation $\sim$loc.

A simplicial presheaf $\mathcal{X}$ on Sm/k satisfies Nisnevich descent if for every upper distinguished square (12), the following diagram is a homotopy cartesian square of simplicial sets:

$$\begin{array}{ccc} \mathcal{X}(X) & \longrightarrow & \mathcal{X}(V) \\ \downarrow & & \downarrow \\ \mathcal{X}(U) & \longrightarrow & \mathcal{X}(W) \end{array} \tag{19}$$

The following fundamental result is the Nisnevich descent theorem which was proven by Morel-Voevodsky in [MV99].

Theorem 5.13 *Suppose $\mathcal{X}$ and $\mathcal{Y}$ satisfy Nisnevich descent on Sm/k, and there is a local weak equivalence*

$$\mathcal{X} \xrightarrow{\sim \mathrm{loc}} \mathcal{Y} \ .$$

Then $\mathcal{X}$ and $\mathcal{Y}$ are schemewise weakly equivalent.

Remark 5.14 *We point out that the Nisnevich descent theorem also holds for the big site of finite type S-schemes, where S denotes a Noetherian scheme of finite Krull dimension.*

Theorem 5.13 follows easily from the following Lemma.

Lemma 5.15 *Suppose $\mathcal{X}$ satisfies Nisnevich descent on* Sm$/k$*, and there is a local weak equivalence*

$$\mathcal{X} \xrightarrow{\sim \text{loc}} * \, .$$

Then $\mathcal{X}$ and $*$ are schemewise weakly equivalent, so that $\mathcal{X}$ is schemewise contractible.

The Lemma implies the Theorem: Our object is to prove that for every $X \in \mathrm{Sm}/k$, there is a weak equivalence of simplicial sets

$$\mathcal{X}(X) \longrightarrow \mathcal{Y}(X) \, .$$

It suffices to show that the schemewise homotopy fiber over any zero-simplex $x \in \mathcal{Y}(X)_0$ is contractible. Since $\mathcal{X}$ and $\mathcal{Y}$ are locally weakly equivalent and satisfy Nisnevich descent, it follows that the schemewise homotopy fiber is locally weakly equivalent to $*$ and satisfies Nisnevich descent on $\mathrm{Sm}/k \downarrow X$. Lemma 5.15 applies to the Nisnevich site of $\mathrm{Sm}/k \downarrow X$. Thus the homotopy fiber is schemewise contractible. We conclude there is a schemewise weak equivalence

$$\mathcal{X} \xrightarrow{\sim \text{sch}} \mathcal{Y} \, .$$

Later in this text, we shall use an alternate form of the Nisnevich descent theorem in the context of constructing model structures for spectra of spaces. Since the proof of 5.13 and its reformulation makes use of model structures on the presheaf category $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}/k)$, we will discuss such model structures in the next section.

5.2 Model Structures for Spaces

This section looks into the construction of a motivic model structure on $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}/k)$. Instead of using simplicial sheaves, we shall work in the setting of simplicial presheaves. The motivic model structure for simplicial sheaves follows immediately from the existence of a Quillen equivalent motivic model structure on the corresponding presheaf category $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$.

There are now several model structures underlying the motivic homotopy category. The classes of weak equivalences coincide in all these motivic models. However, the motivic models differ greatly with respect to the choice of cofibrations. Now, the good news is that having a bit a variety in the choice of foundations gives a more in depth understanding of the whole theory. As our cofibrations we choose monomorphisms of simplicial presheaves. This has the neat effect that all objects are cofibrant. On the other hand, this choice makes it difficult to describe the fibrations defined using the right lifting property with respect to trivial cofibrations. In other models, there are more fibrant objects and the fibrations are easier to describe, but then again not every object is cofibrant.

We do not attempt to give a thorough case by case study of each motivic model. The reader can consult the following list of papers on this subject: Blander [Bla01], Dugger [Dug01], Dugger-Hollander-Isaksen [DHI04], Dundas-Röndigs-Østvær [DRØ03], Isaksen [Isa04], Jardine [Jar03], and Voevodsky [Voe00a].

Recall that a cofibration of simplicial sets is simply an inclusion. Jardine [Jar87] proved the theorem that monomorphisms form an adequate class of cofibrations in the simplicial presheaf setting. Adequate means, in particular, that the classes of local weak equivalences and monomorphisms form a model structure on $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$. This leads to the model structure introduced by Morel-Voevodsky [MV99]. The main innovative idea in their construction of the motivic theory is that the affine line plays the role of the unit interval in topology. In our discussion of the Jardine and Morel-Voevodsky model structures, we follow the approach in the paper of Goerss-Jardine [GJ98].

For the basic notions in homotopical algebra we will use, such as left/right proper and simplicial model categories, see for example [Dun]. First in this section, we deal with Jardine's model structure on $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$. Global fibrations are, by definition, morphisms having the right lifting property with respect to morphisms which are monomorphisms and local weak equivalences. This forces half of the lifting axiom $\mathcal{M}4$ in the model category structure. We refer to the following model as the local injective model structure.

Theorem 5.16 *The classes of local weak equivalences, monomorphisms and global fibrations define a proper, simplicial and cofibrantly generated model structure for simplicial presheaves on the smooth Nisnevich site of k.*

Fibrant objects in the local injective model structure are called globally fibrant. An essential input in the proof of 5.16 is the following list of properties:

P1 The class of local weak equivalences is closed under retracts.

P2 The class of local weak equivalences satisfies the two out of three axiom.

P3 Every schemewise weak equivalence is a local weak equivalence.

P4 The class of trivial cofibrations is closed under pushouts.

P5 Let γ be a limit ordinal, considered as a partially ordered set. Suppose there is a functor

$$F\colon \gamma \longrightarrow \Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)\ ,$$

such that for each morphism $i \leq j$ in γ there is a trivial cofibration

$$F(i) \overset{\sim \mathrm{loc}}{\rightarrowtail} F(j)\ .$$

Then there is a canonically induced trivial cofibration

$$F(i) \overset{\sim \mathrm{loc}}{\rightarrowtail} \operatorname*{colim}_{j\in\gamma} F(j)\ .$$

P6 Suppose there exist trivial cofibrations for $i \in I$

$$F_i \overset{\sim \mathrm{loc}}{\rightarrowtail} G_i \ .$$

Then there is a canonically induced trivial cofibration

$$\coprod_{i \in I} F_i \overset{\sim \mathrm{loc}}{\rightarrowtail} \coprod_{i \in I} G_i \ .$$

P7 There is an infinite cardinal λ which is an upper bound for the cardinality of the set of morphisms of Sm/k such that for every trivial cofibration $\mathcal{X} \overset{\sim \mathrm{loc}}{\rightarrowtail} \mathcal{Y}$ and every λ-bounded subobject $\mathcal{Z}$ of $\mathcal{Y}$ there exists some λ-bounded subobject $\mathcal{W}$ of $\mathcal{Y}$ and a diagram of simplicial presheaves:

$$\begin{array}{ccccc} & & \mathcal{W} \cap \mathcal{X} & \hookrightarrow & \mathcal{X} \\ & & \downarrow{\scriptstyle \sim \mathrm{loc}} & & \downarrow{\scriptstyle \sim \mathrm{loc}} \\ \mathcal{Z} & \hookrightarrow & \mathcal{W} & \hookrightarrow & \mathcal{Y} \end{array}$$

Properties **P**1–**P**3 are clear from our discussion of local weak equivalences in Sect. 5.1. For example, the morphism of presheaves of homotopy groups induced by a schemewise weak equivalence is an isomorphism.

To prove **P**4, we consider a pushout diagram in $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ where i is a cofibration and a local weak equivalence:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Z} \\ {\scriptstyle i}\downarrow{\scriptstyle \sim \mathrm{loc}} & & \downarrow{\scriptstyle j} \\ \mathcal{Y} & \longrightarrow & \mathcal{Y} \cup_{\mathcal{X}} \mathcal{Z} =: \mathcal{W} \end{array}$$

We want to prove that the right vertical morphism is a local weak equivalence. Pushouts along monomorphisms preserves schemewise weak equivalences, so we may assume that all simplicial presheaves are schemewise fibrant; hence, locally fibrant, and moreover that f is a monomorphism. Exercise 5.9 and the characterization of local weak equivalences by combinatorial homotopy groups π^{loc} imply: j is a local weak equivalence if and only if for every $X \in \mathrm{Sm}/k$ and every diagram

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\alpha} & \mathcal{Z}(X) \\ \downarrow & & \downarrow{\scriptstyle j(X)} \\ \Delta^n & \xrightarrow{\beta} & \mathcal{W}(X) \end{array} \tag{20}$$

there exists a covering sieve R of X together with a local homotopy. That is, for every $\phi\colon U \to X$ in R, there is a simplicial homotopy $\Delta^n \times \Delta^1 \to \mathcal{W}(U)$, which is constant on $\partial\Delta^n$, from $\phi^* \circ \beta$ to a map β' with image in $\mathcal{Z}(U)$. Replacing the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ in diagram (20) by an appropriate subdivision $K \hookrightarrow L$, one can assume that the image under β of every simplex σ of L lies either in $\mathcal{Z}(X)$ or in $\mathcal{Y}(X)$ (or in both, meaning that $\beta(\sigma) \in \mathcal{X}(X) = \mathcal{Y}(X) \cap \mathcal{Z}(X)$). Since L is obtained from K by attaching finitely many simplices (of dimension $0 \le d \le n$), one may construct the required simplicial homotopy by induction on these simplices. In case the simplex has image in $\mathcal{Z}(X)$, use a constant local homotopy. Otherwise, one can construct a local homotopy as desired, because i is a local weak equivalence. Observe that this requires passing to a covering sieve as many times as there are non-degenerate simplices in $L \setminus K$.

The first step in the proof of **P**5 is left to the reader as an exercise:

Exercise 5.17 *Note that there is a functor*

$$\mathrm{Ex}^\infty F \colon \gamma \longrightarrow \Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k), \quad i \longmapsto \mathrm{Ex}^\infty F(i)\ ,$$

together with a natural transformation

$$F \longrightarrow \mathrm{Ex}^\infty F\ .$$

By considering the following commutative diagram, show that it suffices to prove **P**5 *when $F(i)$ is a presheaf of Kan complexes for all $i \in \gamma$:*

$$\begin{array}{ccc} F(i) & \longrightarrow & \operatorname{colim}_{j\in\gamma} F(j) \\ \downarrow & & \downarrow \\ \mathrm{Ex}^\infty F(i) & \longrightarrow & \operatorname{colim}_{j\in\gamma} \mathrm{Ex}^\infty F(j) \end{array}$$

(Hint: Schemewise weak equivalences are local weak equivalences.)

Taking the previous exercise for granted, we may now assume that each $F(i)$ is a presheaf of Kan complexes.

Consider the diagram obtained from the Nisnevich sheafification functor:

$$\begin{array}{ccccc} F(i) & \longrightarrow & \operatorname*{colim}_{j\in\gamma} F(j) & \longrightarrow & a_{\mathrm{Nis}}(\operatorname*{colim}_{j\in\gamma} F(j)) \\ \downarrow & & \downarrow & & \downarrow \cong \\ a_{\mathrm{Nis}}F(i) & \longrightarrow & \operatorname*{colim}_{j\in\gamma} a_{\mathrm{Nis}}F(j) & \longrightarrow & a_{\mathrm{Nis}}(\operatorname*{colim}_{j\in\gamma} a_{\mathrm{Nis}}F(j)) \end{array}$$

Concerning the left lower horizontal morphism, note that $a_{\mathrm{Nis}}F(i) \to a_{\mathrm{Nis}}F(j)$ is a local weak equivalence of locally fibrant simplicial presheaves; hence a schemewise weak equivalence, which implies a schemewise and hence a local weak equivalence between $a_{\mathrm{Nis}}F(i)$ and $\operatorname*{colim}_{j\in\gamma} a_{\mathrm{Nis}}F(j)$.

The associated Nisnevich sheaf morphisms are all local weak equivalences, so that starting in the right hand square and using the two out of three property for local weak equivalences, it follows that all the morphisms in the diagram are local weak equivalences. This proves **P**5.

