B-systems[1]

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Abstract

B-systems are algebras (models) of an essentially algebraic theory that is expected to be constructively equivalent to the essentially algebraic theory of C-systems which is, in turn, constructively equivalent to the theory of contextual categories. The theory of B-systems is closer in its form to the structures directly modeled by contexts and typing judgements of (dependent) type theories and further away from categories than contextual categories and C-systems.

1 Introduction

In [7, Def. 2.2] we introduced the concept of a C-system. The type of the C-systems is constructively equivalent to the type of contextual categories defined by Cartmell in [2] and [1] but the definition of a C-system is slightly different from the Cartmell's foundational definition.

The concept of a B-system is introduced in this paper. It provides an abstract formulation of a structure formed by contexts and "typing judgements" of a type theory relative to the operations of context extensions, weakening and substitutions.

We define B-systems in several steps. First we describe pre-B-systems that are models of an essentially algebraic theory with countable families of sorts operations but no relations.

Already at this stage we tart to distinguish between unital and non-unital (pre-)B-systems. This distinction continues throughout the paper. While non-unital B-systems have no direct connection to C-systems and therefore no direct connection to categories they have a definition with interesting symmetries and we believe that they are quite interesting in there own right.

Following the ideas of [7], how to construct a unital pre-B-system from a C-system. This construction is functorial with respect to homomorphisms of C-systems and unital pre-B-systems and moreover defines a full embedding of the category of C-systems to the category of unital pre-B-systems.

It is more or less clear from the proof of the full embedding theorem that the image of this full embedding consists of unital pre-B-systems whose operations satisfy some algebraic conditions. We suggest a form of these conditions in our definition of a non-unital and then unital B-system (Definitions 3.5 and 3.6).

We conclude the first part of the paper with a problem (essentially a conjecture) that the image of the full embedding from C-systems to unital pre-B-systems is precisely the class of unital B-systems. A constructive solution to this problem would also provide an explicit construction of a C-system from a unital B-system.

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In the second part we describe an approach to the definition of non-unital B-systems that can be conveniently formalized in Coq and that provide a possible step towards the definition of higher B-systems that is B-systems whose component types are of higher h-levels.

The work on this paper, especially in the part where the axioms of TT, SS, TS and ST of B-systems are introduced was influenced and facilitated by recent discussions with Richard Garner and Egbert Rijke. Many other ideas of this work go back to [5].

The subject of this paper is closely related to the subject of recent notes by John Cartmell [3]. The most important difference between our exposition and that of Cartmell is that we are using the formalism of *essentially algebraic* theories while Cartmell uses the formalism of *generalized algebraic* theories. While there are important connections between these two kinds of theories there are also important distinctions which we intend to discuss in a future paper.

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2 pre-B-systems

Definition 2.1 *A non-unital pre-B-system a collection of data of the following form:*

1. *for all* $n \in \mathbf{N}$ *two set* B_n *and* $\widetilde{B}_{n+1}$,
2. *for all* $n \in \mathbf{N}$ *maps of the form:*
   (a) $ft : B_{n+1} \to B_n$,
   (b) $\partial : \widetilde{B}_{n+1} \to B_{n+1}$
3. *an element* $pt \in B_0$,
4. *for all* $m, n \in \mathbf{N}$ *such that* $m \geq n$ *maps of the form:*
   (a) $T : (Y \in B_{n+1}, X \in B_{m+1}, ft(Y) = ft^{m+1-n}(X)) \to B_{m+2}$,
   (b) $\widetilde{T} : (Y \in B_{n+1}, r \in \widetilde{B}_{m+1}, ft(Y) = ft^{m+1-n}\partial(r)) \to \widetilde{B}_{m+2}$,
   (c) $S : (s \in \widetilde{B}_{n+1}, X \in B_{m+2}, \partial(s) = ft^{m+1-n}(X)) \to B_{m+1}$,
   (d) $\widetilde{S} : (s \in \widetilde{B}_{n+1}, r \in \widetilde{B}_{m+2}, \partial(s) = ft^{m+1-n}\partial(r)) \to \widetilde{B}_{m+1}$,

Definition 2.2 *A unital pre-B-system is a non-unital pre-B-system together with, for every* $n \geq 0$ *of an operation*

$$\delta : B_{n+1} \to \widetilde{B}_{n+2}$$

Homomorphisms of non-unital and unital pre-B-systems are defined in the obvious way giving us the corresponding categories. Also in the obvious way one defines the concepts of sub-pre-B-systems.

Let CC be a C-system as defined in [7, Def. 2.2]. Recall the following notations. For X such that $l(X) \geq i$ and $f : Y \to ft^i(X)$ denote by by $f^*(X, i)$ the objects and by $q(f, X, i) : f^*(X, i) \to X$ the morphisms defined inductively by the rule

$$f^*(X, 0) = Y \qquad q(f, X, 0) = f,$$

$$f^*(X, i+1) = q(f, ft(X), i)^*(X) \qquad q(f, X, i+1) = q(q(f, ft(X), i), X).$$

If $l(X) < i$, then $q(f, X, i)$ is undefined since $q(-, X)$ is undefined for $X = pt$ and again, as in the case of $p_{X,i}$, *all of the considerations involving $q(f, X, i)$ are modulo the qualification that $l(X) \geq i$.*

For $i \geq 1$, $(s : ft(X) \to X) \in \widetilde{Ob}$ such that $l(X) \geq i$, and $f : Y \to ft^i(X)$ let

$$f^*(s, i) : f^*(ft(X), i-1) \to f^(ft(X), i)$$

be the pull-back of the section $ft(X) \to X$ along the morphism $q(f, ft(X), i-1)$. We again use the agreement that always when $f^*(s, i)$ is used the condition $l(X) \geq i$ is part of the assumptions.

One constructs a unital pre-B-system from CC as follows. The B-sets of CC are:

$$B_n(CC) = Ob_n(CC) = \{X \in Ob(CC) \,|\, l(X) = n\}$$

$$\widetilde{B}_{n+1}(CC) = \widetilde{Ob}_n(CC) = \{(X, s) \in \widetilde{Ob}(CC) \,|\, l(X) = n+1\}$$

The definition of pt, ft and ∂ is obvious. The operations T, $\widetilde{T}$, S, $\widetilde{S}$ and δ on the B-sets of a C-system are as follows:

1. T sends (Y, X) such that $ft(Y) = ft^{m+1-n}(X)$ to $p_Y^*(X, m+1-n)$,
2. $\widetilde{T}$ sends (Y, r) such that $ft(Y) = ft^{m+1-n}\partial(r)$ to $p_Y^*(r, m+1-n)$,
3. S sends (s, X) such that $\partial(s) = ft^{m+1-n}(X)$ to $s^*(X, m+1-n)$,
4. $\widetilde{S}$ sends (s, r) such that $\partial(s) = ft^{m+1-n}\partial(r)$ to $s^*(r, m+1-n)$.
5. δ sends X to the diagonal section of the projection $p_X^* X \to X$.

