C-system of a finitary monad

Vladimir Voevodsky[1,2]

July 2014

Abstract

We construct for any finitary monad M on sets a C-system (contextual category) $CC(M)$ and describe, using the results of [2], the subsystems of $CC(M)$ directly in terms of M. In the special case of the monads of expressions associated with dependent signatures these constructions lead to the contextual categories of computation-free dependent type systems.

1 Finitary monads on sets

A finitary monad (on sets) is a monad $M : Sets \to Sets$ that, as a functor, commutes with filtering colimits. Since any set is, canonically, the colimit of the filtering diagram of its finite subsets, a functor $Sets \to Sets$ that commutes with filtering colimits can be equivalently described as a functor $FSets \to Sets$ where $FSets$ is the category of finite sets.

The monad structure consists of two families of maps:

1. for any $X \in Sets$, a function $X \to M(X)$,
2. for any $X \in Sets$, a function $M(M(X)) \to M(X)$

which satisfy certain conditions. Given a monad structure on a functor M which commutes with filtering colimits, finite sets X, X' and a function $X' \to M(X)$, the composition $M(X') \to M(M(X)) \to M(X)$ is a function $M(X') \to M(X)$. This allows one to describe finitary monads directly in terms of functors $FSets \to Sets$ as follows:

Lemma 1.1 *[**2014.06.30.l1**] The construction outline above defines an equivalence between (the type of) finitary monads on Sets and (the type of) collections of data of the form:*

1. *for every finite set X a set $M(X)$,*
2. *for every finite set X a function $\iota_X : X \to M(X)$,*
3. *for every finite sets X, X' and a function $f : X \to M(X')$, a function $\mu(f) : M(X) \to M(X')$*

which satisfy the following conditions:

1. *for a finite set X, $\mu(\iota_X) = id_{M(X)}$,*
2. *for a function of finite sets $f : X \to X'$, $\iota_X \mu(f) = f$,*
3. *for two functions $f : X \to M(X')$, $g : X' \to M(X'')$, $\mu(f\mu(g)) = \mu(f)\mu(g)$.*

Proof: See [].

The description of finitary monads given by Lemma 1.1 is much more convenient than the direct definition for formalization.

For $T \in M(\{x_1, \ldots, x_n\})$ and $f : X \to M(X')$ such that $f(x_i) = f_i$ we write $\mu(f)(T)$ as $T(f_1/x_1, \ldots, f_n/x_n)$.

For a finitary monad M we let $M - cor$ ("M-correspondences") to be the full subcategory of the Kleisli category of M whose objects are finite sets. Recall, that the set of morphisms from X to Y in $M - cor$ is the set of maps from X to $M(Y)$ i.e. $M(Y)^X$ (in other words, $M - cor$ is the category of free, finitely generated M-algebras).

[1]School of Mathematics, Institute for Advanced Study, Princeton NJ, USA. e-mail: vladimir@ias.edu
[2]Work on this paper was supported by NSF grant 1100938.

We further let $C(M)$ denote the pre-category[3] with

$$Ob(C(M)) = \mathbf{N}$$

$$Mor(C(M)) = \coprod_{m,n\in\mathbf{N}} M(\{1,\ldots,m\})^n$$

which is equivalent, as a category, to $(M - cor)^{op}$.

For $E \in M(\{1,\ldots,m\})$ and $n \geq 1$ we set:

$$t_n(E) = E[n+1/n, n+2/n+1, \ldots, m+1/m]$$

$$s_n(E) = E[n/n+1, n+1/n+2, \ldots, m-1/m]$$

If we were numbering elements of sets with n-elements from 0 then we would have $t_n = M(\partial_{n-1})$ and $s_n = M(\sigma_{n-1})$ where ∂_i and σ_i are the usual generators of the simplicial category.

Remark 1.2 The correspondence $M \mapsto C(M)$ defines an equivalence between the type of the finitary monads on $Sets$ and the type of the pre-category structures on $\mathbf{N}$ which extend the a pre-category structure of finite sets and where the addition remains to be the coproduct.

Remark 1.3 A finitary sub-monad of M is the same as a sub-pre-category in $C(M)$ which contains all objects. Intersection of two sub-monads is a sub-monad which allows one to speak of sub-monads generated by a set of elements.

