Subsystems of C-systems

Vladimir Voevodsky[1,2]

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C-systems where introduced by John Cartmell in [1] and then described in more detail by Thomas Streicher (see [3, Def. 1.2, p.47]). Both authors used the name "contextual categories" for these structures. We feel it to be important to use the word "category" only for constructions which are invariant under equivalences of categories. For the essentially algebraic structure with two sorts "morphisms" and "objects" and operations "source", "target", "identity" and "composition" we suggest to use the word pre-category. Since the additional structures introduced by Cartmell are not invariant under equivalences we can not say that they are structures on categories but only that they are structures on pre-categories. Correspondingly, Cartmell objects should be called "contextual pre-categories". We suggest to use the name C-systems instead.

To any C-system CC we associate two families of sets $B_n(CC)$ and $\widetilde{B}_{n+1}(CC)$, $n \geq 0$ where $B_n(CC)$ is just the set of objects of CC of "length" n and $\widetilde{B}_{n+1}(CC)$ is the set of pairs (G, s) where $G \in B_{n+1}(CC)$ and s is the section of the canonical morphism $p_X : X \to ft(X)$.

The goal of this note is to prove Theorem 3.1 which gives a description of C-subsystems of a given C-system CC in terms of families of subsets in $B_*(CC)$ and $\widetilde{B}_*(CC)$ satisfying explicit algebraic conditions.

This result is the basis for the theory of B-systems on the one hand and an explanation for the "structural" rules of dependent type systems on the other. This note is one of the several short papers based on the material of [4].

1 C-systems

Recall that a pre-category C is a pair of sets $Mor(C)$ and $Ob(C)$ with four maps

$$\partial_0, \partial_1 : Mor(C) \to Ob(C)$$

$$Id : Ob(C) \to Mor(C)$$

and

$$\circ : Mor(C)_{\partial_0} \times_{\partial_1} Mor(C) \to Mor(C)$$

which satisfy the well known conditions (note that we write composition of morphisms in the form $f \circ g$ where $f : Y \to X$ and $g : Z \to Y$).

A C-system is a pre-category CC with additional structure of the form

1. a function $l : Ob(CC) \to \mathbf{N}$,
2. an object pt,
3. a map $ft : Ob(CC) \to Ob(CC)$,

[1]School of Mathematics, Institute for Advanced Study, Princeton NJ, USA. e-mail: vladimir@ias.edu
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4. for each $X \in Ob(CC)$ a morphism $p_X : X \to ft(X)$,
5. for each $X \in Ob(CC)$ such that $X \neq pt$ and each morphism $f : Y \to ft(X)$ an object f^*X and a morphism $q(f, X) : f^*X \to X$,

which satisfies the following conditions:

1. $l^{-1}(0) = \{pt\}$
2. for X such that $l(X) > 0$ one has $l(ft(X)) = l(X) - 1$
3. $ft(pt) = pt$
4. pt is a final object,
5. for $X \in Ob(CC)$ such that $X \neq pt$ and $f : Y \to ft(X)$ one has $ft(f^*X) = Y$ and the square

   [2009.10.14.eq1]
   $$\begin{array}{ccc} f^*X & \xrightarrow{q(f,X)} & X \\ {\scriptstyle p_{f^*X}}\downarrow & & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow{f} & ft(X) \end{array} \qquad (1)$$

   is a pull-back square,
6. for $X \in Ob(CC)$ such that $X \neq pt$ one has $id^*_{ft(X)}(X) = X$ and $q(id_{ft(X)}, X) = id_X$,
7. for $X \in Ob(CC)$ such that $X \neq pt$, $f : Y \to ft(X)$ and $g : Z \to Y$ one has $(fg)^*(X) = g^*(f^*(X))$ and $q(fg, X) = q(f, X)q(g, f^*X)$.

Remark 1.1 Let
$$Ob_n(CC) = \{X \in Ob(CC) \,|\, l(X) = n\}$$
$$Mor_{n,m}(CC) = \{f : Mor(CC) | \partial_0(f) \in Ob_n \, and \, \partial_1(f) \in Ob_m\}$$
One can reformulate the definition of a C-system using $Ob_n(CC)$ and $Mor_{n,m}(CC)$ as the underlying sets together with the obvious analogs of maps and conditions the definition given above. In this reformulation there will be no use of $\neq$ and the only use of the existential qualifier will be as a part of "there exists a unique" condition. This shows that C-systems can be considered as models of an essentially algebraic theory with sorts Ob_n, and $Mor_{n,m}$ and in particular all the results of [2] are applicable to C-systems.