When we need to distinguish between the unital pre-B-system defined by CC and its non-unital analog we will write $uB(CC)$ for the unital version and $nuB(CC)$ for the non-unital one.

One of the main results of [7], Proposition 4.3 can be reformulated as follows:

Theorem 2.3 *There is a natural bijection between C-subsystems of a C-system CC and unital sub-pre-B-systems of $uB(CC)$.*

Another way to define a pre-B-system is from a pair (R, LM) where R is a monad on sets and LM a left module over R with values in sets as in [6] . For the pre-B-system $B(R, LM)$ we have

$$B_n(R, LM) = LM(\emptyset) \times \ldots \times LM(\{1, \ldots, n-1\})$$

$$\widetilde{B}_{n+1}(R, LM) = B_{n+1}(R, LM) \times R(\{1, \ldots, n\})$$

The operations ft and ∂ are obvious. The element pt is the only point of the product of the empty family of sets. The rest of the operations are defined as follows. For $E \in LM(\{1, \ldots, m\})$ or $E \in R(\{1, \ldots, m\})$ and $n \geq 1$ we set:

$$t_n(E) = E[n+1/n, n+2/n+1, \ldots, m+1/m]$$

$$s_n(E) = E[n/n+1, n+1/n+2, \ldots, m-1/m]$$

1. Operations T:
$$T((E_1,\ldots,E_n,F),(E_1,\ldots,E_n,E_{n+1},\ldots,E_{m+1})) =$$
$$(E_1,\ldots,E_n,F,t_{n+1}E_{n+1},\ldots,t_{n+1}E_{m+1})$$

2. Operations $\widetilde{T}$:
$$\widetilde{T}((E_1,\ldots,E_n,F),(E_1,\ldots,E_n,E_{n+1},\ldots,E_{m+1},r)) =$$
$$(E_1,\ldots,E_n,F,t_{n+1}E_{n+1},\ldots,t_{n+1}E_{m+1},t_{n+1}r)$$

3. Operations S:
$$S((E_1,\ldots,E_n,F,s),(E_1,\ldots,E_n,F,E_{n+1},\ldots,E_{m+1})) =$$
$$(E_1,\ldots,E_n,s_n(E_{n+1}[s/n]),\ldots,s_n(E_{m+1}[s/n]))$$

4. Operation $\widetilde{S}$:
$$S((E_1,\ldots,E_n,F,s),(E_1,\ldots,E_n,F,E_{n+1},\ldots,E_{m+1},r)) =$$
$$(E_1,\ldots,E_n,s_n(E_{n+1}[s/n]),\ldots,s_n(E_{m+1}[s/n]),s_n(r[s/n]))$$

5. Operations δ:
$$\delta(E_1,\ldots,E_n,E_{n+1}) = (E_1,\ldots,E_n,E_{n+1},\eta_R(n+1))$$
where η_R is the unit of the monad R.

Note that the unit of R also participates in the definition of operations S and $\widetilde{S}$ since the explicit form of the substitution $E \mapsto E[s/n]$ involves η_R.

We can form non-unital pre-B-systems using this construction by considering non-unital sub-pre-B-systems in $uB(R, LM)$ (cf. Example 3.7 below).

For this pre-B-system as well as for its subsystems and regular quotients we can use notations such as $\Gamma \vdash o : T$ directly since in this case $\Gamma \in B_n$, $T \in LM(\{1,\ldots,n\})$ and $o \in R(\{1,\ldots,n\})$ are elements of types or sets that do not depend on elements of other types or sets and the substitution is defined on the level of these sets.

If $CC(R, LM)$ is the C-system corresponding to (R, LM) then there is a constructive isomorphism
$$B(CC(R,LM)) \cong B(R,LM)$$

The construction $CC \mapsto B(CC)$ is clearly compatible with homomorphisms and defines a functor from the category of C-systems to the category of unital pre-B-systems.

Theorem 2.4 *The functor $CC \mapsto uB(CC)$ is a full embedding.*

The proof follows from the lemmas below that show that a C-system can be reconstructed from the associated unital pre-B-system.

We start by introducing intermediate concepts of a B0-systems.

Definition 2.5 *A non-unital pre-B-system is called a non-unital B0-system if the following conditions hold:*

1. *for all $X \in B_0$ one has $X = pt$.*

2. *for $Y \in B_{n+1}$, $X \in B_{m+1}$ such that $ft(Y) = ft^{m+1-n}(X)$ and $m \geq n \geq 0$ one has:*

$$ft(T(Y,X)) = \begin{cases} T(Y, ft(X)) & \text{if } m > n \\ Y & \text{if } m = n \end{cases} \tag{1}$$

3. *for $Y \in B_{n+1}$, $r \in \widetilde{B}_{m+1}$ such that $ft(Y) = ft^{m+1-n}\partial(r)$ and $m \geq n \geq 0$ one has:*

$$\partial(\widetilde{T}(Y,r)) = T(Y, \partial(r)) \tag{2}$$

4. *for $s \in \widetilde{B}_{n+1}$, $X \in \widetilde{B}_{m+2}$ such that $\partial(s) = ft^{m+1-n}(X)$ and $m \geq n \geq 0$ one has:*

$$ft(S(s,X)) = \begin{cases} S(s, ft(X)) & \text{if } m > n \\ ft(Y) & \text{if } m = n \end{cases} \tag{3}$$

5. *for $s \in \widetilde{B}_{n+1}$, $r \in \widetilde{B}_{m+2}$ such that $\partial(s) = ft^{m+1-n}\partial(r)$ and $m \geq n \geq 0$ one has:*

$$\partial(\widetilde{S}(s,r)) = S(s, \partial(r)) \tag{4}$$

6.

Definition 2.6 *A unital pre-B-system is called a unital B0-system if the underlying non-unital pre-B-system is a non-unital B0-system and for all $i \geq 0$, $X \in B_{n+1}$ one has*

$$\partial(\delta(X)) = T(X,X) \tag{5}$$

Lemma 2.7 *Let B be a unital pre-B-system of the form $uB(CC)$. Then B is a unital B0-system.*

Proof: Straightforward.