2 C-system defined by a monad.

Let $CC(M)$ be the pre-category whose set of objects is $Ob(CC(M)) = \amalg_{n\geq 0} Ob_n$ where

$$Ob_n = M(\emptyset) \times \ldots \times M(\{1,\ldots,n-1\})$$

and the set of morphisms is

$$Mor(CC(M)) = \coprod_{m,n\geq 0} Ob_m \times Ob_n \times M(\{1,\ldots,m\})^n$$

with the obvious domain and codomain maps. The composition of morphisms is defined in the same way as in $C(M)$ such that the mapping $Ob(CC(M)) \to \mathbf{N}$ which sends all elements of Ob_n to n, is a functor from $CC(M)$ to $C(M)$. The associativity of compositions follows immediately from the associativity of compositions in $M - cor$.

Note that if $M(\emptyset) = \emptyset$ then $CC(M) = \emptyset$ and otherwise the functor $CC(M) \to C(M)$ is an equivalence, so that in the second case $C(M)$ and $CC(M)$ are indistinguishable as categories. However, as pre-categories they are quite different.

The pre-category $CC(M)$ is given the structure of a C-system as follows. The final object is the only element of Ob_0, the map ft is defined by the rule

$$ft(T_1,\ldots,T_n) = (T_1,\ldots,T_{n-1}).$$

The canonical pull-back square defined by an object $(T_1,\ldots,T_{n+1})$ and a morphism

$$(f_1,\ldots,f_n) : (R_1,\ldots,R_m) \to (T_1,\ldots,T_n)$$

is of the form:

$$\begin{tikzcd}
(R_1,\ldots,R_m,T_{n+1}(f_1/1,\ldots,f_n/n)) \arrow[r, "{(f_1,\ldots,f_n,m+1)}"] \arrow[d, "{(1,\ldots,m)}"'] & (T_1,\ldots,T_{n+1}) \arrow[d, "{(1,\ldots,n)}"] \\
(R_1,\ldots,R_m) \arrow[r, "{(f_1,\ldots,f_n)}"] & (T_1,\ldots,T_n)
\end{tikzcd} \quad (1)$$

[3] See the introduction to [2].

Proposition 2.1 *[2009.10.01.prop2] With the structure defined above $CC(M)$ is a C-system.*

Proof: Straightforward.

Any morphism of monads $M \to M'$ defines a C-system morphism $CC(M) \to CC(M')$.

Remark 2.2 There is another construction of a pre-category from a finitary monad M which takes as an additional parameter a set Var which is called the set of variables. Let $F_n(Var)$ be the set of sequences of length n of pair-wise distinct elements of Var. Define the pre-category $CC(M, Var)$ as follows. The set of objects of $CC(M, Var)$ is

$$Ob(CC(M, Var)) = \amalg_{n\geq 0} \amalg_{(x_1,\ldots,x_n)\in F_n(Var)} M(\emptyset) \times \ldots \times M(\{x_1, \ldots, x_{n-1}\})$$

For notational compatibility with the traditional type theory we will write the elements of $Ob(CC(M, X))$ as sequences of the form $x_1 : E_1, \ldots, x_n : E_n$. The set of morphisms is given by

$$Hom_{CC(M,,Var)}((x_1 : E_1, \ldots, x_m : E_m), (y_1 : T_1, \ldots, y_n : T_n)) = M(\{x_1, \ldots, x_m\})^n$$

The composition is defined in such a way that the projection

$$(x_1 : E_1, \ldots, x_n : E_n) \mapsto (E_1, E_2(1/x_1), \ldots, E_n(1/x_1, \ldots, n-1/x_{n-1}))$$

is a functor from $CC(M, Var)$ to $CC(M)$.

This functor is clearly an equivalence of categories but not an isomorphism of pre-categories.

There is an obvious final object and ft map on $CC(M, Var)$.