Let $X \in Ob(CC)$ and $i \geq 0$. Denote by $p_{X,i}$ the composition of the canonical projections $X \to ft(X) \to \ldots \to ft^i(X)$ such that $p_{X,0} = Id_X$ and $p_{X,1} = p_X$. For $f : Y \to ft^i(X)$ denote by $q(f, X, i) : f^*(X, i) \to X$ the morphism defined inductively by the rule

$$f^*(X, 0) = Y \qquad q(f, X, 0) = f,$$

$$f^*(X, i+1) = q(f, ft(X), i)^*(X) \qquad q(f, X, i+1) = q(q(f, ft(X), i), X).$$

In other words, $q(f, X, i)$ is the canonical pull-back of the morphism $f : Y \to ft^i(X)$ with respect to the sequence of canonical projections $X \to ft(X) \to \ldots \to ft^i(X)$.

Let $i \geq 1$, $f : Y \to ft^i(X)$ be a morphism and $s : ft(X) \to X$ an element of $\widetilde{Ob}(CC)$. Denote by $f^*(s,i)$ the element of $\widetilde{Ob}(CC)$ of the form $f^*(ft(X), i-1) \to f^*(X,i)$ which is the pull-back of s with respect to $q(f, ft(X), i-1)$.

For a C-system CC let $\widetilde{Ob}(CC)$ be the set of pairs of the form (X,s) where $X \in Ob(CC)$, $X \neq pt$ and s is a section of the canonical morphism $p_X : X \to ft(X)$ i.e. a morphism $s : ft(X) \to X$ such that $p_X \circ s = Id_{ft(X)}$.

2 C-subsystems.

A C-subsystem CC' of a C-system CC is a subcategory of the underlying pre-category which is closed, in the obvious sense under the operations which define the C-system on CC and such that the canonical squares which belong to CC' are pull-back squares in CC'.

A C-subsystem is itself a C-system with respect to the induced structure.

Lemma 2.1 *[2009.10.15.l1] Let CC be a C-system and CC', CC'' be two C-subsystems such that $Ob(CC') = Ob(CC'')$ (as subsets of $Ob(CC)$) and $\widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$ (as subsets of $\widetilde{Ob}(CC)$). Then $CC' = CC''$.*

Proof: Let $f : Y \to X$ be a morphism in CC'. We want to show that it belongs to CC''. Proceed by induction on m where $X \in Ob_m$. For $m = 0$ the assertion is obvious. Suppose that $m > 0$. Since CC is a C-system we have a commutative diagram

[**2009.11.07.oldeq1**]
$$\begin{array}{ccccc} Y & \xrightarrow{s_f} & (p_X f)^* X & \xrightarrow{q(p_X f, X)} & X \\ \downarrow & & \downarrow p' & & \downarrow p \\ Y & \xrightarrow{=} & Y & \xrightarrow{p_X f} & ft(X) \end{array} \qquad (2)$$

such that $f = q(p_X f, X)\, s_f$. Since the right hand side square is a canonical one,

$$((p_X f)^* \Gamma', s_f) \in \widetilde{Ob}(CC)$$

and $ft(X) \in Ob_{m-1}$, the inductive assumption implies that $f \in CC''$.

Remark 2.2 In Lemma 2.1, it is sufficient to assume that $\widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$. The condition $Ob(CC') = Ob(CC'')$ is then also satisfied. Indeed, let $X \in Ob(CC')$. Then $p_X^* X$ is the product $X \times X$ in CC. Consider the diagonal section $\Delta_X : X \to p_X^* X$ of $p_{p_X^*(X)}$. Since CC' is assumed to be a C-subsystem we conclude that $\Delta_X \in \widetilde{Ob}(CC') = \widetilde{Ob}(CC'')$ and therefore $X \in Ob(CC'')$. It is however more convenient to think of C-subsystems in terms of subsets of both Ob and $\widetilde{Ob}$.

Let CC be a C-system. Let us say that a pair of subsets $C \subset Ob(CC)$, $\widetilde{C} \subset \widetilde{Ob}(CC)$ is saturated if there exists a C-subsystem CC' such that $C = Ob(CC')$ and $\widetilde{C} = \widetilde{Ob}(CC')$. By Lemma 2.1 we have a bijection between C-subsystems of CC and saturated pairs $(C, \widetilde{C})$.