From now on in this section we assume that we consider a unital B0-system. Let us denote by

$$T_j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}} (B_{m+1}) \to B_{m+1+j}$$

$$\widetilde{T}_j : (B_{n+j})_{ft^j} \times_{ft^{m+1-n}\partial} (\widetilde{B}_{m+1}) \to \widetilde{B}_{m+1+j}$$

the maps which are defined inductively by

$$T_j(Y,X) = \begin{cases} X & \text{if } j = 0 \\ T(Y, T_{j-1}(ft(Y), X)) & \text{if } j > 0 \end{cases} \qquad \widetilde{T}_j(Y,s) = \begin{cases} s & \text{if } j = 0 \\ \widetilde{T}(Y, \widetilde{T}_{j-1}(ft(Y), s)) & \text{if } j > 0 \end{cases} \tag{6}$$

Note that for any $i = 0, \dots, j$ we have

$$T_j(Y,X) = T_i(Y, T_{j-i}(ft^i(Y), X))$$

and

$$\widetilde{T}_j(Y,s) = \widetilde{T}_i(Y, \widetilde{T}_{j-i}(ft^i(Y), s))$$

Lemma 2.8 *One has*

$$T_j(Y, ft(X)) = ft(T_j(Y, X))$$

Proof: For $n = 0$ the statement is obvious. For $n > 0$ we have by induction on j

$$T_j(Y, ft(X)) = T(Y, T_{j-1}(ft(Y), ft(X))) = T(Y, ft(T_{j-1}(ft(Y), X))) =$$

$$= ft(T(Y, T_{j-1}(ft(Y), X))) = ft(T_j(Y, X)).$$

Let $f : Y \to X$ be a morphism such that $Y \in B_n$ and $X \in B_m$. Define a sequence $(s_1(f), \ldots, s_m(f))$ of elements of $\widetilde{B}_{n+1}$ inductively by the rule

$$(s_1(f), \ldots, s_m(f)) = (s_1(ft(f)), \ldots, s_{m-1}(ft(f)), s_f) = (s_{ft^{m-1}(f)}, \ldots, s_{ft(f)}, s_f)$$

where $ft(f) = p_X f$ and s_f is the s-operation of [7, Def. 2.2]. For $m = 0$ we start with the empty sequence. This construction can be illustrated by the following diagram for $f : Y \to X$ where $X \in B_4$:

$$\begin{array}{ccccccccccc}
Y & \xrightarrow{s_4(f)} & Z_{4,3} & \longrightarrow & Z_{4,2} & \longrightarrow & Z_{4,1} & \longrightarrow & T_n(Y,X) & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & Y & \xrightarrow{s_3(f)} & Z_{3,2} & \longrightarrow & Z_{3,1} & \longrightarrow & T_n(Y,ft(X)) & \longrightarrow & ft(X) \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & Y & \xrightarrow{s_2(f)} & Z_{2,1} & \longrightarrow & T_n(Y,ft^2(X)) & \longrightarrow & ft^2(X) \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & Y & \xrightarrow{s_1(f)} & T_n(Y,ft^3(X)) & \longrightarrow & ft^3(X) \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & Y & \longrightarrow & pt
\end{array} \tag{7}$$

which is completely determined by the condition that the squares are the canonical ones and the composition of morphisms in the i-th arrow from the top is $ft^i(f)$. For the objects Z_i^j we have:

$$\begin{array}{lll}
Z_{4,1} = S(s_1(f), T_n(Y,X)) & Z_{4,2} = S(s_2(f), Z_{4,1}) & Z_{4,3} = S(s_3(f), Z_{4,2}) \\
Z_{3,1} = S(s_1(f), T_n(Y, ft(X))) & Z_{3,2} = S(s_2(f), Z_{3,1}) & \\
Z_{2,1} = S(s_1(f), T_n(Y, ft^2(X))) & &
\end{array} \tag{8}$$

A simple inductive argument similar to the one in the proof of [7, Lemma 4.1] show that if $f, f' : Y \to X$ are two morphisms such that $X \in B_m$ and $s_i(f) = s_i(f')$ for $i = 1, \ldots, m$ then $f = f'$. Therefore, we may consider the set $Mor(CC)$ of morphisms of CC as a subset in $\amalg_{n,m \geq 0} B_n \times B_m \times \widetilde{B}^m_{n+1}$.

Let us show how to describe this subset in terms of the operations introduced above.

Lemma 2.9 *An element* $(Y, X, s_1, \ldots, s_m)$ *of* $B_n \times B_m \times \widetilde{B}^m_{n+1}$ *corresponds to a morphism if and only if the element* $(Y, ft(X), s_1, \ldots, s_{m-1})$ *corresponds to a morphism and* $\partial(s_m) = Z_{m,m-1}$ *where* $Z_{m,i}$ *is defined inductively by the rule:*

$$Z_{m,0} = T_n(Y, X) \qquad Z_{m,i+1} = S(s_{i+1}, Z_{m,i})$$

Proof: Straightforward from the example considered above.

Let us show now how to identify the canonical morphisms $p_{X,i} : X \to ft^i(X)$ and in particular the identity morphisms.

Lemma 2.10 *Let* $X \in B_m$ *and* $0 \leq i \leq m$. *Let* $p_{X,i} : X \to ft^i(X)$ *be the canonical morphism. Then one has:*

$$s_j(p_{X,i}) = \widetilde{T}_{m-j}(X, \delta_{ft^{m-j}(X)}) \qquad j = 1, \ldots, m - i$$

Proof: Let us proceed by induction on $m - i$. For $i = m$ the assertion is trivial. Assume the lemma proved for $i + 1$. Since $ft(p_{X,i}) = p_{X,i+1}$ we have $s_j(p_{X,i}) = s_j(p_{X,i+1})$ for $j = 1, \ldots, m - i - 1$. It remains to show that

$$s_{m-i}(p_{X,i}) = \widetilde{T}_i(X, \delta_{ft^i(X)}) \tag{9}$$

By definition $s_{m-i}(p_{X,i}) = s_{p_{X,i}}$ and (9) follows from the commutative diagram:

$$\begin{array}{ccccc}
X & \longrightarrow & ft^i(X) & & \\
{\scriptstyle s_p}\downarrow & & \downarrow{\scriptstyle \delta_{ft^i(X)}} & & \\
p^*_{X,i+1}(ft^i(X)) & \longrightarrow & p^*_{ft^i(X)}(ft^i(X)) & \longrightarrow & ft^i(X) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{ft^i(X)}} \\
X & \longrightarrow & ft^i(X) & \longrightarrow & ft^{i+1}(X)
\end{array}$$

where $p = p_{X,i}$.