It is time to consider property **P**6. Again, let us start with an exercise.

Exercise 5.18 *Show there is no loss of generality in assuming that F_i and G_i are Kan complexes for all $i \in I$. (Hint:* Ex^∞ *preserves coproducts.)*

The proof proceeds by noting the local weak equivalence between locally fibrant presheaves $a_{\mathrm{Nis}}F_i \to a_{\mathrm{Nis}}G_i$. In effect, we use

$$\coprod_{i\in I} a_{\mathrm{Nis}}F_i \xrightarrow{\sim\mathrm{sch}} \coprod_{i\in I} a_{\mathrm{Nis}}G_i \ .$$

Sheafification induces the commutative diagram:

$$\begin{array}{ccccccc} a_{\mathrm{Nis}} \coprod_{i\in I} F_i & \longleftarrow & \coprod_{i\in I} F_i & \longrightarrow & \coprod_{i\in I} G_i & \longrightarrow & a_{\mathrm{Nis}} \coprod_{i\in I} G_i \\ \cong\downarrow & & \downarrow & & \downarrow & & \downarrow\cong \\ a_{\mathrm{Nis}} \coprod_{i\in I} a_{\mathrm{Nis}}F_i & \longleftarrow & \coprod_{i\in I} a_{\mathrm{Nis}}F_i & \xrightarrow{\sim\mathrm{loc}} & \coprod_{i\in I} a_{\mathrm{Nis}}G_i & \longrightarrow & a_{\mathrm{Nis}} \coprod_{i\in I} a_{\mathrm{Nis}}G_i \end{array}$$

By starting with the outer squares, an easy check shows that all morphisms in the diagram are local weak equivalences. The part of property **P**6 dealing with cofibrations is clear.

In the formulation of property **P**7 or the 'bounded cofibration condition', we implicitly use that Sm/k is skeletally small. The latter means that isomorphism classes of objects in Sm/k form a set. If κ is an infinite cardinal and $X \in \mathrm{Sm}/k$, then the cardinality of X is less than κ, written $\mathbf{card}(X) < \kappa$, if the following hold:

C1 The cardinality of the underlying topological space of X is smaller than the cardinal κ.

C2 For all Zariski open affine patches $\mathrm{Spec}(A)$ of X we have $\mathbf{card}(A) < \kappa$.

Suppose that A is a commutative ring with unit, such that $\mathbf{card}(A) < \kappa$. Then, as a ring, A is isomorphic to the quotient by an ideal of a polynomial ring $\mathbb{Z}[T]$ on a set T of generators such that $\mathbf{card}(T) < \kappa$. This implies the inequality $\mathbf{card}(\mathbb{Z}[T]) < \kappa$. Hence, the cardinality of the collection of ideals of $\mathbb{Z}[T]$ is bounded above by 2^κ. It follows that the collection of isomorphisms classes of all affine schemes $\mathrm{Spec}(A)$ such that $\mathbf{card}(A) < \kappa$ forms a set. To generalize to schemes, use that isomorphism classes of schemes are bounded above by isomorphism classes of diagrams of affine schemes. Fixing an infinite cardinal κ such that $\mathbf{card}(k) < \kappa$, implies, from what we have just observed, that Sm/k is skeletally small; thus, the formulation of **P**7 makes sense.

On a related matter, a cofibration in $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$

$$\mathcal{X} \rightarrowtail \mathcal{Y}$$

is λ-bounded if the object $\mathcal{Y}$ is λ-bounded, i.e. for all $X \in \mathrm{Sm}/k$, $n \geq 0$, each set $\mathcal{Y}_n(X)$ has smaller cardinality than λ. For each object $X \in \mathrm{Sm}/k$, there is the X-section functor

$$\mathcal{X} \longmapsto \mathcal{X}(X) .$$

It has a left adjoint whose value on the standard simplicial n-simplex $\Delta[n]$ is the λ-bounded simplicial presheaf $h_X\Delta[n]$ defined by

$$Y \longmapsto \coprod_{\phi\colon Y \to X} \Delta[n] .$$

Using adjointness yields bijections between morphisms of simplicial sets and morphisms of simplicial presheaves,

$$\Delta[n] \longrightarrow \mathcal{Y}(X), h_X\Delta[n] \longrightarrow \mathcal{Y} .$$

It follows that any simplicial presheaf on the smooth Nisnevich site of k is a filtered colimit of its λ-bounded subobjects because the generating simplicial presheaves $h_X\Delta[n]$ are all λ-bounded.

Suppose now that $\mathcal{X} \overset{\sim\mathrm{loc}}{\rightarrowtail} \mathcal{Y}$ is given, and choose a λ-bounded subpresheaf $\mathcal{Z} \subseteq \mathcal{Y}$. By applying the functor Ex^∞, we may assume that all simplicial presheaves in sight are locally fibrant. The proof of **P**7 proceeds by constructing inductively a sequence of λ-bounded subobjects

$$\mathcal{W}_0 := \mathcal{Z} \subseteq \mathcal{W}_1 \subseteq \mathcal{W}_2 \cdots$$

such that, for each $X \in \mathrm{Sm}/k$, all local lifting problems of the form

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{W}_i \cap \mathcal{X}(X) \\ \big\downarrow & & \big\downarrow \\ \Delta^n & \longrightarrow & \mathcal{W}_i(X) \end{array}$$

have solutions over $\mathcal{W}_{i+1}$. Such a local lifting problem amounts to an element e in a relative local homotopy group $\pi_n^{\mathrm{loc}}(\mathcal{W}_i \cap \mathcal{X}, \mathcal{W}_i)$. This element maps to zero in $\pi_n^{\mathrm{loc}}(\mathcal{X}, \mathcal{Y})$. Since local homotopy groups commute with filtered colimits, and since $\mathcal{Y}$ is the filtered colimit of its λ-bounded subobjects by assumption on λ, there exists a λ-bounded subobject $\mathcal{W}_i^e$ such that e maps to zero in the group $\pi_n^{\mathrm{loc}}(\mathcal{W}_i^e \cap \mathcal{X}, \mathcal{W}_i^e)$. The relative local homotopy group $\pi_n^{\mathrm{loc}}(W_i \cap \mathcal{X}, \mathcal{W}_i)$ is λ-bounded as well, thus $\mathcal{W}_{i+1}$ is the union of all the $\mathcal{W}_i^e$'s. This completes the inductive step.

Set $\mathcal{W} := \cup \mathcal{W}_i$, which is again λ-bounded. It follows, using properties of morphisms having the local right lifting property with respect to the inclusions $\partial\Delta^n \subseteq \Delta^n$, that there is a local weak equivalence

$$\mathcal{W} \cap \mathcal{X} \xrightarrow{\sim \mathrm{loc}} \mathcal{W} .$$

This finishes the sketch proof of **P**7.

In the statement that the local injective model structure is simplicial, we made implicitly use of the fact that the presheaf category $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ is enriched in the category of simplicial sets. The simplicial structure of a function complex

$$\mathbf{hom}(\mathcal{X}, \mathcal{Y})$$

is determined by

$$\mathbf{hom}(\mathcal{X}, \mathcal{Y})_n := \Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)(\mathcal{X} \times \Delta[n], \mathcal{Y}) .$$

As a simplicial presheaf, the tensor object

$$\mathcal{X} \times \Delta[n]$$

is given by

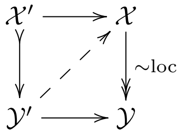
$$(\mathcal{X} \times \Delta[n])(X) := \mathcal{X}(X) \times \Delta[n] .$$

Pointed function complexes and tensor objects are defined similarly making pointed simplicial presheaves into a category enriched in pointed simplicial sets.

Proof. (Theorem 5.16). In the lectures [Lev] we learned that small limits and small colimits exist for the presheaf category $\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$. Hence, the limit axiom $\mathcal{M}1$ holds for $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ [Dun]. We have already noted that the two out of three axiom $\mathcal{M}2$ holds for the class of local weak equivalences.

The retract axiom $\mathcal{M}3$ holds trivially for both local weak equivalences and cofibrations. Global fibrations are defined by the right lifting property with respect to trivial cofibrations; using this, it follows that global fibrations are closed under retracts.

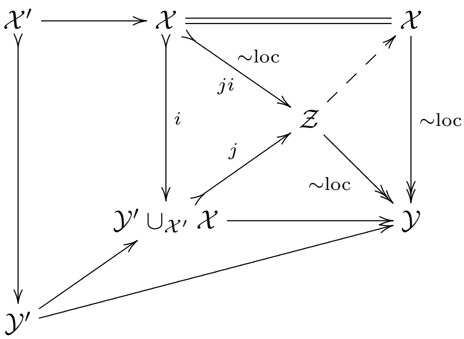
Consider the lifting axiom $\mathcal{M}4$. In our case, global fibrations are rigged so that the right lifting property part of $\mathcal{M}4$ holds. For the part of $\mathcal{M}4$ which is not true by definition, consider the diagram where i is a cofibration and p is a trivial global fibration:



We want to prove that the indicated filler exists. In the following, let us assume the factorization axiom $\mathcal{M}5$ holds for the canonical morphism

$$\mathcal{Y}' \cup_{\mathcal{X}'} \mathcal{X} \longrightarrow \mathcal{Y}\,.$$

With this standing assumption, we obtain the commutative diagram:



Concerning this diagram we make two remarks:

(i) Note that ji is a cofibration being the composition of the cofibrations.
(ii) Commutativity implies there is a local weak equivalence

$$\mathcal{X} \overset{i}{\rightarrowtail} \mathcal{Y}' \cup_{\mathcal{X}'} \mathcal{X} \longrightarrow \mathcal{Y}\,.$$

Hence, there is a local weak equivalence

$$\mathcal{X} \overset{ji}{\rightarrowtail} \mathcal{Z} \overset{\sim\text{loc}}{\rightarrowtail} \mathcal{Y}\,.$$

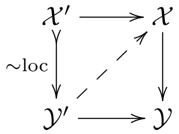
Thus ji is a trivial cofibration according to $\mathcal{M}2$.

We conclude that the filler with source $\mathcal{Z}$ exists rendering the diagram commutative. This uses the definition of global fibrations in terms of the right lifting property with respect to trivial cofibrations. Note that the above immediately solves our original lifting problem. At this stage of the proof, we have not used the properties **P**4–**P**7.

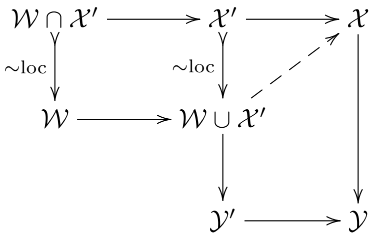
The serious part of the proof is to prove the factorization axiom $\mathcal{M}5$. Consider an infinite cardinal λ as in **P**7. We claim that a morphism

$$\mathcal{X} \longrightarrow \mathcal{Y}$$

is a global fibration if it has the right lifting property with respect to all trivial cofibrations with λ-bounded targets. In other words, for morphisms as above, we claim there exists a filler in every commutative diagram of the form:



Here, we may of course assume that the left vertical morphism is not an isomorphism. In effect, there exists a λ-bounded subobject $\mathcal{Z}$ of $\mathcal{Y}'$ which is not a subobject of $\mathcal{X}'$. By property **P**7, there exists a λ-bounded subobject $\mathcal{W}$ of $\mathcal{Y}'$ containing $\mathcal{Z}$, and a diagram:



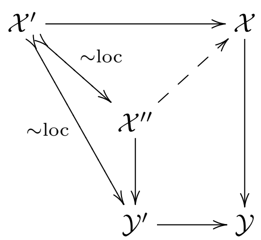
Concerning this diagram we make two remarks:

(i) Property **P**4 implies the trivial cofibration

$$\mathcal{X}' \overset{\sim \mathrm{loc}}{\rightarrowtail} \mathcal{W} \cup \mathcal{X}' \,.$$

(ii) By the assumption on the right vertical morphism, the partial filler exists.

Consider now the inductively ordered non-empty category of partial lifts where we assume $\mathcal{X}' \neq \mathcal{X}''$:



From **P**5 and Zorn's lemma, there exists at least one maximal partial lift $\mathcal{X}'' \longrightarrow \mathcal{Y}'$. Maximality implies $\mathcal{X}'' = \mathcal{Y}'$. This solves our lifting problem.