There is however a real problem in making it into a C-system which is due to the following. Consider an object $(y_1 : T_1, \ldots, y_{n+1} : T_{n+1})$ and a morphism $(f_1, \ldots, f_n) : (x_1 : R_1, \ldots, x_m : R_m) \to (y_1 : T_1, \ldots, y_n : T_n)$. In order for the functor to $CC(M)$ to be a C-system morphism the canonical square build on this pair should have the form

$$\begin{array}{ccc} (x_1 : R_1, \ldots, x_m : R_m, x_{m+1} : T_{n+1}(f_1/1, \ldots, f_n/n)) & \xrightarrow{(f_1,\ldots,f_n,x_{n+1})} & (y_1 : T_1, \ldots, y_{n+1} : T_{n+1}) \\ \downarrow & & \downarrow \\ (x_1 : R_1, \ldots, x_m : R_m) & \xrightarrow{(f_1,\ldots,f_n)} & (y_1 : T_1, \ldots, y_n : T_n) \end{array}$$

where x_{m+1} is an element of Var which is distinct from each of the elements $x_1, \ldots, x_m$. Moreover, we should choose x_{m+1} in such a way the the resulting construction satisfies the C-system axioms for $(f_1, \ldots, f_n) = Id$ and for the compositions $(g_1, \ldots, g_m) \circ (f_1, \ldots, f_n)$. One can easily see that no such choice is possible for a finite set Var. At the moment it is not clear to me whether or not it is possible for an infinite Var.

Remark 2.3 The pre-category $C(M)$ also extends to a C-system which is defined as follows. The final object is 0. The map ft is given by

$$ft(n) = \begin{cases} 0 & \text{if } n = 0 \\ n-1 & \text{if } n > 0 \end{cases}$$

The canonical projection $n \to n-1$ is given by the sequence $(1, \ldots, n-1)$. For $f = (f_1, \ldots, f_m) : n \to m$ the canonical square build on f and the canonical projection $m + 1 \to m$ is of the form

$$\begin{array}{ccc} n+1 & \xrightarrow{(f_1,\ldots,f_m,n+1)} & m+1 \\ \downarrow & & \downarrow \\ n & \xrightarrow{(f_1,\ldots,f_m)} & m \end{array}$$

Any morphism of monads $M \to M'$ defines a C-system morphism $C(M) \to C(M')$. Non-trivial C-subsystems of $C(M)$ are in one-to-one correspondence with finitary sub-monads of M.

The natural projection $CC(M) \to C(M)$ is a C-system homomorphism. It's C-system sections are in one-to-one correspondence with $M(\emptyset)$ such that $U \in M(\emptyset)$ corresponds to the section which takes the object n of $C(M)$ to the object $(U, \ldots, U)$ of $CC(M)$

Let us describe now the B-sets of the C-system $CC(M)$. Recall from [2] the B-sets of a C-system CC are the sets of two families:

$$B_n(CC) = \{X \in Ob(C) | l(X) = n\}$$

$$\widetilde{B}_{n+1}(CC) = \{(X, s) | X \in B_{n+1}(CC), s : ft(X) \to X, s \circ p_X = Id_{ft(X)}\}$$

Lemma 2.4 *[***2014.06.30.l2***] One has:*

$$B_n(CC(M)) = M(\emptyset) \times \ldots \times M(\{1, \ldots, n-1\})$$

$$\widetilde{B}_{n+1}(CC(M)) = M(\emptyset) \times \ldots \times M(\{1, \ldots, n-1\}) \times M(\{1, \ldots, n\})^2$$

Proof: By definition we have

$$Ob(CC(M)) = \coprod_{n \geq 0} \prod_{i=0}^{n-1} M(\{1, \ldots, i\})$$

which implies the first equality.

An element of $\widetilde{Ob}(CC(M))$ is given by a pair (Γ, s) where $\Gamma \in Ob(CC(M))$ is an object and $s : ft(\Gamma) \to \Gamma$ is a section of the canonical morphism $p_\Gamma : \Gamma \to ft(\Gamma)$. It follows immediately from the definition of $CC(M)$ that for $\Gamma = (E_1, \ldots, E_{n+1})$, a morphism $(f_1, \ldots, f_{n+1}) \in M(\{1, \ldots, n\})^{n+1}$ from $ft(\Gamma)$ to Γ is a section of p_Γ if an only if $f_i = i$ for $i = 1, \ldots, n$. Therefore, any such section is determined by its last component f_{n+1} and mapping $((E_1, \ldots, E_{n+1}), (f_1, \ldots, f_{n+1}))$ to $(E_1, \ldots, E_n, E_{n+1}, f_{n+1})$ we get a bijection

$$[\mathbf{2009.10.15.eq2}]\widetilde{Ob}(CC(M)) \cong \coprod_{n \geq 0} (\prod_{i=0}^{n-1} M(\{1, \ldots, i\})) \times M(\{1, \ldots, n\})^2 \tag{2}$$

which implies the second equality (bijection).