Lemma 2.11 *Let* $(X, s) \in \widetilde{B}_{m+1}$, $Y \in B_n$ *and* $f : Y \to ft(X)$. *Define inductively* $(f, i)^*(s) \in \widetilde{B}_{n+m+1-i}$ *by the rule*

$$(f, 0)^*(s) = \widetilde{T}_n(Y, s)$$

$$(f, i + 1)^*(s) = \widetilde{S}(s_{i+1}(f), (f, i)^*(s))$$

Then $f^*(s) = (f, m)^*(s)$.

Proof: It follows from the diagram:

$$
\begin{array}{ccccccccccc}
Y & \xrightarrow{s_m(f)} & * & \longrightarrow & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(X) \\
{\scriptstyle f^*(s)}\downarrow & & \downarrow{\scriptstyle (f,m-1)^*(s)} & & & & \downarrow{\scriptstyle (f,1)^*(s)} & & \downarrow{\scriptstyle (f,0)^*(s)} & & \downarrow{\scriptstyle s} \\
* & \longrightarrow & * & \longrightarrow & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & X \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
Y & \xrightarrow{s_m(f)} & * & \longrightarrow & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft(X) \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
 & & Y & \xrightarrow{s_{m-1}(f)} & \dots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & ft^2(X) \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & \dots & & \dots & & \dots \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & Y & \xrightarrow{s_1(f)} & * & \longrightarrow & ft^{m-1}(X) \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & Y & \longrightarrow & pt
\end{array}
$$

Lemma 2.12 *Let $g : Z \to Y$, $f : Y \to X$ and $X \in B_m$. Then $s_i(fg) = g^* s_i(f)$.*

Proof: It follows immediately from the equations $s_{fg} = g^* s_f$ and $ft(fg) = ft(f)g$.

Lemma 2.13 *Let $f : Y \to ft(X)$ be a morphism, $Y \in B_n$ and $X \in B_{m+1}$. Define $(f,i)^*(X)$ inductively by the rule:*

$$(f,0)^*(X) = T_n(Y,X)$$

$$(f,i+1)^*(X) = S(s_{i+1}(f),(f,i)^*(X))$$

Then $f^*(X) = (f,m)^*(X)$.

Proof: Similar to the proof of Lemma 2.11.

Lemma 2.14 *Let $f : Y \to ft(X)$ be a morphism, $Y \in B_n$ and $X \in B_{m+1}$. Then*

$$
s_i(q(f,X)) = \begin{cases} \widetilde{T}(f^*X, s_i(f)) & \text{if } i \leq m \\ \\ \widetilde{T}(f^*X, \delta_X) & \text{if } i = m+1 \end{cases}
$$

Proof: We have $s_i(q(f,X)) = s_{ft^{m+1-i}(q(f,X))}$. For $i \leq m$ we have

$$ft^{m+1-i}(q(f,X)) = ft^{m-i}(f) p_{f^*X}$$

Therefore,

$$s_{ft^{m+1-i}(q(f,X))} = s_{ft^{m-i}(f)p_{f^*X}} = p^*_{f^*X}s_{ft^{m-i}(f)} = \widetilde{T}(f^*X, s_i(f))$$

and for $i = m+1$ we have

$$s_i(q(f,X)) = s_{q(f,X)} = p^*_{f^*X}(\delta_X) = \widetilde{T}(f^*X, \delta_X).$$

The lemmas proved above show that a C-system can be reconstructed from the sets $B_n, \widetilde{B}_{n+1}$ and operations ft, ∂, δ, T, $\widetilde{T}$, S and $\widetilde{S}$. This completes our proof of Theorem 2.4.

3 B-systems

The next question that we want to address is the description of the image of the functor $CC \mapsto uB(CC)$. To make this question more precise we introduce below the concepts of non-unital and unital B-systems and formulate a problem whose solution would imply that the functor $CC \mapsto uB(CC)$ defines an equivalence between the category of C-systems and the full subcategory of the category of unital pre-B-systems that consists of unital B-systems.

For $Y \in B_i$ let $B(Y)_j$ denote the subset of B_{i+j} that consists of X such that $ft^j(X) = Y$. In particular $B(Y)_0$ is the one point subset $\{Y\}$. Let also $\widetilde{B(Y)}_j$ denote the subset of $\widetilde{B}_{i+j}$ that consists of r such that $ft^j(\partial(r)) = Y$.

Then the operations T, $\widetilde{T}$, S and $\widetilde{S}$ can be seen as follows:

$$T(Y, -) : B(ft(Y))_* \to B(Y)_*$$

$$\widetilde{T}(Y, -) : \widetilde{B}(ft(Y))_* \to \widetilde{B}(Y)_*$$

$$S(s, -) : B(\partial(s))_* \to B(ft(\partial(s)))_*$$

$$\widetilde{S}(s, -) : \widetilde{B}(\partial(s))_* \to \widetilde{B}(ft(\partial(s)))_*$$

Definition 3.1 *Let B be a non-unital B0-system. Define the following conditions on B:*

1. *The TT-condition. For all $GT \in B_{i+1}$, $GDT' \in B(ft(GT))_{j+1}$ one has*
   (a) *for all $R \in B(ft(GDT'))_*$*
   $$T(T(GT, GDT'), T(GT, R)) = T(GT, T(GDT', R))$$
   (b) *for all $r \in \widetilde{B}(ft(GDT'))_*$*
   $$\widetilde{T}(T(GT, GDT'), \widetilde{T}(GT, r)) = \widetilde{T}(GT, \widetilde{T}(GDT', r))$$
2. *The SS-condition. For all $s \in \widetilde{B}_{i+1}$, $s' \in \widetilde{B(\partial(s))}_{j+1}$ one has*
   (a) *for all $R \in B(\partial(s'))_*$*
   $$S(\widetilde{S}(s, s'), S(s, R)) = S(s, S(s', R))$$

(b) *for all* $r \in \widetilde{B}(\partial(s'))_*$
$$\widetilde{S}(\widetilde{S}(s,s'),\widetilde{S}(s,r)) = \widetilde{S}(s,\widetilde{S}(s',r))$$

3. *The TS-condition. For any* $GT \in B_{i+1}$ *and* $s' \in B(\widetilde{ft(GT)})_{j+1}$ *one has*

   (a) *for all* $R \in B(\partial(s'))_*$
$$S(\widetilde{T}(GT,s'),T(GT,R)) = T(GT,S(s',R))$$

   (b) *for all* $r \in \widetilde{B}(\partial(s'))_*$
$$\widetilde{S}(\widetilde{T}(GT,s'),\widetilde{T}(GT,r)) = \widetilde{T}(GT,\widetilde{S}(s',r))$$

4. *The ST-condition. For any* $s \in \widetilde{B}_{i+1}$ *and* $GTDT' \in \widetilde{B(\partial(s))}_{j+1}$ *one has*