So far, we have made the key observation that morphisms having the right lifting property with respect to all trivial cofibrations with λ-bounded targets are global fibrations. The converse statement holds by definition of global fibrations. We recall from [Dun] that – in more technical terms – this means there exists a set of morphisms, called generating trivial cofibrations, which detects global fibrations. Alas, the argument gives no explicit description of the generators.

We can now set out to construct factorizations of the form:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\ & {\scriptstyle i_f}\searrow^{\sim \mathrm{loc}} \quad \nearrow\!\!\!\nearrow {\scriptstyle p_f} & \\ & \mathcal{Z}_f & \end{array}$$

The proof is a transfinite small object argument.

Given a cardinal $\beta > 2^\lambda$ we define inductively a functor

$$F\colon \beta \longrightarrow \Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k) \downarrow \mathcal{Y} \ ,$$

by setting

(i) $F(0)\colon = f$ and $X(0) = \mathcal{X}$,
(ii) For a limit ordinal ζ,

$$\mathcal{X}(\zeta)\colon \ = \operatorname*{colim}_{\gamma<\zeta} \mathcal{X}(\gamma) \ .$$

Transitions morphisms are obtained via pushout diagrams

$$\begin{array}{ccc} \coprod_D \mathcal{Z}_D & \xrightarrow{\coprod_D i_D} & \coprod_D \mathcal{W}_D \\ \downarrow & & \downarrow \\ \mathcal{X}(\gamma) & \longrightarrow & \mathcal{X}(\gamma+1) \end{array}$$

indexed by the set of all diagrams D where the left vertical morphism is a λ-bounded trivial cofibration:

$$\begin{array}{ccc} \mathcal{Z}_D & \overset{\sim \mathrm{loc}}{\rightarrowtail} & \mathcal{W}_D \\ \downarrow & & \downarrow \\ \mathcal{X}(\gamma) & \longrightarrow & \mathcal{Y} \end{array}$$

We note the following trivial cofibrations:

(i) Property **P**6 implies

$$\textstyle\coprod_D i_D \colon \coprod_D \mathcal{Z}_D \overset{\sim\text{loc}}{\rightarrowtail} \coprod_D \mathcal{W}_D \ .$$

(ii) Part (i) and **P**4 imply

$$\mathcal{X}(\gamma) \overset{\sim\text{loc}}{\rightarrowtail} \mathcal{X}(\gamma + 1) \ .$$

Using these constructions, we may now consider the induced factorization:

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ & {\scriptstyle i(\beta)} \searrow \quad \nearrow {\scriptstyle F(\beta)} & \\ & \operatorname*{colim}_{\gamma<\beta} \mathcal{X}(\gamma) & \end{array}$$

Property **P**5 lets us conclude that the first morphism in the factorization is a trivial cofibration.

For the second morphism, one has to solve lifting problems of the form

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \operatorname*{colim}_{\gamma<\beta} \mathcal{X}(\gamma) \\ {\scriptstyle \sim\text{loc}} \big\downarrow & \nearrow & \big\downarrow {\scriptstyle F(\beta)} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

where the left vertical morphism is λ-bounded. To obtain the lifting, note that since $\beta > 2^{\lambda}$, the upper horizontal morphism factors through some lower stage $\mathcal{X}(\gamma)$ of the colimit.

It remains to prove the second part of the factorization axiom $\mathcal{M}5$. Functorial factorization in the model structure on simplicial sets allows us to factor any morphism in the presheaf category

$$\mathcal{X} \longrightarrow \mathcal{Y}$$

into a cofibration and a schemewise weak equivalence

$$\mathcal{X} \rightarrowtail \mathcal{Z} \xrightarrow{\sim\text{sch}} \mathcal{Y} \ .$$

If we factor the schemewise weak equivalence into a trivial cofibration and a fibration, we obtain a commutative diagram:

$$\begin{array}{cccc} \mathcal{X} \rightarrowtail & \mathcal{Z} & \xrightarrow{\sim\text{sch}} & \mathcal{Y} \\ & {\scriptstyle \sim\text{loc}} \searrow & & \nearrow \\ & & \mathcal{W} & \end{array}$$

The only comments needed here are:

(i) There is a cofibration obtained by composition of morphisms

$$\mathcal{X} \rightarrowtail \mathcal{W} .$$

(ii) There is a local weak equivalence obtained by **P**2 and **P**3

$$\mathcal{W} \xrightarrow{\sim \mathrm{loc}} \mathcal{Y} .$$

Hence items (i) and (ii) imply the desired factorization

$$\mathcal{X} \rightarrowtail \mathcal{W} \overset{\sim \mathrm{loc}}{\twoheadrightarrow} \mathcal{Y} .$$

An alternate and more honest way of proving the second part of $\mathcal{M}5$ resembles the transfinite small object argument given in the first part. This implies that there exists a set of generating cofibrations.

For the second part of $\mathcal{M}5$, we note a stronger type of factorization result: Given a presheaf morphism

$$\mathcal{X} \longrightarrow \mathcal{Y} ,$$

there exists a factorization

$$\mathcal{X} \rightarrowtail \mathcal{W} \overset{\sim \mathrm{sch}}{\twoheadrightarrow} \mathcal{Y} .$$

Consider Sm/k in the indiscrete topology, i.e. the only covering sieves are maximal ones [Lev]. One can construct the local injective model structure for the indiscrete topology. This is a simplicial cofibrantly generated model structure on $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ where the weak equivalences are schemewise weak equivalences and cofibrations are monomorphisms. We refer to it as the injective model structure.

Applying $\mathcal{M}5$ in the injective model structure to the morphism

$$\mathcal{Z} \xrightarrow{\sim \mathrm{sch}} \mathcal{Y} ,$$

yields the factorization:

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\sim \mathrm{sch}} & \mathcal{Y} \\ & {\scriptstyle \sim \mathrm{sch}} \searrow \; \nearrow & \\ & \mathcal{W} & \end{array}$$

Since $\mathcal{M}2$ holds for schemewise weak equivalences, we immediately obtain the refined form of factorization in the local injective model structure.

Left properness is the assertion that local weak equivalences are preserved under pushouts along cofibrations:

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow[\Downarrow]{\sim \mathrm{loc}} & \mathcal{Y} \\
\big\downarrow & & \big\downarrow \\
\mathcal{Z} & \xrightarrow{\sim \mathrm{loc}} & \mathcal{Z} \cup_{\mathcal{X}} \mathcal{Y}
\end{array}$$

All objects are cofibrant in the local injective model structure. Since in any model category, pushouts of weak equivalences between cofibrant objects along cofibrations are weak equivalences [Hov99, Cube Lemma 5.2.6], left properness follows.

Right properness is the assertion that local weak equivalences are preserved under pullbacks along global fibrations:

$$\begin{array}{ccc}
\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} & \xrightarrow{\sim \mathrm{loc}} & \mathcal{Z} \\
\big\downarrow & & \big\downarrow \\
\mathcal{X} & \xrightarrow{\Uparrow \atop \sim \mathrm{loc}} & \mathcal{Y}
\end{array}$$

However, even a stronger property holds, because local weak equivalences are closed under pullback along local fibrations. The reason is that pullbacks commute with filtered colimits, which implies that it suffices to consider pullback diagrams of the form:

$$\begin{array}{ccc}
\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}(\mathcal{O}^h_{X,x}) & \xrightarrow{\sim} & \mathcal{Z}(\mathcal{O}^n_{X,x}) \\
\big\downarrow & & \big\downarrow \\
\mathcal{X}(\mathcal{O}^h_{X,x}) & \xrightarrow{\sim} & \mathcal{Y}(\mathcal{O}^h_{X,x})
\end{array}$$

The result follows because the category of simplicial sets is right proper. □

Making analogous definitions of local weak equivalences, monomorphisms and global fibrations for simplicial sheaves, we infer that there exists a local injective model structure for spaces. The proof consists mostly of repeating arguments we have seen in the simplicial presheaf setting. Details are left to the reader.

Theorem 5.19 *The classes of local weak equivalences, monomorphisms and global fibrations define a proper, simplicial and cofibrantly generated model structure for simplicial sheaves on the smooth Nisnevich site of k.*

Existence of local injective model structures on the categories of pointed simplicial presheaves and pointed simplicial sheaves follow immediately from existence of the respective local injective model structures on $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ and $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}/k)$.

A globally fibrant model for a simplicial presheaf $\mathcal{X}$ consists of a local weak equivalence

$$\mathcal{X} \longrightarrow G\mathcal{X} ,$$

where $G\mathcal{X}$ is globally fibrant. We have seen that globally fibrant models are well-defined up to schemewise weak equivalence. Globally fibrant models exist, and can be chosen functorially, because of the factorization axiom $\mathcal{M}5$. Note, however, that the morphism from $\mathcal{X}$ to $G\mathcal{X}$ is not necessarily a cofibration, so that we will not be tied down to any particular choice of $G\mathcal{X}$. The letter G stands for 'global' or for 'Godement' in the civilized example of the smooth Nisnevich site of k where Godement resolutions yield globally fibrant models. Globally fibrant models in the simplicial sheaf category are defined similarly. Note that a globally fibrant simplicial sheaf is globally fibrant in the simplicial presheaf category.

Exercise 5.20 *Nisnevich descent holds for any globally fibrant simplicial sheaf. (Hint: Use the characterization of sheaves in the Nisnevich topology, see 5.3.)*

We may reformulate the Nisnevich descent Theorem 5.13 in terms of globally fibrant models.

Theorem 5.21 *A simplicial presheaf $\mathcal{X}$ satisfies Nisnevich descent on* Sm/k *if and only if any globally fibrant model $G\mathcal{X}$ is schemewise weakly equivalent to $\mathcal{X}$.*

Proof. We consider the commutative diagram obtained by sheafifying:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{a_{\mathrm{Nis}}} & a_{\mathrm{Nis}}\mathcal{X} \\ \downarrow & & \downarrow \\ G\mathcal{X} & \xrightarrow{Ga_{\mathrm{Nis}}} & Ga_{\mathrm{Nis}}\mathcal{X} \end{array}$$

All morphisms are local weak equivalences. In addition, Ga_{Nis} is a schemewise weak equivalence, since locally weakly equivalent globally fibrant models are schemewise weakly equivalent. Exercise 5.20 implies, since $Ga_{\mathrm{Nis}}\mathcal{X}$ is a sheaf, that $G\mathcal{X}$ satisfies Nisnevich descent.

Theorem 5.13 implies, provided Nisnevich descent holds for $\mathcal{X}$, that there is a schemewise weak equivalence

$$\mathcal{X} \xrightarrow{\sim\mathrm{sch}} G\mathcal{X} .$$

Conversely, if $\mathcal{X}$ is schemewise weakly equivalent to any of its globally fibrant models; which we have already shown satisfies Nisnevich descent, it follows easily that $\mathcal{X}$ satisfies Nisnevich descent. □

The model structure we shall discuss next is the $\mathbb{A}^1$- or motivic model structure introduced by Morel-Voevodsky [MV99]. Precursors are the local injective model structure and localization techniques developed in algebraic topology. What results is a homotopy theory having deep connections with algebraic geometry.

The motivic model structure arises as the localization theory obtained from the local injective model structure by "formally inverting" any rational point of the affine line

$$* \longrightarrow \mathbb{A}^1_k \; .$$

Since any two rational points correspond to each other under k-automorphisms of $\mathbb{A}^1_k$, it suffices to consider the zero section $0\colon \mathrm{Spec}(k) \to \mathbb{A}^1_k$. Getting the motivic theory off the ground involves to a great extend manipulations with function complexes of simplicial presheaves. The main innovative idea is now to replace the local weak equivalences by another class of simplicial presheaf morphisms making the affine line contractible, which we will call motivic weak equivalences, and prove properties **P**1-**P**7 for the motivic weak equivalences. With this input, proceeding as in the construction of the local injective model structure, we get a new cofibrantly generated model structure for simplicial presheaves $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)$. This is the motivic model structure.