Proposition 2.3 *[2009.10.15.prop2] A pair $(C, \widetilde{C})$ is saturated if and only if it satisfies the following conditions:*

1. $pt \in C$,
2. if $X \in C$ then $ft(X) \in C$,
3. if $(s : ft(X) \to X) \in \widetilde{C}$ then $X \in C$,
4. if $(s : ft(X) \to X) \in \widetilde{C}$, $X' \in C$, $i \geq 1$ and $ft^i(X) = ft(X')$ then $q(p_{X'}, ft(X), i-1)^*(s) \in \widetilde{C}$,
5. if $(s_1 : ft(X) \to X) \in \widetilde{C}$, $i \geq 1$ and $(s_2 : ft^{i+1}(X) \to ft^i(X)) \in \tilde{C}$ then $q(s_2, ft(X), i-1)^*(s_1) \in \widetilde{C}$,
6. if $X \in C$ then the diagonal $s_{id_X} : X \to (p_X)^*(X)$ is in $\widetilde{C}$.

Conditions (4) and (5) are illustrated by the following diagrams:

$$
\begin{array}{ccc}
p^*_{X'}(ft(X), i-1) & \xrightarrow{q(p_{X'}, ft(X), i-1)} & ft(X) \\
\downarrow{\scriptstyle q(p_{X'}, ft(X), i-1)^*(s)} & & \downarrow{\scriptstyle s} \\
p^*_{X'}(X, i) & \xrightarrow{q(p_{X'}, X, i)} & X \\
\downarrow & & \downarrow{\scriptstyle p_X} \\
p^*_{X'}(ft(X), i-1) & \xrightarrow{q(p_{X'}, ft(X), i-1)} & ft(X) \\
\downarrow & & \downarrow \\
\dots & & \dots \\
\downarrow & & \downarrow \\
X' & \xrightarrow{p_{X'}} & ft^i(X)
\end{array}
\qquad
\begin{array}{ccc}
s^*_2(ft(X), i-1) & \xrightarrow{q(s_2, ft(X), i-1)} & ft(X) \\
\downarrow{\scriptstyle q(s_2, ft(X), i-1)^*(s_1)} & & \downarrow{\scriptstyle s_1} \\
s^*_2(X, i) & \xrightarrow{q(s_2, X, i)} & X \\
\downarrow & & \downarrow{\scriptstyle p_X} \\
s^*_2(ft(X), i-1) & \xrightarrow{q(s_2, ft(X), i-1)} & ft(X) \\
\downarrow & & \downarrow \\
\dots & & \dots \\
\downarrow & & \downarrow \\
ft^{i+1}(X) & \xrightarrow{s_2} & ft^i(X)
\end{array}
$$

Proof: The "only if" part of the proposition is straightforward. Let us prove that for any $(C, \widetilde{C})$ satisfying the conditions of the proposition there exists a C-subsystem CC' of CC such that $C = Ob(CC')$ and $\widetilde{C} = \widetilde{Ob}(CC')$.

For a morphism $f : Y \to X$ let $ft(f) = p_X f : Y \to ft(X)$. Any morphism $f : Y \to X$ in CC has a canonical representation of the form $Y \stackrel{s_f}{\to} X_f \stackrel{q_f}{\to} X$ where $X_f = ft(f)^*(X)$, $q_f = q(ft(f), X)$ and $s_f : Y \to X_f$ is the section of the canonical projection $X_f \to Y$ corresponding to f.

Define a candidate subcategory CC' setting $Ob(CC') = C$ and defining the set $Mor(CC')$ of morphisms of CC' inductively by the conditions:

1. $Y \to pt$ is in $Mor(CC')$ if and only if $Y \in C$,
2. $f : Y \to X$ is in $Mor(CC')$ if and only if $X \in Ob(C)$, $ft(f) \in Mor(CC')$ and $s_f \in \widetilde{C}$.

(note that the for $(f : Y \to X) \in Mor(CC')$ one has $Y \in C$ since $s_f : Y \to X_f$).

Let us show that if the condition of the proposition are satisfied then $(Ob(CC'), Mor(CC'))$ form a C-subsystem of CC.

The subset $Ob(CC')$ contains pt and is closed under ft map by the first two conditions. The following lemma shows that $Mor(CC')$ contains identities and the compositions of canonical projections.