   (a) *for all* $R \in B(ft(GTDT'))_*$
$$T(S(s,GTDT'),S(s,R)) = S(s,T(GTDT',R))$$

   (b) *for all* $r \in \widetilde{B}(ft(GTDT'))_*$
$$\widetilde{T}(S(s,GTDT'),\widetilde{S}(s,r)) = \widetilde{S}(s,\widetilde{T}(GTDT',r))$$

5. *The STid-condition. For any* $s \in \widetilde{B}_{i+1}$ *one has*

   (a) *for all* $R \in B(ft(\partial(s)))_*$
$$S(s,T(\partial(s),R)) = R$$

   (b) *for all* $r \in \widetilde{B}(ft(\partial(s)))_*$
$$\widetilde{S}(s,\widetilde{T}(\partial(s),r)) = r$$

Definition 3.2 *Let* B *be a unital B0-system. Define the following conditions on* B*:*

1. *The* δT*-condition. For any* $GT \in B_{i+1}$ *and* $GDT' \in B(ft(GT))_{j+1}$ *one has*
$$\widetilde{T}(GT,\delta(GDT')) = \delta(T(GT,GDT'))$$

2. *The* δS*-condition. For any* $s \in \widetilde{B}_{i+1}$ *and* $GTDT' \in B(\partial(s))_{j+1}$ *one has*
$$\widetilde{S}(s,\delta(GTDT')) = \delta(S(s,GTDT'))$$

3. *The* δSid*-condition. For any* $s \in \widetilde{B}_{i+1}$ *one has*
$$\widetilde{S}(s,\delta(\partial(s))) = s$$

4. *The* $S\delta T$*-condition. For any* $GT \in B_{i+1}$ *one has*

   (a) *for* $R \in B(GT)_*$ *one has:*
$$S(\delta(GT),T(GT,R)) = R$$

(b) for $r \in \widetilde{B(GT)}_*$ one has

$$\widetilde{S}(\delta(GT), \widetilde{T}(GT, r)) = r$$

Remark 3.3 *The conditions defined above can be shown as follows:*

1. *The TT-condition:*

$$\frac{\Gamma, T \rhd \quad \Gamma, \Delta, T' \rhd \quad \Gamma, \Delta \vdash \mathcal{J}}{\frac{\Gamma, T \rhd \quad \Gamma, \Delta, T' \vdash \mathcal{J}}{\Gamma, T, \Delta, T' \vdash \mathcal{J}} \quad \frac{\Gamma, T, \Delta, T' \rhd \quad \Gamma, T, \Delta \vdash \mathcal{J}}{\Gamma, T, \Delta, T' \vdash \mathcal{J}}}$$

2. *The SS-condition:*

$$\frac{\Gamma \vdash s : T \quad \Gamma, T, \Delta \vdash s' : T' \quad \Gamma, T, \Delta, T' \vdash \mathcal{J}}{\frac{\Gamma \vdash s : T \quad \Gamma, T, \Delta \vdash \mathcal{J}[s]}{\Gamma, \Delta[s] \vdash \mathcal{J}[s'][s]} \quad \frac{\Gamma, \Delta[s] \vdash s'[s] : T'[s] \quad \Gamma, \Delta[s], T'[s] \vdash \mathcal{J}[s]}{\Gamma, \Delta[s] \vdash \mathcal{J}[s][s']}}$$

3. *The TS-condition:*

$$\frac{\Gamma, T \rhd \quad \Gamma, \Delta \vdash s' : T' \quad \Gamma, \Delta, T' \vdash \mathcal{J}}{\frac{\Gamma, T \rhd \quad \Gamma, \Delta \vdash \mathcal{J}[s']}{\Gamma, T, \Delta \vdash \mathcal{J}[s']} \quad \frac{\Gamma, T, \Delta \vdash s' : T' \quad \Gamma, T, \Delta, T' \vdash \mathcal{J}}{\Gamma, T, \Delta \vdash \mathcal{J}[s']}}$$

4. *The ST-condition:*

$$\frac{\Gamma \vdash s : T \quad \Gamma, T, \Delta, T' \rhd \quad \Gamma, T, \Delta \vdash \mathcal{J}}{\frac{\Gamma \vdash s : T \quad \Gamma, T, \Delta, T' \vdash \mathcal{J}[s]}{\Gamma, \Delta[s], T'[s] \vdash \mathcal{J}[s]} \quad \frac{\Gamma, \Delta[s], T'[s] \rhd \quad \Gamma, \Delta[s] \vdash \mathcal{J}[s]}{\Gamma, \Delta[s], T'[s] \vdash \mathcal{J}[s]}}$$

5. *The $STid$-condition:*

$$\frac{\Gamma \vdash s : T \quad \Gamma, T \rhd \quad \Gamma \vdash \mathcal{J}}{\frac{\Gamma \vdash s : T \quad \Gamma, T \vdash \mathcal{J}}{\Gamma \vdash \mathcal{J}[s]}}$$

6. *The δT-condition:*

$$\frac{\Gamma, T \rhd \quad \Gamma, \Delta, x : T' \rhd}{\frac{\Gamma, T \rhd \quad \Gamma, \Delta, x : T' \vdash x : T'}{\Gamma, T, \Delta, x : T' \vdash x : T'} \; \frac{\Gamma, T, \Delta, x : T' \rhd}{\Gamma, T, \Delta, x : T' \vdash x : T'}}$$

7. *The δS-condition:*

$$\frac{\Gamma \vdash s : T \quad \Gamma, T, \Delta, x : T' \rhd}{\frac{\Gamma \vdash s : T \quad \Gamma, T, \Delta, x : T' \vdash x : T'}{\Gamma, \Delta[s], x : T'[s] \vdash x : T'[s]} \; \frac{\Gamma, \Delta[s], x : T[s]' \rhd}{\Gamma, \Delta[s], x : T'[s] \vdash x : T'[s]}}$$

8. *The δSid-condition:*

$$\frac{\Gamma \vdash s : T \quad \Gamma, x : T \rhd}{\frac{\Gamma \vdash s : T \quad \Gamma, x : T \vdash x : T}{\Gamma \vdash s : T}}$$

9. *The $S\delta T$-condition:*

$$\frac{\Gamma, y : Y, \Delta \vdash \mathcal{J}}{\frac{\Gamma, y_1 : Y, y : Y, \Delta \vdash \mathcal{J} \quad \Gamma, y_1 : Y \vdash y_1 : Y}{\Gamma, y_1 : Y, \Delta[y_1/y] \vdash \mathcal{J}[y_1/y]}}$$

Lemma 3.4 *Let B be a unital B0-system and let δ_1, δ_2 be two families of operations as in Definition 2.2. Suppose that both δ_1 and δ_2 satisfy the δT, δSid and $S\delta T$ conditions. Then $\delta_1 = \delta_2$.*

Proof: We have:

$$\delta_1(GT) = \widetilde{S}(\delta_2(GT), \widetilde{T}(GT, \delta_1(GT))) = \widetilde{S}(\delta_2(GT), \delta_1(T(GT, GT))) = \delta_2(GT)$$

where the first equality is the $S\delta T$-condition for δ_2, the second equality is the δT-condition for δ_1 and the third equality is the δSid-condition for δ_1.