Definition 5.22 *The classes of motivic weak equivalences and fibrations are defined as follows.*

(i) A simplicial presheaf $\mathcal{Z}$ is motivically fibrant if it is globally fibrant and for every cofibration

$$\mathcal{X} \rightarrowtail \mathcal{Y} \, ,$$

the canonical morphism from $\mathcal{Z}$ to $*$ has the right lifting property with respect to all presheaf inclusions

$$(\mathcal{X} \times \mathbb{A}^1_k) \cup_{\mathcal{X}} \mathcal{Y} \rightarrowtail (\mathcal{Y} \times \mathbb{A}^1_k)$$

induced by the zero section of the affine line.

(ii) A simplicial presheaf morphism

$$\mathcal{X} \longrightarrow \mathcal{Y}$$

is a motivic weak equivalence if for any motivically fibrant simplicial presheaf $\mathcal{Z}$ there is an induced weak equivalence of simplicial sets

$$\mathbf{hom}(\mathcal{Y}, \mathcal{Z}) \longrightarrow \mathbf{hom}(\mathcal{X}, \mathcal{Z}) \; .$$

(iii) A simplicial presheaf morphism is a motivic fibration if it has the right lifting property with respect to morphisms which are simultaneously motivic weak equivalences and monomorphisms.

The lifting property in item (i) is equivalent to having a trivial global fibration

$$\mathbf{Hom}(\mathbb{A}^1_k, \mathcal{Z}) \xrightarrow{\sim_{\mathrm{loc}}} \mathbf{Hom}(*, \mathcal{Z}) \; .$$

Note that the above morphism is always a global fibration.

It follows that a globally fibrant simplicial presheaf $\mathcal{Z}$ is motivically fibrant if and only if all projections

$$\mathbb{A}^1_X \longrightarrow X$$

induce weak equivalences of simplicial sets

$$\mathcal{Z}(X) \xrightarrow{\sim} \mathcal{Z}(\mathbb{A}^1_X) \ .$$

In general, there is no explicit description of motivic fibrations.

The following are examples of motivic weak equivalences:

$$\mathcal{X} \times * \longrightarrow \mathcal{X} \times \mathbb{A}^1_k \ ,$$

$$(\mathcal{X} \times \mathbb{A}^1_k) \cup_{(\mathcal{X} \times *)} (\mathcal{Y} \times *) \rightarrowtail (\mathcal{Y} \times \mathbb{A}^1_k) \ .$$

Every local weak equivalence is a motivic weak equivalence for trivial reasons.

Exercise 5.23 *Show that a vector bundle* $p\colon X \longrightarrow Y$ *in* Sm/k *is a motivic weak equivalence. Proceed by induction on the number of elements in an open cover of* Y *which trivializes* p.

Theorem 5.24 *There exists a functor*

$$\mathcal{L}\colon \Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k) \longrightarrow \Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k) \ ,$$

and a monomorphism of simplicial presheaves

$$\eta_{\mathcal{X}}\colon \mathcal{X} \longrightarrow \mathcal{L}(\mathcal{X})$$

such that the following holds:

(i) $\mathcal{L}(\mathcal{X})$ *is motivically fibrant.*
(ii) For every motivically fibrant $\mathcal{Z}$*, there is an induced weak equivalence of simplicial sets*

$$\mathbf{hom}(\mathcal{L}(\mathcal{X}), \mathcal{Z}) \xrightarrow{\sim} \mathbf{hom}(\mathcal{X}, \mathcal{Z}) \ .$$

In the construction of the functorial motivic fibrant replacement functor $\mathcal{L}$, we shall make use of the fact that there exists a continuous functorial fibrant replacement functor $\mathcal{L}_G$ in the local injective model structure [GJ98]. That $\mathcal{L}_G$ is continuous simply says that the natural maps of hom-sets extend to natural maps of hom-simplicial sets

$$\mathcal{L}_G\colon \mathbf{hom}(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathbf{hom}(\mathcal{L}_G\mathcal{X}, \mathcal{L}_G\mathcal{Y})$$

which are compatible with composition.

Let I be the set of simplicial presheaf morphisms

$$\mathcal{X} \times h_X \Delta[n] \cup_{\mathcal{X}\times\mathcal{Z}} \mathcal{Y} \times \mathcal{Z} \hookrightarrow \mathcal{Y} \times h_X \Delta[n] \ .$$

A typical element in I will be denoted

$$\mathcal{C}_\alpha \hookrightarrow \mathcal{D}_\alpha \ .$$

Choose a cardinal $\beta > 2^\lambda$. Then the first step in the construction of $\mathcal{L}$ is setting

$$\mathcal{L}_0\mathcal{X} \colon = \mathcal{L}_G\mathcal{X} \ .$$

At a limit ordinal $\zeta < \beta$, set

$$\mathcal{L}_\zeta\mathcal{X} \colon = \mathcal{L}_G(\operatorname*{colim}_{\gamma<\zeta} \mathcal{L}_\gamma\mathcal{X}) \ .$$

At successor ordinals, consider the pushout diagram

$$\begin{array}{ccc} \coprod_{\alpha\in I} \mathcal{C}_\alpha \times \mathbf{hom}(\mathcal{C}_\alpha, \mathcal{L}_\zeta\mathcal{X}) & \longrightarrow & \mathcal{L}_\zeta\mathcal{X} \\ \downarrow & & \downarrow \\ \coprod_{\alpha\in I} \mathcal{D}_\alpha \times \mathbf{hom}(\mathcal{C}_\alpha, \mathcal{L}_\zeta\mathcal{X}) & \longrightarrow & P_I\mathcal{L}_\zeta\mathcal{X} \end{array}$$

and set

$$\mathcal{L}_{\zeta+1}\mathcal{X} \colon = \mathcal{L}_G(P_I\mathcal{L}_\zeta\mathcal{X}) \ .$$

These constructions give the natural definition

$$\mathcal{L}\mathcal{X} \colon = \operatorname*{colim}_{\zeta<\beta} \mathcal{L}_\zeta\mathcal{X} \ .$$

Recall that we choose $\beta > 2^\lambda$ so that any morphism with target $\mathcal{L}\mathcal{X}$ factors through some $\mathcal{L}_\zeta\mathcal{X}$. On account of this observation, we leave it as an exercise to finish the proof of 5.24.

We have the following characterizations of motivic weak equivalences.

Lemma 5.25 *The following assertions are equivalent.*

(i) There is a motivic weak equivalence

$$\mathcal{X} \xrightarrow{\sim \mathrm{mot}} \mathcal{Y} \ .$$

(ii) For every motivically fibrant simplicial presheaf $\mathcal{Z}$, there is an isomorphism in the local injective homotopy category

$$\mathbf{Ho}_{\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)_{\sim\mathrm{loc}}}(\mathcal{Y},\mathcal{Z}) \xrightarrow{\cong} \mathbf{Ho}_{\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)_{\sim\mathrm{loc}}}(\mathcal{X},\mathcal{Z}) \ .$$

(iii) There is a local weak equivalence

$$\mathcal{L}(\mathcal{X}) \xrightarrow{\sim\text{loc}} \mathcal{L}(\mathcal{Y}) \ .$$

Proof. Recall that motivically fibrant objects are in particular globally fibrant, so that (i) implies (ii).

Let $\mathcal{Z}$ be motivically fibrant. By abstract homotopy theory there is an isomorphism in the local injective homotopy category between

$$\mathbf{Ho}_{\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)_{\sim\mathrm{loc}}}(\mathcal{X}, \mathcal{Z}) \ ,$$

and

$$\mathbf{Ho}_{\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)_{\sim\mathrm{loc}}}(\mathcal{L}(\mathcal{X}), \mathcal{Z}) \ .$$

When (ii) holds, this implies an isomorphism

$$\mathbf{Ho}_{\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)_{\sim\mathrm{loc}}}(\mathcal{L}(\mathcal{Y}), \mathcal{Z}) \xrightarrow{\cong} \mathbf{Ho}_{\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)_{\sim\mathrm{loc}}}(\mathcal{L}(\mathcal{X}), \mathcal{Z}) \ .$$

Theorem 5.24 shows that $\mathcal{L}(\mathcal{X})$ and $\mathcal{L}(\mathcal{Y})$ are motivically fibrant; this implies an isomorphism in the local injective homotopy category

$$\mathcal{L}(\mathcal{X}) \xrightarrow{\cong} \mathcal{L}(\mathcal{Y}) \ .$$

The latter is equivalent to (iii).

When (iii) holds, (i) follows by contemplating the simplicial set diagram:

$$\begin{array}{ccc} \mathbf{hom}(\mathcal{L}(\mathcal{Y}), \mathcal{Z}) & \xrightarrow{\sim} & \mathbf{hom}(\mathcal{Y}, \mathcal{Z}) \\ {\scriptstyle\sim}\downarrow & & \downarrow \\ \mathbf{hom}(\mathcal{L}(\mathcal{X}), \mathcal{Z}) & \xrightarrow{\sim} & \mathbf{hom}(\mathcal{X}, \mathcal{Z}) \end{array}$$

The horizontal morphisms are weak equivalences according to Theorem 5.24. Our assumption implies without much work that the left vertical morphism is a weak equivalence. □

We are ready to state the existence of the motivic model structure.

Theorem 5.26 *The classes of motivic weak equivalences, motivic fibrations and monomorphisms define a proper, simplicial and cofibrantly generated model structure for simplicial presheaves on the smooth Nisnevich site of k.*

The proof of 5.26 follows the same script as we have seen for the local injective model structure. Properties **P**1-**P**7 are shown to hold for the class of motivic weak equivalences rather than the class of local weak equivalences. Note that **P**1-**P**3 hold trivially, while **P**4-**P**6 follow from 5.25 using that trivial fibrations of simplicial sets are closed under base change. Finally, the proof of **P**7 follows the sketch proof of the same property in the local injective structure, using the motivic fibrant replacement functor [GJ98, 4.7].

Remark 5.27 *In the Morel-Voevodsky paper [MV99] the notions of left and right proper model structures are reversed. However, proper model structure means the usual thing.*

In the following, we discuss the motivic model structure for the category of spaces $\mathrm{Spc}(k)$, i.e. simplicial sheaves on the smooth Nisnevich site of k.

A morphism in $\mathrm{Spc}(k)$ is a motivic weak equivalence if it is a motivic weak equivalence in the simplicial presheaf category. Motivic fibrations are defined similarly. The cofibrations are the monomorphisms.

Concerning sheafified simplicial presheaves and local weak equivalences, there is the following useful result.

Lemma 5.28 *Suppose that $\mathcal{X}$ is a simplicial presheaf and $\mathcal{Y}$ is a simplicial sheaf. Then*

$$\mathcal{X} \longrightarrow \mathcal{Y}$$

is a local weak equivalence if and only if the same holds true for the morphism

$$a_{\mathrm{Nis}}\mathcal{X} \longrightarrow \mathcal{Y}\,.$$

We note the local weak equivalence

$$\mathcal{X} \xrightarrow{\sim \mathrm{loc}} a_{\mathrm{Nis}}\mathcal{X}\,.$$

Since the Nisnevich sheafification functor is idempotent up to isomorphism [Lev], an easy consequence of Lemma 5.28 is that any simplicial presheaf is both local and motivic weakly equivalent to its sheafification.

Exercise 5.29 *Show that a morphism between simplicial sheaves is a motivic fibration if and only if it has the right lifting property with respect to motivic trivial cofibrations of simplicial sheaves.*

Theorem 5.30 *Let* $\mathrm{Spc}(k)$ *be the category of simplicial Nisnevich sheaves on* Sm/k.

(i) Motivic weak equivalences, motivic fibrations and cofibrations define a proper, simplicial and cofibrantly generated model structure on $\mathrm{Spc}(k)$.

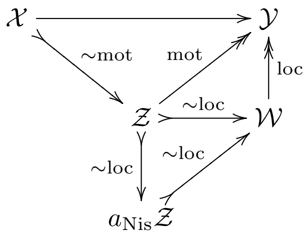
(ii) The Nisnevich sheafification functor induces a Quillen equivalence

$$\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)_{\sim \mathrm{mot}} \rightleftarrows \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}/k)_{\sim \mathrm{mot}}.$$

Proof. The limit axiom $\mathcal{M}1$ holds for simplicial sheaves, see e.g. [Lev]. The two out of three axiom $\mathcal{M}2$ and the retract axiom $\mathcal{M}3$ follow immediately since the corresponding statements hold for simplicial presheaves.