Using the notations of type theory we can write elements of $B_n(CC(M))$ as

$$\Gamma = (T_1, \ldots, T_n \rhd)$$

where $T_i \in M(\{1, \ldots, i-1\})$ and the elements of $\widetilde{B}_{n+1}(CC(M))$ as

$$\mathcal{J} = (T_1, \ldots, T_n \vdash t : T)$$

where $T_i \in M(\{1, \ldots, i-1\})$ and $t, T \in M(\{1, \ldots, n\})$.

In this notation the operations $T, \widetilde{T}, S, \widetilde{S}$ and δ on the B-sets which were introduced in [2] take the form:

1. $T((\Gamma, T_{n+1} \rhd), (\Gamma, \Delta \rhd)) = (\Gamma, T_{n+1}, t_{n+1}(\Delta) \rhd)$ when $l(\Gamma) = n$,
2. $\widetilde{T}((\Gamma, T_{n+1} \rhd), (\Gamma, \Delta \vdash r : R)) = (\Gamma, T_{n+1}, t_{n+1}(\Delta) \vdash t_{n+1}(r : R))$ when $l(\Gamma) = n$,
3. $S((\Gamma \vdash s : S), (\Gamma, S, \Delta \rhd)) = (\Gamma, s_{n+1}(\Delta[s/n+1]) \rhd)$ when $l(\Gamma) = n$,
4. $\widetilde{S}((\Gamma \vdash s : S), (\Gamma, S, \Delta \vdash r : R)) = (\Gamma, s_{n+1}(\Delta[s/n+1]) \vdash s_{n+1}((r : R)[s/n+1])$ when $l(\Gamma) = n$,
5. $\delta(\Gamma, T \rhd) = (\Gamma, T \vdash (n+1) : T)$ when $l(\Gamma) = n$.

3 C-subsystems of $CC(M)$.

Let TM be a C-subsystem of $CC(M)$. By [2] TM is determined by the subsets $B = Ob(TM)$ and $\widetilde{B} = \widetilde{Ob}(TM)$ in $Ob(CC(M))$ and $\widetilde{Ob}(CC(M))$.

For $\Gamma = (E_1, \ldots, E_n)$ we write $(\Gamma \rhd_{TM})$ if $(E_1, \ldots, E_n)$ is in B and $(\Gamma \vdash_{TM} t : T)$ if $(E_1, \ldots, E_n, T, t)$ is in $\widetilde{B}$.

The following result is an immediate corollary of [2, Theorem 4.1] together with the description of the B-sets and operations $T, \widetilde{T}, S, \widetilde{S}$ and δ on them for $CC(M)$ which is given above.

Proposition 3.1 *[**2009.10.16.prop3**] Let M be a finitary monad on Sets. A pair of subsets*

$$B \subset \coprod_{n \geq 0} \prod_{i=0}^{n-1} M(\{1, \ldots, i\})$$

$$\widetilde{B} \subset \coprod_{n \geq 0} (\prod_{i=0}^{n-1} M(\{1, \ldots, i\})) \times M(\{1, \ldots, n\})^2$$

corresponds to a C-subsystem TM of $CC(M)$ if and only if the following conditions hold:

1. $(\rhd_{TM})$
2. $(\Gamma, T \rhd_{TM}) \Rightarrow (\Gamma \rhd_{TM})$
3. $(\Gamma \vdash_{TM} r : R) \Rightarrow (\Gamma, R \rhd_{TM})$
4. $(\Gamma, T \rhd_{TM}) \wedge (\Gamma, \Delta, \vdash_{TM} r : R) \Rightarrow (\Gamma, T, t_{n+1}(\Delta) \vdash_{TM} t_{n+1}(r : R))$ *where* $n = l(\Gamma_1)$
5. $(\Gamma \vdash_{TM} s : S) \wedge (\Gamma, S, \Delta \vdash_{TM} r : R) \Rightarrow (\Gamma, s_{n+1}(\Delta[s/n+1]) \vdash_{TM} s_{n+1}((r : R)[s/n+1]))$ *where* $n = l(\Gamma_1)$,
6. $(\Gamma, T \rhd_{TM}) \Rightarrow (\Gamma, T \vdash_{TM} n+1 : T)$ *where* $n = l(\Gamma)$.