Definition 3.5 *A non-unital B-system is a non-unital B0-system that satisfy the conditions TT, SS, TS, ST and $STid$ of Definition 3.1.*

Definition 3.6 *A unital B-system is a unital B0-system that satisfy the conditions TT, SS, TS, ST, $STid$ of Definition 3.1 and the conditions δT, δS, δSid and $S\delta T$ of Definition 3.2.*

Equivalently, a unital B-system is non-unital B-system such that there exists a family of operations δ satisfying the conditions δT, δS, δSid and $S\delta T$ of Definition 3.2.

Example 3.7 While being unital is a property of non-unital B-systems not any homomorphism of non-unital B-systems preserves units. Here is a sketch of an example of a homomorphism that does not preserve units.

Consider the following pairs of a monad and a left module over it. In both cases pt is the constant functor corresponding to the one point set $\{T\}$ that has a unique left module structure over any monad.

1. (R_1, pt) where R_1 is the monad corresponding to one unary operation $s_1(x)$ and the relation

$$s_1(s_1(x)) = s_1(x)$$

2. (R_2, pt) where R_2 is the monad corresponding to two unary operations $s_1(x)$ and $s_2(x)$ and relations:

$$s_1(s_1(x)) = s_1(x) \quad s_1(s_2(x)) = s_1(x) \quad s_2(s_1(x)) = s_1(x) \quad s_2(s_2(x)) = s_2(x)$$

Consider the unital B-systems $uB(R_1, pt)$ and $uB(R_2, pt)$. In $uB(R_1, pt)$ consider the non-unital sub-B-system nuB_1 generated by $(T \vdash s_1(1) : T)$. In $uB(R_2, pt)$ consider the non-unital sub-B-system nuB_2 generated by $(T \vdash s_1(1) : T)$ and $(T \vdash s_2(1) : T)$.

Observe that both nuB_1 and nuB_2 are in fact unital with the unit in the first one given by $(T, \ldots, T \vdash s_1(n) : T)$ and unit in the second one is given by $(T, \ldots, T \vdash s_2(n) : T)$ where n is the number of T's before the turnstile $\vdash$ symbol.

We also have an obvious (unital) homomorphism from $uB(R_1, pt)$ to $uB(R_2, pt)$ that defines a homomorphism $nuB_1 \to nuB_2$ and that latter homomorphism is not unital.

Remark 3.8 For a unital B-systems operations S and T can be expressed as follows.

$$T(Y, X) = \begin{cases} Y & \text{if } l(X) = l(Y) - 1 \\ ft(\partial(\widetilde{T}(Y, \delta(X)))) & \text{if } l(X) \geq l(Y) \end{cases} \tag{10}$$

$$S(s, X) = \begin{cases} ft(\partial(s)) & \text{if } l(X) = l(\partial(s)) \\ ft(\partial(\widetilde{S}(s, \delta(X)))) & \text{if } l(X) > l(\partial(s)) \end{cases} \tag{11}$$

I would like to end this section with the formulation of the following problem. I am reasonably sure that it has a straightforward solution.

Problem 3.9 *To show that a unital B0-system is isomorphic to a unital B0-system of the form $uB(CC)$ if and only if it is a unital B-system.*

4 B-systems in Coq

While our main interest is in pre-B-systems and B-systems in sets we would like to be able to formalize their definitions in Coq without assuming that B_n and $\widetilde{B}_{n+1}$ are of h-level 2.

This suggests the following reformulation of our definitions. In what follows we give a presentation of non-unital B-systems in "functional terms". The presentation of the axioms related to the δ-operations is more complex as can be see already in the case of the δT-axiom and we leave it for the future.

Let us define a tower as a sequence of functions $T := (\ldots \to T_{i+1} \stackrel{p_i}{\to} T_i \to \ldots \to T_0)$.

For a tower T and $i, j \geq 0$ define $ft_i^j : T_{i+j} \to T_i$ as the composition of the functions p_k for $k = i, \ldots, i+j-1$. When no ambiguity can arise we will write ft^j instead of ft_i^j and we will write ft instead of ft^1.

For a tower T, $i \geq 0$ and $G \in T_i$ define a new tower $T(G)$ setting:

$$T(G)_j = \{GD \in T_{i+j} | ft_i^j(x) = G\}$$

and defining the functions $T(G)_{j+1} \to T(G)_j$ in the obvious way. More categorically this can expressed by saying that $T(G)_j$ is defined by the standard (homotopy) pull-back square

$$\begin{array}{ccc} T(G)_j & \longrightarrow & T_{i+j} \\ \downarrow & & \downarrow {\scriptstyle ft_i^j} \\ pt & \xrightarrow{G} & T_i \end{array}$$

For $G \in T_{i+j}$ we let $\phi_j(G) \in T(ft^{i+j}(G))_j$ denote the obvious element.

For towers T and T' define a function or morphism of towers $F : T \to T'$ as a sequence of morphisms $F_i : T_i \to T'_i$ which commute in the obvious sense with the functions p_i and p'_i.

The identity function of towers id_T and the composition of functions of towers are defined in the obvious way.

For T, $i, j, k \geq 0$, $G \in T_i$ and $GD \in T_j(G)$ we have the digrams:

$$\begin{array}{ccccc} T(G)(GD)_k & \longrightarrow & T(G)_{j+k} & \longrightarrow & T_{i+(j+k)} \\ \downarrow & & \downarrow {\scriptstyle ft^k_{T(G),j}} & & \downarrow \\ pt & \xrightarrow{GD} & T(G)_j & \xrightarrow{u_{G,j}} & T_{i+j} \\ & & \downarrow & & \downarrow {\scriptstyle ft^j_{T,i}} \\ & & pt & \xrightarrow{G} & T_i \end{array} \qquad \begin{array}{ccc} T(u_{G,i}(GD))_k & \longrightarrow & T_{(i+j)+k} \\ \downarrow & & \downarrow \\ pt & \xrightarrow[u_{G,j}(GD)]{} & T_{i+j} \end{array}$$

which shows that we have natural equivalences (isomorphisms)

$$T(G)(GD)_k \cong T(u_{G,j}(GD))_k \tag{12}$$

The equivalences (12) commute with the functions $p(G)(GD)_i$ and $p(u_{G,j}(GD))$ in the obvious sense and define an equivalence of towers

$$T(G)(GD) \cong T(u_{G,j}(GD)) \tag{13}$$

Remark 4.1 In the case when standard pull-backs are pull-backs in a category, the functions $u_{G,j}$ from $T_j(G)$ to T_{i+j} are pull-backs of (split) monomorphisms and therefore are monomorphisms. In this case $T_k(G)(GD)$ is a sub-object of $T_{i+(j+k)}$ and $T_k(u_{G,j}(GD))$ is a sub-object of $T_{(i+j)+k}$ which are canonically equal. Then we can say that

$$T(G)(GD)_k = T(u_{G,j}(GD))_k \tag{14}$$

where the equality is the equality of sub-objects of $T_{(i+j)+k}$.