Exercise 5.29 shows that the right lifting property part of axiom $\mathcal{M}4$ holds. Given the motivic model structure for simplicial presheaves, the second part of the lifting axiom $\mathcal{M}4$ holds tautologically.

Let $\mathcal{X} \to \mathcal{Y}$ be a morphism in the simplicial sheaf category. We consider the motivic trivial cofibration and motivic fibration factorization part of $\mathcal{M}5$. The motivic model structure shows there is a simplicial presheaf $\mathcal{Z}$ together with morphisms having the required factorization in the simplicial presheaf category. Since $\mathcal{Y}$ is a simplicial sheaf, we may sheafify $\mathcal{Z}$, and employ axiom $\mathcal{M}5$ for the local injective model structure for the simplicial sheaf category. This gives the diagram:



From the above diagram, we deduce:

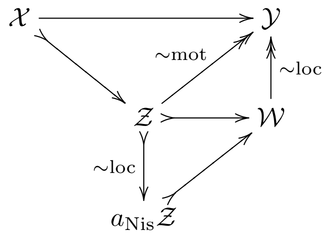
(i) There is a motivically trivial cofibration of simplicial sheaves

$$\mathcal{X} \xrightarrow{\sim\text{mot}} \mathcal{W} .$$

(ii) To obtain the factorization, it suffices to show that the global fibration between $\mathcal{W}$ and $\mathcal{Y}$ is a motivic fibration.

As objects of the site $\Delta^{\text{op}}\text{Pre}_{\text{Nis}}(\text{Sm}/k) \downarrow \mathcal{Y}$, note that $\mathcal{Z}$ and $\mathcal{W}$ are both cofibrant and globally fibrant. We claim that a local weak equivalence between globally fibrant simplicial presheaves is a schemewise weak equivalence. In fact, a standard trick in simplicial homotopy theory shows the morphism in question is a homotopy equivalence. A global fibration of simplicial sheaves is also a global fibration of simplicial presheaves, and whether a global fibration of simplicial presheaves is also a motivic fibration can be tested schemewise. This implies the statement in (ii).

For the second half of axiom $\mathcal{M}5$, we proceed as above by forming the diagram:



We want to show that the morphism between $\mathcal{W}$ and $\mathcal{Y}$ is a motivic fibration. Since the morphism between $\mathcal{Z}$ and $\mathcal{Y}$ is a motivic fibration of simplicial presheaves, we may conclude by noting that the motivic trivial cofibration between $\mathcal{Z}$ and $\mathcal{W}$ is a schemewise weak equivalence.

The right adjoint in the adjunction (ii) preserves motivic fibrations and trivial motivic fibrations. Hence we are dealing with a Quillen pair. To show that Nisnevich sheafification is a left Quillen equivalence, let $\mathcal{X}$ be a simplicial presheaf, $\mathcal{Y}$ a motivically fibrant simplicial sheaf, and $\mathcal{Z}$ a motivically fibrant simplicial presheaf. We claim that a morphism in $\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k)_{\sim\mathrm{mot}}$, say

$$\mathcal{X} \longrightarrow \mathcal{Y} \tag{21}$$

is a motivic weak equivalence if and only if Nisnevich sheafification yields a motivic weak equivalence of simplicial sheaves

$$a_{\mathrm{Nis}}\mathcal{X} \longrightarrow \mathcal{Y} . \tag{22}$$

In effect, Lemma 5.28 shows that the canonical map

$$\mathcal{X} \longrightarrow a_{\mathrm{Nis}}\mathcal{X} \tag{23}$$

is a local weak equivalence, hence a motivic weak equivalence. This implies the claim. Moreover, a more refined result holds. Assuming the morphism in (21) is a motivic weak equivalence, we have the diagram of function complexes:

$$\begin{array}{ccc} \mathbf{hom}(\mathcal{Y},\mathcal{Z}) & \longrightarrow & \mathbf{hom}(a_{\mathrm{Nis}}\mathcal{X},\mathcal{Z}) \\ & {\scriptstyle\sim}\searrow \quad \nearrow{\scriptstyle\sim} & \\ & \mathbf{hom}(\mathcal{X},\mathcal{Z}) & \end{array} \tag{24}$$

Now we use the following facts: The local injective model structure is simplicial, every simplicial presheaf is cofibrant in the local injective model structure, the morphism in (23) is a local weak equivalence, and finally $\mathcal{Z}$ is globally fibrant. Then, by a standard result in homotopical algebra, we get the second weak equivalence indicated in (23). We have just shown for every motivically fibrant simplicial presheaf $\mathcal{Z}$ that there is a weak equivalence

$$\mathbf{hom}(\mathcal{Y},\mathcal{Z}) \longrightarrow \mathbf{hom}(a_{\mathrm{Nis}}\mathcal{X},\mathcal{Z}) .$$

In particular, the morphism in (22) is a motivic weak equivalence.

Clearly, the motivic model structur is left proper. For right properness, consider Exercise 5.34. □

Analogously to the work in this section, one shows there exist motivic model structures on the categories of pointed simplicial presheaves and pointed simplicial sheaves on the smooth Nisnevich site of k. The relevant morphisms may be defined via forgetful functors to the unpointed categories.

We will need the following consequences of Nisnevich descent.

Lemma 5.31 *Consider morphisms of motivically fibrant simplicial presheaves*

$$\mathcal{X}_0 \longrightarrow \mathcal{X}_1 \longrightarrow \mathcal{X}_2 \longrightarrow \cdots .$$

(i) Any globally fibrant model

$$\operatorname*{colim}_{n\in\mathbb{N}} \mathcal{X}_n \longrightarrow G \operatorname*{colim}_{n\in\mathbb{N}} \mathcal{X}_n$$

is motivically fibrant.

(ii) Any motivically fibrant model

$$\operatorname*{colim}_{n\in\mathbb{N}} \mathcal{X}_n \longrightarrow \mathcal{X}$$

is a schemewise weak equivalence.

Proof. Nisnevich descent 5.21 shows that the morphism in (i) is a schemewise weak equivalence. The weak equivalences of simplicial sets

$$\mathcal{X}_n(X) \longrightarrow \mathcal{X}_n(X \times_k \mathbb{A}^1_k)$$

induced from the projection map $X \times_k \mathbb{A}^1_k \to X$ induce weak equivalences in the filtered colimit. This implies (i). Item (ii) follows directly from (i). □

We will briefly discuss the notion of 'motivic flasque simplicial presheaves'. Such presheaves occupy a central role in the motivic stable theory presented here. Another motivation is that, based in this notion, Isaksen has constructed motivic flasque model structures for simplicial presheaves [Isa04]. First, we shall recall the notion of flasque presheaves. This goes back to the pioneering work of Brown and Gersten on flasque model structures for simplicial sheaves [BG73]. Recently, Lárusson worked in this theme in [Lár04], demonstrating the applicability of abstract homotopy theoretic methods in complex analysis.

A simplicial presheaf $\mathcal{X}$ of Kan complexes is flasque if every finite collection of subschemes X_i of a scheme X induces a Kan fibration

$$\mathbf{hom}(X, \mathcal{X}) \longrightarrow \mathbf{hom}(\cup X_i, \mathcal{X}) .$$

The union is formed in the presheaf category, i.e. $\cup X_i$ is the coequalizer of the diagram of representable presheaves

$$\textstyle\coprod_{i,j} X_i \times_X X_j \rightrightarrows \coprod_i X_i .$$

There is a canonical monomorphism from $\cup X_i$ to X. In particular, the empty collection of subschemes of X induces the morphism

$$\mathbf{hom}(X, \mathcal{X}) \longrightarrow \mathbf{hom}(\emptyset, \mathcal{X}) .$$

The class of flasque simplicial presheaves is closed under filtered colimits.

Example 5.32 *Globally fibrant simplicial presheaves are flasque.*

Pointed simplicial presheaves $\mathcal{X}$ and $\mathcal{Y}$ have an internal hom

$$\mathbf{Hom}_*(\mathcal{X},\mathcal{Y})\ .$$

In X-sections there is the defining equation

$$\mathbf{Hom}_*(\mathcal{X},\mathcal{Y})(X)=\mathbf{hom}_*(\mathcal{X}|X,\mathcal{Y}|X)\ .$$

Here, the simplicial presheaf $\mathcal{X}|X$ is the restriction of $\mathcal{X}$ to the site $\mathrm{Sm}/k\downarrow X$ along the forgetful functor $\mathrm{Sm}/k\downarrow X\to\mathrm{Sm}/k$. For objects X and Y of Sm/k there is a natural isomorphism

$$\mathbf{Hom}_*(X,\mathcal{Y})(Y)\cong\mathcal{Y}(X\times_k Y)\ .$$

Since the Tate sphere T is the quotient $\mathbb{A}^1_k/(\mathbb{A}^1_k\smallsetminus\{0\})$, it follows that the internal hom $\mathbf{Hom}_*(T,\mathcal{X})$ sits in the pullback square where the right vertical morphism is induced by the inclusion:

$$\begin{array}{ccc}\mathbf{Hom}_*(T,\mathcal{X}) & \longrightarrow & \mathbf{Hom}(\mathbb{A}^1_k,\mathcal{X})\\ \downarrow & & \downarrow\\ * & \longrightarrow & \mathbf{Hom}((\mathbb{A}^1_k\smallsetminus\{0\}),\mathcal{X})\ .\end{array}$$

Jardine uses this fact to prove that when $\mathcal{X}$ is flasque, then so is the internal hom $\mathbf{Hom}_*(T,\mathcal{X})$, and moreover, that $\mathbf{Hom}_*(T,-)$ preserves filtered colimits of simplicial presheaves, and schemewise weak equivalences between flasque simplicial presheaves [Jar00, §1.4].

A flasque simplicial presheaf $\mathcal{X}$ is motivically flasque if for all objects $X\in\mathrm{Sm}/k$ the projection

$$X\times_k\mathbb{A}^1_k\longrightarrow X$$

induces a weak equivalence of simplicial sets

$$\mathcal{X}(X)\longrightarrow\mathcal{X}(X\times_k\mathbb{A}^1_k)\ .$$

If $\mathcal{X}$ is motivically flasque, we have noted that $\mathbf{Hom}_*(T,\mathcal{X})$ is flasque; to see that $\mathbf{Hom}_*(T,\mathcal{X})$ is motivically flasque, it remains to show homotopy invariance. For every $X\in\mathrm{Sm}/k$ we have the fiber sequence

$$\mathbf{Hom}_*(T,\mathcal{X})(X)\longrightarrow\mathcal{X}(X\times_k\mathbb{A}^1_k)\longrightarrow\mathcal{X}(X\times_k(\mathbb{A}^1_k\smallsetminus\{0\}))\ .$$

Comparing with the corresponding fiber sequence for $X\times_k\mathbb{A}^1_k$, it follows that $\mathbf{Hom}_*(T,\mathcal{X})$ is homotopy invariant.

The next lemma summarizes some properties of the internal hom functor $\mathbf{Hom}_*(T,-)$.

Lemma 5.33 *The Tate sphere satisfies the following properties.*

(i) For sequential diagrams of pointed simplicial presheaves, we have

$$\mathbf{Hom}_*(T, \operatorname*{colim}_{n\in\mathbb{N}} \mathcal{X}_n) \cong \operatorname*{colim}_{n\in\mathbb{N}} \mathbf{Hom}_*(T, \mathcal{X}_n) .$$

(ii) If $\mathcal{X}$ is motivically flasque, then so is the internal hom $\mathbf{Hom}_*(T,\mathcal{X})$.

(iii) $\mathbf{Hom}_*(T,-)$ preserves schemewise equivalences between motivic flasque simplicial presheaves.

That the Tate sphere is 'compact' refers to the combination of all the properties listed in 5.33.