Lemma 2.4 *[***2009.10.16.l1***] Under the assumptions of the proposition, if $X \in C$ and $i \geq 0$ then $p_{X,i} : X \to ft^i(X)$ is in $Mor(CC')$.*

Proof: By definition of C-systems there exists n such that $ft^n(X) = pt$. Then $p_{X,n} \in Mor(CC')$ by the first constructor of $Mor(CC')$. By induction it remains to show that if $X \in C$ and $p_{X,i} \in Mor(CC')$ then $p_{X,i-1} \in Mor(CC')$. We have $ft(p_{X,i-1}) = p_{X,i}$ and $s_{p_{X,i-1}}$ is the pull-back of the diagonal $ft^{i-1}(X) \to (p_{ft^{i-1}(X)})^*(ft^{i-1}(X))$ with respect to the canonical morphism $X \to ft^{i-1}(X)$. The diagonal is in $\widetilde{C}$ by condition (6) and therefore $s_{p_{X,i-1}}$ is in $\widetilde{C}$ by repeated application of condition (4).

Lemma 2.5 *[***2009.10.16.l3***] Under the assumptions of the proposition, let $X \in C$, $(s : ft(X) \to X) \in \widetilde{C}$, $i \geq 0$, and $(f : Y \to ft^i(X)) \in Mor(CC')$. Then $q(f, ft(X), i-1)^*(s) : ft(f^*(X,i)) \to f^*(X,i)$ is in $Mor(CC')$.*

Proof: Suppose first that $ft^i(X) = pt$. Then $f = p_{Y,n}$ for some n and the statement of the lemma follows from repeated application of condition (4). Suppose that the lemma is proved for all morphisms to objects of length $j-1$ and let the length of $ft^i(X)$ be j. Consider the canonical decomposition $f = q_f s_f$. The morphism q_f is the canonical pull-back of $ft(f)$ and therefore the pull-back of s relative to q_f coincides with its pull-back relative to $ft(f)$ which is $\widetilde{C}$ by the inductive assumption. The pull-back of an element of $\widetilde{C}$ with respect to s_f is in $\widetilde{C}$ by condition (5).

Lemma 2.6 *[***2009.10.16.l4***] Under the assumptions of the proposition, let $g : Z \to Y$ and $f : Y \to X$ be in $Mor(CC')$. Then $fg \in Mor(CC')$.*

Proof: If $X = pt$ the the statement is obvious. Assume that it is proved for all f whose codomain is of length $< j$ and let X be of length j. We have $ft(fg) = ft(f)g$ and therefore $ft(fg) \in Mor(CC')$ by the inductive assumption. It remains to show that $s_{fg} \in \widetilde{C}$. We have the following diagram whose squares are canonical pull-back squares

$$\begin{array}{ccccc}
X_{fg} & \longrightarrow & X_f & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow p_X \\
Z & \xrightarrow{g} & Y & \xrightarrow{ft(f)} & ft(X)
\end{array}$$

which shows that $s_{fg} = g^*(s_f)$. Therefore, $s_{fg} \in Mor(CC')$ by Lemma 2.5.

Lemma 2.7 *[***2009.10.16.l5***] Under the assumptions of the proposition, let $X \in C$ and let $f : Y \to ft(X)$ be in $Mor(CC')$, then $f^*(X) \in C$ and $q(f, X) \in Mor(CC')$.*

Proof: Consider the diagram

$$\begin{array}{ccccc}
f^*(X) & \xrightarrow{q(f,X)} & X & & \\
{\scriptstyle s_{q(f,X)}}\downarrow & & \downarrow{\scriptstyle s_{Id_X}} & & \\
q(f,X)^*(X) & \longrightarrow & p_X^*(X) & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
f^*(X) & \xrightarrow{q(f,X)} & X & \longrightarrow & ft(X) \\
{\scriptstyle p_{f^*(X)}}\downarrow & & \downarrow{\scriptstyle p_X} & & \\
Y & \xrightarrow{f} & ft(X) & &
\end{array}$$

where the squares are canonical. By condition (6) we have $s_{Id} \in \widetilde{C}$. Therefore, by Lemma 2.5, we have $s_{q(f,X)} \in \widetilde{C}$. In particular, $q(f,X)^*(X) \in C$ and therefore $f^*(X) = ft(q(f,X)^*(X)) \in C$. The fact that $q(f,X) \in Mor(CC')$ follows from the fact that $s_{q(f,X)} \in \widetilde{C}$ and $ft(q(f,X)) = f \circ p_{f^*(X)}$ is in $Mor(CC')$ by previous lemmas.