More generally, if T_i are objects of h-level 2, the functions $u_{G,j}$ are of h-level 1 (monic inclusions) and we again can say that the equality (14) holds as the unique equality of monic sub-objects of $T_{(i+j)+k}$.

For a function $F : T \to T'$ and $G \in T_i$ we obtain a function $F(G) : T(G) \to T'(G)$ using functoriality of standard pull-backs.

Define a B-system carrier or a B-carrier as a pair $\mathbf{B} = (B, \widetilde{B})$ where B is a tower and $\widetilde{B}$ is a family $\widetilde{B}_{i+1}$, $i \geq 0$ together with functions $\partial_i : \widetilde{B}_{i+1} \to B_i$. The B-system carriers in sets are the same as the "type-and-term structures" of [4].

We will denote the standard fiber of ∂_i over $GT \in B_{i+1}$ by $\widetilde{B}_{GT}$.

For a B-carrier $\mathbf{B}$, $i \geq 0$ and $G \in B_i$, define a B-carrier $\mathbf{B}(G)$ as the pair $(B(G), \widetilde{B(G)})$ where

$$\widetilde{B(G)}_{j+1} = \{s \in \widetilde{B}_{i+j+1} | \partial(s) \in B(G)_{j+1}\}$$

or, categorically, $\widetilde{B(G)}_{j+1}$ is defined by the standard pull-back square

$$\begin{array}{ccc} \widetilde{B(G)}_{j+1} & \xrightarrow{\widetilde{u}_{G,j+1}} & \widetilde{B}_{i+(j+1)} \\ {\scriptstyle \partial(G)} \downarrow & & \downarrow {\scriptstyle \partial} \\ B(G)_{j+1} & \xrightarrow{u_{G,j+1}} & B_{i+(j+1)} \end{array}$$

For a B-carrier $\mathbf{B}$, $i, j \geq 0$, $G \in B_i$ and $GD \in B_{i+j}$ the equivalence (13) clearly extends to an equivalence

$$\mathbf{B}(G)(GD) \cong \mathbf{B}(u_G(GD)) \tag{15}$$

For B-carriers $\mathbf{B}$ and $\mathbf{B}'$ define a function of B-carriers $\mathbf{F} : \mathbf{B} \to \mathbf{B}'$ as a pair $\mathbf{F} = (F, \widetilde{F})$ where $F : B \to B'$ is a function of towers and for every $i \geq 0$, $\widetilde{F}_{i+1}$ is a function $\widetilde{B}_{i+1} \to \widetilde{B}'_{i+1}$ which commutes in the obvious sense with the functions ∂', F_{i+1} and ∂.

The identity function of B-carriers $id_{\mathbf{B}}$ and the composition of functions of B-carriers are defined in the obvious way.

For a function of B-carriers $\mathbf{F} : \mathbf{B} \to \mathbf{B}'$ and $G \in B_i$ we obtain a function of B-carriers $\mathbf{F}(G) : \mathbf{B}(G) \to \mathbf{B}'(F(G))$ using functoriality of standard pull-backs.

Definition 4.2 *Non-unital B-system data is given by the following:*

1. *a B-system carrier* $\mathbf{B}$,

2. *an isomorphism* $pt \to B_0$*,*

3. *for every* $m \geq 0$*,* $Y \in B_{n+1}$ *a B-carrier function* $\mathbf{T}_Y : \mathbf{B}(p_n(Y)) \to \mathbf{B}(Y)$*,*

4. *for every* $m \geq 0$*,* $s \in \widetilde{B}_{n+1}$*, a B-carrier function* $\mathbf{S}_s : \mathbf{B}(\partial(s)) \to \mathbf{B}(p_n(\partial(s)))$*,*

Problem 4.3 *Construct an equivalence between the type of non-unital B0-systems the type of non-unital B-system data such that the types* B_* *and* $\widetilde{B}$ *are sets.*

Construction 4.4 A non-unital B-system carrier is the same as two families of sets B_n, $\widetilde{B}_{n+1}$ together with maps $p_n : B_{n+1} \to B_n$ and $\partial : \widetilde{B_{n+1}} \to B_{n+1}$.

An isomorphism $pt \to B_0$ is the same as an element $pt \in B_0$ such that for all $X \in B_0$, $X = pt$.

For a given $Y \in B_{n+1}$ a B-carrier function $\mathbf{T}_Y : \mathbf{B}(ft(Y)) \to \mathbf{B}(Y)$ is the same as:

1. for all $i \geq 0$, $X \in B_{n+i}$ such that $ft^i(X) = ft(Y)$, an element $T(Y, X) \in B_{n+i+1}$ such that $ft^i(T(Y, X)) = Y$,

2. for all $i \geq 0$, $r \in \widetilde{B}_{n+i+1}$ such that $ft^{i+1}(\partial(r)) = ft(Y)$, an element $\widetilde{T}(Y, r)$ such that $ft^{i+1}(\partial(r)) = ft(Y)$.

For $i = 0$, the operation T is uniquely determined by the condition $ft^i(T(Y, X)) = Y$ which leaves us with the operations T and $\widetilde{T}$ as in Definition 2.1 satisfying the conditions of Lemma 2.7.

The same reasoning applies to S, $\widetilde{S}$.

From this point on everything is assumed to be non-unital. Let $\mathbf{BD} = (\mathbf{B}, \mathbf{T}, \mathbf{S}, \delta)$ be B-data and $G \in B_i$. Define B-data $\mathbf{BD}(G)$ over G as follows. The B-carrier of $\mathbf{BD}(G)$ is $\mathbf{B}(G)$.