Note that conditions (4) and (5) together with condition (6) and condition (3) imply the following

4a $(\Gamma, T \rhd_{TM}) \wedge (\Gamma, \Delta \rhd_{TM}) \Rightarrow (\Gamma, T, t_{n+1}(\Delta) \rhd_{TM})$ where $n = l(\Gamma_1)$,

5a $(\Gamma \vdash_{TM} s : S) \wedge (\Gamma, S, \Delta \rhd_{TM}) \Rightarrow (\Gamma, s_{n+1}(\Delta[s/n+1]) \rhd_{TM})$ where $n = l(\Gamma_1)$.

Note also that modulo condition (2), condition (1) is equivalent to the condition that $B \neq \emptyset$.

Remark 3.2 [**2010.08.07.rem1**] If one re-writes the conditions of Proposition 3.1 in the more familiar in type theory form where the variables introduced in the context are named rather than directly numbered one arrives at the following rules:

$$\frac{}{\rhd_{TM}} \qquad \frac{x_1 : T_1, \ldots, x_n : T_n \rhd_{TM}}{x_1 : T_1, \ldots, x_{n-1} : T_{n-1} \rhd_{TM}} \qquad \frac{x_1 : T_1, \ldots, x_n : T_n \vdash_{TM} t : T}{x_1 : T_1, \ldots, x_n : T_n, y : T \rhd_{TM}}$$

$$\frac{x_1 : T_1, \ldots, x_n : T_n, y : T \rhd_{TM} \qquad x_1 : T_1, \ldots, x_n : T_n, \ldots, x_m : T_m \vdash_{TM} r : R}{x_1 : T_1, \ldots, x_n : T_n, y : T, x_{n+1} : T_{n+1}, \ldots, x_m : T_m \vdash_{TM} r : R}$$

$$\frac{x_1 : T_1, \ldots, x_n : T_n \vdash_{TM} s : S \qquad x_1 : T_1, \ldots, x_n : T_n, y : S, x_{n+1} : T_{n+1}, \ldots, x_m : T_m \vdash_{TM} r : R}{x_1 : T_1, \ldots, x_n : T_n, x_{n+1} : T_{n+1}[s/y], \ldots, x_m : T_m[s/y] \vdash_{TM} (r : R)[s/y]}$$

$$\frac{x_1 : E_1, \ldots, x_n : E_n \rhd_{TM}}{x_1 : E_1, \ldots, x_n : E_n \vdash_{TM} x_n : E_n}$$

which are similar (and probably equivalent) to the "basic rules of DTT" given in [1, p.585]. The advantage of the rules given here is that they are precisely the ones which are necessary and sufficient for a given collection of contexts and judgements to define a C-system.

Lemma 3.3 *[**2009.11.05.l1**] Let TM be as above and let $(E_1, \ldots, E_m), (T_1, \ldots, T_n) \in B_*(TM)$ and $(f_1, \ldots, f_n) \in M(\{1, \ldots, m\})^n$. Then*

$$(f_1, \ldots, f_n) \in Hom_{TM}((E_1, \ldots, E_m), (T_1, \ldots, T_n))$$

if and only if $(f_1, \ldots, f_{n-1}) \in Hom_{TM}((E_1, \ldots, E_m), (T_1, \ldots, T_{n-1}))$ and

$$(E_1, \ldots, E_m, T_n(f_1/1, \ldots, f_{n-1}/n-1), f_m) \in \widetilde{B}_{m+1}(TM)$$

Proof: Straightforward using the fact that the canonical pull-back squares in $CC(M)$ are given by (1).

References

[1] Bart Jacobs. *Categorical logic and type theory*, volume 141 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[2] Vladimir Voevodsky. Subsystems of C-systems. *http://arxiv.org/abs/1406.7413*, 2014.