Lemma 2.8 *[***2009.10.16.l6***] Under the assumptions of Lemma 2.7, the square*

$$\begin{array}{ccc}
f^*(X) & \xrightarrow{q(f,X)} & X \\
{\scriptstyle p_{f^*(X)}}\downarrow & & \downarrow{\scriptstyle p_X} \\
Y & \xrightarrow{f} & ft(X)
\end{array}$$

is a pull-back square in CC'.

Proof: We need to show that for a morphism $g : Z \to f^*(X)$ such that $p_{f^*(X)}g$ and $q(f,X)g$ are in $Mor(CC')$ one has $g \in Mor(CC')$. We have $ft(g) = p_{f^*(X)}g$, therefore by definition of $Mor(CC')$ it remains to check that $s_g \in \widetilde{C}$. The diagram

$$\begin{array}{ccccc}
(f^*Y)_g & \longrightarrow & f^*Y & \xrightarrow{q(f,X)} & X \\
\downarrow & & \downarrow & & \downarrow \\
Z & \xrightarrow{ft(g)} & Y & \xrightarrow{f} & ft(X)
\end{array}$$

shows that $s_g = s_{q(f,X)g}$ and therefore $s_g \in Mor(CC')$.

To finish the proof of the proposition it remains to show that $Ob(CC') = C$ and $\widetilde{Ob}(CC') = \widetilde{C}$. The first assertion is tautological. The second one follows immediately from the fact that for $(s : ft(X) \to X) \in \widetilde{Ob}(CC)$ one has $ft(s) = Id_{ft(X)}$ and $s_s = s$.

3 Subsystems in terms of B-sets

Define the B-sets of CC as follows:

$$B_n(CC) = Ob_n(CC) = \{X \in Ob(CC) \,|\, l(X) = n\}$$

$$\widetilde{B}_{n+1}(CC) = \widetilde{Ob}_n(CC) = \{(X,s) \in \widetilde{Ob}(CC) \,|\, l(X) = n+1\}$$

(note that $\widetilde{B}_0(CC)$ is undefined, our numbering for $\widetilde{B}$ starts with 1). We will also use the following notations:

1. $B(X) = \{Y \in Ob(CC) \,|\, ft(Y) = X \, and \, Y \neq pt\}$,
2. $\widetilde{B}(X) = \partial^{-1}(X)$ (note that $\widetilde{B}(pt) = \emptyset$).

In addition to the sets B_n and $\widetilde{B}_n$ and maps $ft : B_{n+1} \to B_n$ and $\partial : \widetilde{B}_{n+1} \to B_{n+1}$ let us consider the following maps given for all $m \geq n \geq 0$:

1. $T : (B_{n+1})_{ft} \times_{ft^{m+1-n}} (B_{m+1}) \to B_{m+2}$, which sends (Y, X) such that $ft(Y) = ft^{m+1-n}(X)$ to $p_Y^*(X, m+1-n)$,
2. $\widetilde{T} : (B_{n+1})_{ft} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \to \widetilde{B}_{m+2}$, which sends (Y, s) such that $ft(Y) = ft^{m+1-n}\partial(s)$ to $p_Y^*(s, m+1-n)$,
3. $S : (\widetilde{B}_{n+1})_{\partial} \times_{ft^{m+1-n}} (B_{m+2}) \to B_{m+1}$, which sends (r, X) such that $\partial(r) = ft^{m+1-n}(X)$ to $r^*(X, m+1-n)$,
4. $\widetilde{S} : (\widetilde{B}_{n+1})_{\partial} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+2}) \to \widetilde{B}_{m+1}$, which sends (r, s) such that $\partial(r) = ft^{m+1-n}\partial(s)$ to $r^*(s, m+1-n)$.
5. $\delta : B_{n+1} \to \widetilde{B}_{n+2}$ which sends X to the diagonal section of the projection $p_X^* X \to X$.

For the discussion of relations which these operations satisfy see [5].

Theorem 3.1 *[***2014.06.26.th1***] There is a natural bijection between C-subsystems of a C-system CC and families of subsets $H_n \subset B_n(CC)$, $\widetilde{H}_n \subset \widetilde{B}_n(CC)$ such that $H_0 = pt$ and which are closed under the operations ft, ∂, $\widetilde{T}$, $\widetilde{S}$ and δ.*

Proof: This is a direct reformulation of Proposition 2.3.

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