Modern formulations of homotopical algebra allow for different approaches to the local injective and motivic model structures. One of these approaches is via Bousfield localization. In the context of cellular model categories, the authorative reference on this subject is Hirschhorn's book [Hir03]. The notion of combinatorial model categories, as introduced by Jeff Smith [Smi], provide acceptable inputs for Bousfield localization. A model category is combinatorial if the model structure is cofibrantly generated, and the underlying category is locally presentable. This makes it quite plausible that all the model structures on simplicial presheaves on the smooth Nisnevich site of k that we discussed are indeed combinatorial. Starting with the schemewise model structure, with schemewise weak equivalences and schemewise cofibrations, the local and the motivic model structure can be constructed using Bousfield localizations. In general, right properness is not preserved under Bousfield localization of model structures. However, the motivic model structure is right proper.

Exercise 5.34 *Compare the proofs of right properness of the motivic model structure in [Jar00, Appendix A] and [MV99, Theorem 2.7].*

This finishes our synopsis of basic motivic unstable homotopy theory.

5.3 Model Structures for Spectra of Spaces

This section deals with the nuts and bolts of the model structures underlying the motivic stable homotopy theory introduced by Voevodsky [Voe98].

The original reference for the material presented in this section is [Jar00]. We will not attempt to cover the motivic symmetric spectra part of Jardine's paper. The main point of working with the category of motivic symmetric spectra is that it furnishes a model category for the motivic stable homotopy theory with an internal symmetric monoidal smash product. These issues, and some other deep homotopical structures, are discussed from an enriched functor point of view in [Dun]. Using a Quillen equivalent model structure for the motivic unstable homotopy category, Hovey [Hov01] constructed a model structure similar to the one we will discuss here. A major difference between the approaches in [Hov01] and in [Jar00] is that Hovey does not use the Nisnevich descent theorem in the construction of the model structure. At any rate, using the internal smash product for symmetric spectra and comparing with ordinary spectra, it follows without much fuss that $\mathrm{SH}(k)$ has the structure of a closed symmetric monoidal and triangulated category. The homotopy categories $\mathrm{SH}_s(k)$ and $\mathrm{SH}_s^{\mathbb{A}^1}(k)$ acquire the exact same type of structure.

First, we discuss the level model structures, and second the stable model structure. There are two level model structures. These structures share the same class of weak equivalences, but their classes of cofibrations and fibrations do not coincide. This is reminiscent of the situation with different models for the motivic unstable homotopy category. The interplay between the level models are important for the construction of the more interesting stable model structure, whose associated homotopy category is the motivic stable homotopy category.

The motivic spectra we consider are suspended with respect to the Tate sphere T, i.e. sequences of pointed simplicial presheaves $E = \{\mathcal{E}_n\}_{n\geq 0}$ on the smooth Nisnevich site of k together with structure maps

$$\sigma\colon T \wedge \mathcal{E}_n \longrightarrow \mathcal{E}_{n+1} \ .$$

The usual compatibility conditions are required for morphisms of motivic spectra. Note that, in the smash product, the Tate sphere is placed on the left hand side.

An optimistic, but homotopy theoretic correct definition of the smash product of two motivic spectra is given by

$$(E \wedge E')_n \ : \ = \begin{cases} \mathcal{E}_i \wedge \mathcal{E}'_i & n = 2i, \\ T \wedge (\mathcal{E}_i \wedge \mathcal{E}'_i) & n = 2i+1. \end{cases}$$

In even degrees, the structure map is the identity, while in degrees $n = 2i+1$ one makes the choice

$$T \wedge (T \wedge (\mathcal{E}_i \wedge \mathcal{E}'_i)) \xrightarrow{\cong} (T \wedge \mathcal{E}_i) \wedge (T \wedge \mathcal{E}'_i) \xrightarrow{\sigma\wedge\sigma'} \mathcal{E}_{i+1} \wedge \mathcal{E}'_{i+1} \ .$$

Then the following diagram commutes, where, up to sign, the left vertical twist isomorphism is homotopic to the identity:

$$\begin{array}{ccccc}
T\wedge(T\wedge(\mathcal{E}_i\wedge\mathcal{E}'_i)) & \xrightarrow[(23)]{\cong} & (T\wedge\mathcal{E}_i)\wedge(T\wedge\mathcal{E}'_i) & \longrightarrow & (T\wedge\mathcal{E}_i)\wedge\mathcal{E}'_{i+1} \\
\cong\big\downarrow(12) & & & & \big\downarrow \\
T\wedge(T\wedge(\mathcal{E}_i\wedge\mathcal{E}'_i)) & & & & \\
\| & & & & \\
T\wedge((T\wedge\mathcal{E}_i)\wedge\mathcal{E}'_i) & & & & \\
\big\downarrow & & & & \\
T\wedge(\mathcal{E}_{i+1}\wedge\mathcal{E}'_i) & \xrightarrow{\cong} & \mathcal{E}_{i+1}\wedge(T\wedge\mathcal{E}'_i) & \longrightarrow & \mathcal{E}_{i+1}\wedge\mathcal{E}'_{i+1}
\end{array}$$

It follows that the suggested smash product of spectra is neither associative nor commutative before passing to the homotopy category. Hence, the smash products and actions are only given up to homotopy. See also Remark 2.14.

There are some set theoretic problems involved in inverting a class of morphisms in a category. Once the model structure has been constructed, we may define the motivic stable homotopy category. Quillen's theory of model structures, or homotopical algebra, provides the foundation for any treatment of motivic stable homotopy theory.

In the motivic levelwise model structures, the weak equivalences are levelwise motivic weak equivalences. We may choose levelwise cofibrations or levelwise fibrations. Both choices induce model structures on motivic spectra.

To construct the motivic stable model structure, we employ the T-loops functor. It is right adjoint to smashing with the Tate sphere functor. This leads to the process of T-stabilization, and a proof of the model axioms for motivic spectra which avoids reference to Nisnevich sheaves of homotopy groups. The cofibrations are defined levelwise. We end the discussion by relating motivic stable weak equivalences to Nisnevich sheaves of bigraded stable homotopy groups of (s,t)-spectra, as in Sect. 2.3.

Although the construction of the motivic stable model structure is more involved, we note that formal techniques originating in the study of spectra of simplicial sets can be hoisted to motivic spectra. We will make use of the approach set forth by Bousfield-Friedlander [BF78], and of injective motivic spectra as a notion for fibrant objects in the motivic level model structure; the latter uses ideas introduced in the Hovey-Shipley-Smith paper on symmetric spectra of simplicial sets [HSS00].

From now on, all simplicial presheaves are pointed.

Definition 5.35 *A morphism of motivic spectra*

$$E \longrightarrow E'$$

is a levelwise equivalence if for every non-negative integer $n \geq 0$*, there is a motivic weak equivalence of simplicial presheaves*

$$\mathcal{E}_n \xrightarrow{\sim \mathrm{mot}} \mathcal{E}'_n .$$

Levelwise cofibrations and levelwise fibrations are defined likewise.

A cofibration is a morphism having the left lifting property with respect to all levelwise equivalences which are levelwise fibrations.

An injective fibration is a morphism having the right lifting property with respect to all levelwise equivalences which are levelwise cofibrations.

Lemma 5.36 *Let $n \geq 1$ and consider a morphism of motivic spectra*

$$i\colon E \longrightarrow E' ,$$

having the additional properties that there are canonically induced cofibrations of simplicial presheaves on the smooth Nisnevich site of k

$$\mathcal{E}_0 \rightarrowtail \mathcal{E}'_0 ,$$

$$\mathcal{E}_n \cup_{T \wedge \mathcal{E}_{n-1}} T \wedge \mathcal{E}'_{n-1} \rightarrowtail T \wedge \mathcal{E}'_n .$$

Then i is a cofibration of motivic spectra.

Proof. Consider the lifting problems where the right hand vertical morphism in the diagram of motivic spectra is a levelwise equivalence and levelwise fibration:

$$\begin{array}{ccc} E & \longrightarrow & F \\ {\scriptstyle i}\downarrow & \overset{s}{\nearrow} & \downarrow \\ E' & \longrightarrow & F' \end{array} \qquad \begin{array}{ccc} \mathcal{E}_n & \longrightarrow & \mathcal{F}_n \\ {\scriptstyle i_n}\downarrow & \overset{s_n}{\nearrow} & \downarrow \\ \mathcal{E}'_n & \longrightarrow & \mathcal{F}'_n \end{array}$$

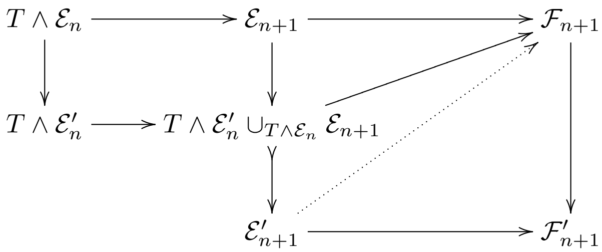
We construct fillers s_n and s by using an induction argument.

If $n = 0$, then since the right hand vertical morphism is a motivically trivial fibration and cofibrations in the motivic model structure are monomorphisms, the filler s_0 exists according to axiom $\mathcal{M}4$ for the motivic model structure.

Suppose that the n-th filler s_n has been constructed. Then, since we are dealing with morphisms of spectra, there is the commutative diagram:

$$\begin{array}{ccccc} T \wedge \mathcal{E}_n & \longrightarrow & T \wedge \mathcal{E}'_n & \xrightarrow{\Sigma_T s_n} & T \wedge \mathcal{F}_n \\ \downarrow & & & & \downarrow \\ \mathcal{E}_{n+1} & & \longrightarrow & & \mathcal{F}_{n+1} \end{array}$$

This allows us to consider the commutative diagram:



Now, the lower central vertical morphism is a cofibration by our assumptions, which implies, using the argument for $n = 0$, that the indicated filler exists. By commutativity of the diagram this morphism is also the $n+1$-th filler.

We leave it to the reader to verify the fact that the fillers assemble into a morphism of motivic spectra. □

Exercise 5.37 *With the same notations and assumptions as in the previous Lemma, show that if the cofibrations of simplicial presheaves are motivic weak equivalences, then i is a level equivalence and cofibration.*

We have collected the crux ingredients needed in the proof of:

Proposition 5.38 *The category of motivic spectra together with the classes of level equivalences, cofibrations, and level fibrations has the structure of a proper simplicial model category.*

The simplicial model structure arises from the smash products $E \wedge K$, where K is a pointed simplicial set, and the function complexes $\mathbf{hom}_*(E, E')$ with n-simplices all morphisms $E \wedge \Delta[n]_-+ \rightarrow E'$. In this definition, we consider the standard n-simplicial set with an added disjoint base-point as a constant pointed space.

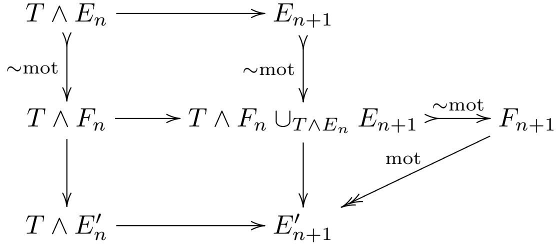
Suppose we want to factor a morphism of motivic spectra

$$E \longrightarrow E'$$

into a cofibration and a level equivalence, followed by a level fibration.

In level zero, this follows from the motivic model structure.

Assume there exist such factorizations up to level n, and consider the commutative diagram:



The left vertical cofibration and motivic weak equivalence are both part of the induction hypothesis. Trivial cofibrations are closed under pushouts in any model category, so that we get the right vertical cofibration and motivic weak equivalence. There is a canonical morphism from the pushout to E'_{n+1} which we may factor in the motivic model structure, as depicted in the diagram. Using 5.36, this clearly produces a motivic spectrum consisting of the F_i's together with the factorization we wanted.

Exercise 5.39 *Finish the proof of 5.38.*

There is the following analogous result involving injective fibrations. We will not dwell into the details of the proof, which uses a transfinite small object argument, since the techniques are reminiscent of what we have seen for simplicial presheaves. The method of proof is to show for motivic spectra properties analogous of **P**1-**P**7.