For $GDT \in B(G)_{i+1}$ we need to define a B-carrier function

$$\mathbf{T}(G)_{GDT} : \mathbf{B}(G)(p_i(GDT)) \to \mathbf{B}(G)(GDT)$$

We define it through the condition of commutativity of the pentagon:

$$\begin{array}{ccc} \mathbf{B}(G)(p_i(GDT)) & \xrightarrow{\mathbf{T}(G)_{GDT}} & \mathbf{B}(G)(GDT) \\ \cong\downarrow & & \downarrow\cong \\ \mathbf{B}(u_G(p_i(GDT))) \cong \mathbf{B}(p_i(u_G(GDT))) & \xrightarrow{\mathbf{T}_{-}} & \mathbf{B}(u_G(GDT)) \end{array} \tag{16}$$

where the vertical equivalences are from (15).

Similarly for $s \in \widetilde{B(G)}_{j+1}$ we define a B-carrier function

$$\mathbf{S}(G)_s : \mathbf{B}(G)(\partial(s)) \to \mathbf{B}(G)(p_j(\partial(s)))$$

by the diagram:

$$\begin{array}{ccc} \mathbf{B}(G)(\partial(s)) & \xrightarrow{\mathbf{S}(G)_s} & \mathbf{B}(G)(p_j(\partial(s))) \\ \downarrow & & \downarrow \\ \mathbf{B}(u_G(\partial(s))) \cong \mathbf{B}(\partial(\widetilde{u}_G(s))) & \xrightarrow{\mathbf{S}(\widetilde{u}_G(s))} & \mathbf{B}(p_j(\partial(\widetilde{u}_G(s)))) \cong \mathbf{B}(u_G(p_j(\partial(s)))) \end{array} \tag{17}$$

We can now give formulations for the conditions TT, SS, TS, ST and STid.

Definition 4.5 *Let us define the following conditions on a B-system data* $(\mathbf{B}, \mathbf{T}, \mathbf{S}, \delta)$*:*

1. *The TT-condition. For any* $GT \in B_{i+1}$, $GDT' \in B_{j+1}(p_i(GT))$ *the pentagon of B-carrier functions*

$$\begin{array}{ccc}
\mathbf{B}(p_i(GT))(p_j(GDT')) & \xrightarrow{\mathbf{T}(p_i(GT))_{GDT'}} & \mathbf{B}(p_i(GT))(GDT') \\
\mathbf{T}_{GT}(p_j(GDT'))\downarrow & & \\
\mathbf{B}(GT)(T_{GT}(p_j(GDT'))) & & \downarrow \mathbf{T}_{GT}(GDT') \\
\cong \downarrow & & \\
\mathbf{B}(GT)(p_j(T_{GT}(GDT'))) & \xrightarrow{\mathbf{T}_{T_{GT}(GDT')}(GT)} & \mathbf{B}(GT)(\mathbf{T}_{GT}(GDT'))
\end{array} \qquad (18)$$

commutes.

2. *The SS-condition. For any* $s \in \widetilde{B}_{i+1}$, $s' \in \widetilde{B}_{j+1}(\partial(s))$ *the diagram of B-carrier functions*

$$\begin{array}{ccc}
\mathbf{B}(\partial(s))(\partial(s')) & \xrightarrow{\mathbf{S}(\partial(s))_{s'}} & \mathbf{B}(\partial(s))(p_j(\partial(s'))) \\
\mathbf{S}_s(\partial(s'))\downarrow & & \downarrow \mathbf{S}_s(p_j(\partial(s'))) \\
\mathbf{B}(p_i(\partial(s)))(S_s(\partial(s'))) & & \mathbf{B}(p_i(\partial(s))(S_s(p_j(\partial(s')))) \\
\cong \downarrow & & \downarrow \cong \\
\mathbf{B}(p_i(\partial(s)))(\partial(\widetilde{S}_s(s'))) & \xrightarrow{\mathbf{S}(p_i(\partial(s)))_{\widetilde{S}_s(s')}} & \mathbf{B}(p_i(\partial(s)))(p_j(\partial(\widetilde{S}_s(s'))))
\end{array} \qquad (19)$$

commutes.

3. *The TS-condition. For any* $GT \in B_{i+1}$, $s' \in \widetilde{B}_{j+1}(p_i(GT))$ *the diagram of B-carrier functions*

$$\begin{array}{ccc}
\mathbf{B}(p_i(GT))(\partial(s')) & \xrightarrow{\mathbf{S}(p_i(GT))_{s'}} & \mathbf{B}(p_i(GT))(p_j(\partial(s'))) \\
\mathbf{T}_{GT}(\partial(s'))\downarrow & & \downarrow \mathbf{T}_{GT}(p_j(\partial(s'))) \\
\mathbf{B}(GT)(T_{GT}(\partial(s'))) & & \mathbf{B}(GT)(T_{GT}(p_j(\partial(s')))) \\
\cong \downarrow & & \downarrow \cong \\
\mathbf{B}(GT)(\partial(\widetilde{T}_{TG}(s'))) & \xrightarrow{\mathbf{S}(GT)_{\widetilde{T}_{GT}(s')}} & \mathbf{B}(GT)(p_j(\partial(\widetilde{T}_{GT}(s'))))
\end{array} \qquad (20)$$

4. *The ST-condition. For any* $s \in \widetilde{B}_{i+1}$, $GTDT' \in B_{j+1}(\partial(s))$ *the diagram of B-carrier functions*

$$\begin{array}{ccc}
\mathbf{B}(\partial(s))(p_j(GTDT')) & \xrightarrow{\mathbf{T}(\partial(s))_{GTDT'}} & \mathbf{B}(\partial(s))(GTDT') \\
\mathbf{S}_s(p_j(GTDT'))\downarrow & & \\
\mathbf{B}(p_i(\partial(s)))(S_s(p_j(GTDT'))) & & \downarrow \mathbf{S}_s(GTDT') \\
\cong \downarrow & & \\
\mathbf{B}(p_i(\partial(s)))(p_j(S_s(GTDT'))) & \xrightarrow{\mathbf{T}(p_i(\partial(s)))_{S_s(GTDT')}} & \mathbf{B}(p_i(\partial(s)))(S_s(GTDT'))
\end{array} \qquad (21)$$

5. *The STid-condition. For any* $s \in \widetilde{B}_{i+1}$ *one has*

$$(\mathbf{B}(p_i(\partial(s))) \stackrel{T_{\partial(s)}}{\to} \mathbf{B}(\partial(s)) \stackrel{S_s}{\to} \mathbf{B}(p_i(\partial(s)))) = id_{\mathbf{B}(p_i(\partial(s))}$$

Formulation of the remaining four conditions that involve δ is more difficult since their formulation using this approach leads to conditions that depend on the conditions from the first group. We leave their study for the future.

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