Proposition 5.40 *The category of T-spectra together with the classes of level equivalences, level cofibrations, and injective fibrations is a proper simplicial model category.*

If $\mathcal{X}$ is a simplicial presheaf on Sm$/k$, its T-loops functor is defined by setting

$$\Omega_T \mathcal{X} := \mathbf{Hom}_*(T, \mathcal{X}) .$$

Taking T-loops is right adjoint to smashing with T.

The T-loops $\Omega_T E$ of a motivic spectrum E is defined by setting

$$\begin{aligned}(\Omega_T E)_n &:= \Omega_T \mathcal{E}_n \\ &= \mathbf{Hom}_*(T, \mathcal{E}_n) .\end{aligned}$$

The structure maps

$$\sigma_T \colon T \wedge (\Omega_T E)_n \longrightarrow (\Omega_T E)_{n+1}$$

are defined, and this is a possible source for confusion, by taking the adjoint of the composite morphism

$$T \wedge \Omega_T \mathcal{E}_n \wedge T \longrightarrow T \wedge \mathcal{E}_n \longrightarrow \mathcal{E}_{n+1} .$$

The T-loops functor

$$E \longmapsto \Omega_T E$$

is right adjoint to smashing with the Tate sphere on the right

$$E \longmapsto E \wedge T .$$

This gives an alternate way of describing the structure maps of $\Omega_T E$ by taking adjoints

$$\sigma_T^*\colon \mathcal{E}_n \longrightarrow \mathbf{Hom}_*(T, \mathcal{E}_{n+1}) \ .$$

There is another functor Ω_T^ℓ which Jardine calls the 'fake T-loop functor' [Jar00]. By definition, there is the T-spectrum

$$(\Omega_T E)_n = (\Omega_T^\ell E)_n \ ,$$

with structure maps adjoint to the morphisms obtained by applying Ω_T to σ_T^*. That is,

$$\Omega_T(\sigma_T^*)\colon \Omega_T(\mathcal{E}_n) \longrightarrow \Omega_T(\mathbf{Hom}_*(T, \mathcal{E}_{n+1})) \ .$$

The reason for the letter ℓ is that the fake T-loops functor

$$E \longmapsto \Omega_T^\ell E$$

is right adjoint to smashing with the Tate sphere on the left

$$E \longmapsto T \wedge E \ ,$$

where $\sigma_n^{T\wedge E} = T \wedge \sigma_n^E$.

Suppose that m is an integer and E is a motivic spectrum. Then a shifted motivic spectrum $E[m]$ is obtained, in the range where it makes sense, by setting

$$E[m]_n \ : \ = \begin{cases} \mathcal{E}_{m+n} & m+n \geq 0, \\ * & m+n < 0. \end{cases}$$

The structure maps are reindexed accordingly. Note that $E[m]$ gives iterated suspensions when m is positive, and iterated loops when m is negative.

Exercise 5.41 *Let E be a motivic spectrum. Is it true that the morphisms*

$$(E \wedge T)_n = \mathcal{E}_n \wedge T \xrightarrow{\cong} T \wedge \mathcal{E}_n \to \mathcal{E}_{n+1} = E[1]_n$$

form a morphism of motivic spectra between $E \wedge T$ and $E[1]$?

The case $m = 1$ is particularly important because the morphisms σ_T^* determine a morphism of motivic spectra

$$\sigma_T^*\colon E \longrightarrow \Omega_T^\ell E[1] \ .$$

By iterating the above, as many times as there are natural numbers, we get the sequence

$$E \xrightarrow{\sigma_T^*} \Omega_T^\ell E[1] \xrightarrow{\Omega_T^\ell \sigma_T^*[1]} (\Omega_T^\ell)^2 E[2] \xrightarrow{(\Omega_T^\ell)^2 \sigma_T^*[2]} \cdots \ . \qquad (25)$$

Let $Q_T E$ denote the colimit of the diagram (25), and consider the canonically induced morphism

$$\eta_E \colon E \longrightarrow Q_T E$$

The functor Q_T is called the stabilization functor for the Tate sphere.

We will have occasions to consider the level fibrant model of E obtained from the motivic levelwise model structure

$$j_E \colon E \longrightarrow JE \ ,$$

and the composite morphism

$$\tilde{\eta}_E \colon E \xrightarrow{j_E} JE \xrightarrow{\eta_{JE}} Q_T JE \ .$$

We are ready to define stable equivalences and stable fibrations.

Definition 5.42 *Let*

$$\phi \colon E \longrightarrow E'$$

be a morphism of motivic spectra. Then

(i) ϕ is a stable equivalence if it induces a level equivalence

$$Q_T J(\phi) \colon Q_T JE \longrightarrow Q_T JE' \ .$$

(ii) ϕ is a stable fibration if it has the right lifting property with respect to all morphisms which are cofibrations and stable equivalences.

A first observation is

Lemma 5.43 *Level equivalences are stable equivalences.*

Proof. Taking the level fibrant model of a level equivalence yields a level equivalence between level fibrant motivic spectra. Now, in each level there is a motivic weak equivalence of motivically fibrant objects, so that a standard argument for simplicial model categories shows that we are dealing with a schemewise weak equivalence of motivically flasque objects. We may conclude since the Tate sphere is compact according to 5.33. □

Lemma 5.43 shows that every stable fibration is a levelwise fibration.

The main theorem in this section is:

Theorem 5.44 *The category of motivic spectra together with the classes of stable equivalences, cofibrations, and stable fibrations forms a proper simplicial model category.*

In the proof, we make use of the following Lemma.

Lemma 5.45 *Let E and E' be motivic spectra.*

(i) A levelwise fibration

$$\phi\colon E \longrightarrow E'$$

is a stable fibration if there is a level homotopy cartesian diagram:

$$\begin{array}{ccc} E & \longrightarrow & Q_TJE \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle Q_TJ(\phi)} \\ E' & \longrightarrow & Q_TJE' \end{array}$$

(ii) Stable equivalences are closed under pullbacks along level fibrations.
(iii) If E is stably fibrant, then E is level fibrant, and there are schemewise weak equivalences

$$\sigma_T^*\colon \mathcal{E}_n \longrightarrow \mathbf{Hom}_*(T, \mathcal{E}_{n+1}) .$$

In particular, from (i) and (iii), we note that E is stably fibrant if and only if E is level fibrant and for all $n \geq 0$, there are schemewise equivalences

$$\sigma_T^*\colon \mathcal{E}_n \longrightarrow \mathbf{Hom}_*(T, \mathcal{E}_{n+1}) \ .$$

Proof. Given 5.43, item (i) follows provided there are levelwise equivalences

$$Q_TJ(\tilde{\eta}_E) = Q_TJ(\eta_{JE}) \circ Q_TJ(j_E)\colon Q_TJE \longrightarrow Q_TJ^2E \longrightarrow (Q_TJ)^2E$$

$$\tilde{\eta}_{Q_TJE} = \eta_{JQ_TJE} \circ j_{Q_TJE}\colon Q_TJE \longrightarrow JQ_TJE \longrightarrow (Q_TJ)^2E \ .$$

Let us consider the first level equivalence. The morphism $Q_TJ(j_E)$ is a level equivalence by construction. To show that $Q_TJ(\eta_{JE})$ is a level equivalence, we consider the commutative diagram:

$$\begin{array}{ccc} Q_TJE & \xrightarrow{Q_T(\eta_{JE})} & Q_TQ_TJE \\ {\scriptstyle Q_T(j_{JE})}\downarrow & & \downarrow{\scriptstyle Q_T(j_{Q_TJE})} \\ Q_TJ^2E & \xrightarrow{Q_TJ(\eta_{JE})} & (Q_TJ)^2E \end{array}$$

A cofinality argument shows $Q_T(\eta_{JE})$ is an isomorphism since $\mathbf{Hom}_*(T, -)$ commutes with sequential colimits of pointed simplicial presheaves; cp. 5.33. Partially by definition, the morphism j_{JE} is a level equivalence which in each level consists of motivically flasque simplicial presheaves. Lemma 5.33, see (i) and (ii), shows that stabilizing with respect to the Tate sphere T preserves this property, so that $Q_T(j_{JE})$ is a level equivalence.

Now the crux of the proof is that Nisnevich descent and compactness of the Tate sphere imply that the right vertical morphism is a level equivalence, see 5.31(ii), applied to $j_{Q_T JE}$, and 5.33(i), (ii). The above implies that the composition $Q_T J(\tilde{\eta}_E)$ is a level equivalence.

Next, we consider the second level equivalence. We have already noted the left vertical schemewise weak equivalence and upper horizontal isomorphism in the commutative diagram:

$$\begin{array}{ccc} (Q_T JE)_n & \xrightarrow{\sigma_T^*} & \mathbf{Hom}_*(T, (Q_T JE)_{n+1}) \\ \Big\downarrow{\scriptstyle j_{Q_T JE}} & & \Big\downarrow{\scriptstyle \mathbf{Hom}_*(T, j_{Q_T JE})} \\ (Q_T JE)_{n+1} & \xrightarrow{\sigma_T^*} & \mathbf{Hom}_*(T, (JQ_T JE)_{n+1}) \end{array}$$

Lemma 5.33(i), and (ii) applied to $j_{Q_T JE}$ implies the level equivalence

$$\eta_{JQ_T JE} \colon JQ_T JX \longrightarrow Q_T JQ_T JX \ .$$

This implies the level equivalence $\tilde{\eta}_{Q_T JE}$.

Item (ii) follows from properness of the motivic levelwise model structure with level fibrations, together with a straight-forward argument.

In the proof of (iii), we employ the motivic levelwise model structure with injective fibrations. There is a natural level cofibration and level equivalence of motivic spectra

$$i_E \colon E \longrightarrow IE,$$

where IE is injective. Generally, a level equivalence with an injective target is called an injective model for the source.

Exercise 5.46 *Show that a morphism of motivic spectra*

$$\phi \colon E \longrightarrow E'$$

is a stable equivalence if and only if it induces a level equivalence

$$IQ_T J(\phi) \colon IQ_T JE \longrightarrow IQ_T JE' \ .$$

Show that $IQ_T JE$ is stably fibrant.

We have shown that the composite morphism

$$E \xrightarrow{j_E} JE \xrightarrow{\eta_{JE}} Q_T JE \xrightarrow{i_{Q_T JE}} IQ_T JE$$

is a stable equivalence. We may factor the latter morphism into a cofibration, followed by a level equivalence and level fibration:

$$\begin{array}{ccc} E & \longrightarrow & IQ_TJE \\ & \searrow \quad \nearrow & \\ & E' & \end{array} \tag{26}$$

Note that the morphism

$$E' \longrightarrow IQ_TJE$$

is a stable fibration because it has the right lifting property with respect to all cofibrations. Hence E' is stably fibrant, and there are schemewise weak equivalences in all levels

$$\sigma_T^* \colon \mathcal{E}'_n \longrightarrow \mathbf{Hom}_*(T, \mathcal{E}'_{n+1}).$$

Moreover, the cofibration in diagram (26) is also a stable equivalence according to 5.43 and the two out of three property of stable equivalences, so that E is a retract of E'. The result follows. □

Corollary 5.47 *The following hold.*

(i) A morphism of motivic spectra is a stable fibration and stable equivalence if and only if it is a level fibration and a level equivalence.
(ii) Every level fibration between two stably fibrant motivic spectra is a stable fibration.

Following the script for ordinary spectra, our aim is now to finish the proof of 5.44. There is really only axiom $\mathcal{M}5$ which requires a comment.

Proof. First, we note that the category of motivic spectra

$$\mathrm{Spt}(\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k), T)$$

is bicomplete: If F is a functor from a small category I to motivic spectra, one puts

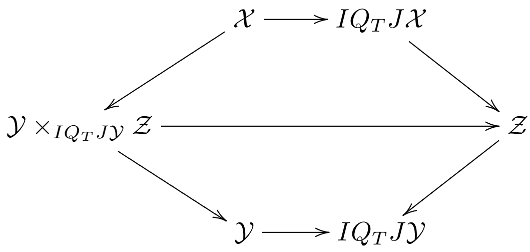
$$(\operatorname*{colim}_{i\in I} F(i))_n \colon = \operatorname*{colim}_{i\in I} (F(i))_n,$$
$$(\lim_{i\in I} F(i))_n \colon = \lim_{i\in I} (F(i))_n \ .$$

When forming colimits, the structure maps are given by

$$\operatorname*{colim}_{i\in I} \sigma : T \wedge (\operatorname*{colim}_{i\in I} (F(i))_n) \cong \operatorname*{colim}_{i\in I} (T \wedge F(i)_n) \longrightarrow \operatorname*{colim}_{i\in I} (F(i)_n) \ .$$

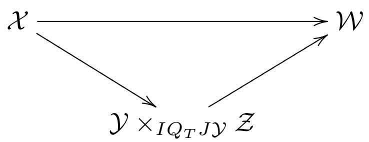
The isomorphism we use above arises from the canonical morphism from the colimit of the suspension with T functor to the same functor applied to the colimit; since the suspension is a left adjoint – which we have inverted – this is an isomorphism. When forming limits, the structure maps are defined similarly using the adjoint structure maps σ_T^*. Axiom $\mathcal{M}1$ for $\mathrm{Spt}(\Delta^{\mathrm{op}}\mathrm{Pre}_{\mathrm{Nis}}(\mathrm{Sm}/k), T)$ follows immediately.

What remains to be proven is the trivial stable cofibration and stable fibration part of axiom $\mathcal{M}5$. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of motivic spectra and form the commutative diagram:



The right hand side makes use of the cofibration–level equivalence and level fibration factorization axiom. Then, $\mathcal{Z}$ is level fibrant and in each level, the cofibration–level equivalence is a schemewise equivalence of motivically flasque simplicial presheaves; it follows that $\mathcal{Z}$ is stably fibrant, and the level fibration is a stable fibration.

Since stable fibrations are closed under pullbacks, $\mathcal{Y} \times_{IQ_TJ\mathcal{Y}} \mathcal{Z} \longrightarrow \mathcal{Z}$ is a stable fibration as well. Via part (ii) of Lemma 5.45 and $\mathcal{M}2$, the morphism $\mathcal{X} \longrightarrow \mathcal{Y} \times_{IQ_TJ\mathcal{Y}} \mathcal{Z}$ is a stable equivalence. Factor this stable equivalence into a cofibration composed with a level fibration–level equivalence:



On account of 5.43, the cofibration in the diagram is a stable equivalence. On the other hand, the level fibration–level equivalence is a stable fibration according to 5.47. This proves the, in our case, non-trivial part of axiom $\mathcal{M}5$. □

The same definitions and arguments give the stable model structure for motivic spectra of spaces. We will leave the analogous formulations for spaces to the reader.

Finally, we shall interpret stable equivalences in terms of Nisnevich sheaves of bigraded stable homotopy groups of (s,t)-spectra. The first step is to make sense of the following statement:

Lemma 5.48 *There is a stable equivalence of motivic spectra*

$$E \longrightarrow E'$$

if and only if there is an isomorphism of motivic stable homotopy presheaves

$$\pi_{p,q}E \cong \pi_{p,q}E' \ .$$

When E is level fibrant, the starting point for defining motivic stable homotopy groups is the filtered colimit

$$\mathcal{E}_n \xrightarrow{\sigma_T^*} \Omega_T\mathcal{E}_{n+1} \xrightarrow{\Omega_T\sigma_T^*} \Omega_T^2\mathcal{E}_{n+2} \xrightarrow{\Omega_T^2\sigma_T^*} \cdots \ .$$

In X-sections, we get the groups $\pi_p Q_T \mathcal{E}_n(X)$ defined as the filtered colimit of the diagram

$$\pi_p(\mathcal{E}_n)(X) \xrightarrow{\pi_p(\sigma_T^*)(X)} \pi_p(\Omega_T\mathcal{E}_{n+1})(X) \xrightarrow{\pi_p(\Omega_T\sigma_T^*)(X)} \pi_p(\Omega_T^2\mathcal{E}_{n+2})(X) \xrightarrow{\pi_p(\Omega_T^2\sigma_T^*)(X)} \cdots \ .$$

Passing to the homotopy category associated to the motivic stable model structure over the scheme X, we can recast the latter as

$$[S_s^p, \mathcal{E}_n|X] \longrightarrow [S_s^p \wedge T, \mathcal{E}_{n+1}|X] \longrightarrow [S_s^p \wedge T^{\wedge 2}, \mathcal{E}_{n+2}|X] \longrightarrow \cdots \ .$$

Next, we want to rewrite this colimit taking into account the unstable version of 2.21.

Lemma 5.49 *There is a motivic weak equivalence between the Tate sphere T and the smash product $S_s^1 \wedge S_t^1$.*

Now, working in the homotopy category, so that we no longer need to impose the fibrancy condition, one obtains an alternative way of considering the groups in X-sections by taking the filtered colimit of the diagram

$$[S_s^p, \mathcal{E}_n|X] \longrightarrow [S_s^{p+1} \wedge S_t^1, \mathcal{E}_{n+1}|X] \longrightarrow [S_s^{p+2} \wedge S_t^2, \mathcal{E}_{n+2}|X] \longrightarrow \cdots \ .$$

Definition 5.50 *Let E be a motivic spectrum. The degree p and weight q motivic stable homotopy presheaf $\pi_{p,q}E$ is defined in X-sections by setting*

$$\pi_{p,q}E(X) := \operatorname*{colim}_{p,q\in\mathbb{Z}} \left([S_s^{p+n} \wedge S_t^{q+m}, \mathcal{E}_n|X] \longrightarrow [S_s^{p+n+1} \wedge S_t^{q+m+1}, \mathcal{E}_{n+1}|X] \cdots\right) .$$

Exercise 5.51 *Define*

$$\Omega_{S_s^1}(-) := \mathbf{Hom}_*(S_s^1, -) \ ,$$
$$\Omega_{S_t^1}(-) := \mathbf{Hom}_*(S_t^1, -) \ .$$

Show the presheaf isomorphisms

$$\pi_{p,q}E \cong \begin{cases} \pi_0 \Omega^{p-q}_{S^1_s} Q_T(JE[-q])_0 & p \geq q \ , \\ \pi_0 \Omega^{q-p}_{S^1_t} Q_T(JE[-p])_0 & p \leq q \ . \end{cases}$$

Unraveling the indices, one finds the identification

$$\pi_{p,q}E(X) \cong \pi_{p-q} Q_T J\mathcal{E}_{-q}(X) \ . \tag{27}$$

Next we look into the proof of 5.48.

Proof. A stable equivalence between E and E' induces for all integers $m \in \mathbb{Z}$ the levelwise schemewise weak equivalence

$$Q_T JE[m] \longrightarrow Q_T JE'[m] \ .$$

Hence, in all sections, the induced maps between motivic stable homotopy groups of E and E' are isomorphisms (27).

Conversely, if $\pi_{p,q}E$ and $\pi_{p,q}E'$ are isomorphic presheaves for $p \geq q \leq 0$, then there is a levelwise weak equivalence

$$Q_T JE \longrightarrow Q_T JE' \ .$$

This shows that $E \to E'$ is a stable equivalence. □

Because of the motivic weak equivalence between T and $S^1_s \wedge S^1_t$, we may switch between motivic spectra and $S^1_s \wedge S^1_t$ spectra [Jar00, 2.13]. In other words, a motivic spectrum consists of pointed simplicial presheaves $\{\mathcal{E}_n\}_{n\geq 0}$ and structure maps

$$S^1_s \wedge S^1_t \wedge \mathcal{E}_n \longrightarrow \mathcal{E}_{n+1} \ .$$

Using this description, we shall see that a motivic spectrum E yields an (s,t)-bispectrum $E_{*,*}$ as discussed in the beginning of Sect. 2.3:

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \\ S^2_t \wedge \mathcal{E}_0 & S^1_t \wedge \mathcal{E}_1 & \mathcal{E}_2 & \cdots \\ S^1_t \wedge \mathcal{E}_0 & \mathcal{E}_1 & S^1_s \wedge \mathcal{E}_2 & \cdots \\ \mathcal{E}_0 & S^1_s \wedge \mathcal{E}_1 & S^2_s \wedge \mathcal{E}_2 & \cdots \end{array}$$

In the s-direction there are structure maps

$$\sigma_s : S^1_s \wedge E_{m,n} \longrightarrow E_{m+1,n} \ .$$

If $m \geq n$, we use the identity morphism. If $m < n$, we use the morphism obtained from switching smash factors

$$S_s^1 \wedge S_t^n \wedge \mathcal{E}_m \xrightarrow{\tau\wedge 1} S_t^{n-1} \wedge S_s^1 \wedge S_t^1 \wedge \mathcal{E}_m \xrightarrow{1\wedge\sigma} S_t^{n-1} \wedge \mathcal{E}_{m+1} \ .$$

Similarly, in the t-direction there are structure maps

$$\sigma_s : S_t^1 \wedge E_{m,n} \longrightarrow E_{m,n+1} \ .$$

If $m > n$, we use the morphism obtained from switching smash factors

$$S_t^1 \wedge S_s^m \wedge \mathcal{E}_n \xrightarrow{\tau\wedge 1} S_s^{m-1} \wedge S_t^1 \wedge S_s^1 \wedge \mathcal{E}_n \xrightarrow{1\wedge\sigma} S_s^{m-1} \wedge \mathcal{E}_{n+1} \ .$$

If $m \leq n$, we use the identity morphism.

Associated to an (s,t)-bispectrum $E_{*,*}$, there are presheaves of bigraded stable homotopy groups $\pi_{p,q}E$. In X-sections, one considers the colimit of the diagram:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\uparrow{\scriptstyle (\sigma_t)_*} & & \uparrow{\scriptstyle (\sigma_t)_*} & & \\
[S_s^{p+m} \wedge S_t^{q+n+1}, E_{m,n+1}|X] & \xrightarrow{(\sigma_s)_*} & [S_s^{p+m+1} \wedge S_t^{q+n+1}, E_{m+1,n+1}|X] & \longrightarrow & \cdots \\
\uparrow{\scriptstyle (\sigma_t)_*} & & \uparrow{\scriptstyle (\sigma_t)_*} & & \\
[S_s^{p+m} \wedge S_t^{q+n}, E_{m,n}|X] & \xrightarrow{(\sigma_s)_*} & [S_s^{p+m+1} \wedge S_t^{q+n}, E_{m+1,n}|X] & \longrightarrow & \cdots
\end{array}$$

Exercise 5.52 *In the above diagram, explain why there is no loss of generality in assuming that $E_{m,n}$ is motivically fibrant for all $m, n \in \mathbb{Z}$.*

Explicate the maps $(\sigma_s)_*$ and $(\sigma_t)_*$.

A cofinality argument shows the colimit of the above diagram of X-sections can be obtained by taking the diagonal and employing the transition maps $(\sigma_s)_*$ and $(\sigma_t)_*$ in either order. In particular, starting with a motivic spectrum, its degree p and weight q motivic stable homotopy presheaf is isomorphic to the bigraded presheaf $\pi_{p,q}$ of its associated (s,t)-bispectrum.

Lemma 5.48 and the previous observation show that

$$E \longrightarrow E'$$

is a stable equivalence if and only if there is an isomorphism of bigraded presheaves

$$\pi_{p,q}E_{*,*} \longrightarrow \pi_{p,q}E'_{*,*} \ .$$

The structure maps in the t-direction determine the sequence of morphisms of s-spectra

$$E_{*,0} \xrightarrow{(\sigma_t)_*} \Omega_{S^1_t} E_{*,1} \xrightarrow{\Omega_{S^1_t}(\sigma_t)_*} \Omega^2_{S^1_t} E_{*,1} \xrightarrow{\Omega^2_{S^1_t}(\sigma_t)_*} \cdots .$$

The presheaf $\pi_{p,q}E$ is the filtered colimit of the presheaves in the diagram

$$s\pi_p \Omega^{q+n}_{S^1_t} JE_{*,n} \longrightarrow \pi_p \Omega^{q+n+1}_{S^1_t} JE_{*,n+1} \longrightarrow \cdots .$$

To conclude the discussion of homotopy groups, let E be a motivic spectrum, and note that a cofinality argument implies there is a natural isomorphism of bigraded presheaves

$$\pi_{p,q}E \cong \pi_{p,q}E_{*,*} .$$

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