Diffeomorphisms and PL Homeomorphisms (Lecture 6)

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Let M be a smooth manifold. In the previous lectures, we showed that M admits a Whitehead compatible triangulation, so that we can regard M as having an underlying piecewise linear manifold. Moreover, this piecewise linear manifold is unique up to piecewise linear homeomorphism. Our goal for the next few lectures is to obtain a more precise form of this statement. For example, we would like to show that every diffeomorphism of smooth manifolds determines a PL homeomorphism, every smooth isotopy of diffeomorphisms determines a piecewise linear isotopy, and so forth. We can summarize the situation by saying that there is a classifying space for smooth manifolds which maps to a suitable classifying space for PL manifolds. Our goal in this lecture is to define the relevant classifying spaces and to outline the relationship between them.

We begin with the smooth case. Let M be a compact smooth manifold. We let $C^\infty(M, M)$ denote the set of smooth maps from M to itself, and $\mathrm{Diff}(M)$ the subset consisting of diffeomorphisms. The set $C^\infty(M, M)$ can be endowed with a topology, where a sequence of functions $f_1, f_2, \ldots : M \to M$ converges to a function $f : M \to M$ if all of the derivatives of $\{f_i\}$ converge uniformly to the derivatives of f. With respect to this topology, $C^\infty(M, M)$ is a Frechet manifold, and the collection of diffeomorphisms $\mathrm{Diff}(M)$ is an open subset (hence also a Frechet manifold).

We will generally not be interested in the exact definition of $\mathrm{Diff}(M)$ (such as the analytic details of what constitutes a convergent sequence of diffeomorphisms), but only the underlying homotopy type. It is therefore convenient to discard the topological space $\mathrm{Diff}(M)$ and work instead with its singular complex $\mathrm{Sing}_\bullet(\mathrm{Diff}(M))$. This is a simplicial set whose n-simplices are given by the formula

$$\mathrm{Sing}_n(\mathrm{Diff}(M)) = \mathrm{Hom}(\Delta^n, \mathrm{Diff}(M)).$$

By general nonsense, we can recover a space homotopy equivalent to $\mathrm{Diff}(M)$ by passing to the geometric realization $|\mathrm{Sing}_n \mathrm{Diff}(M)|$.

Unwinding the definitions, we can describe the simplices of $\mathrm{Sing}_\bullet \mathrm{Diff}(M)$ more explicitly as follows: an n-simplex of $\mathrm{Sing}_\bullet \mathrm{Diff}(M)$ is a homeomorphism

$$f : M \times \Delta^n \to M \times \Delta^n$$

with the following properties:

(1) The function f commutes with the projection to Δ^n.

(2) The function f is smooth in the first variable. In other words, if we write f as $f(m, t)$, then f has arbitrarily many derivatives in the first variable, and these derivatives are continuous in both variables.

(3) For every $t \in \Delta^n$, the induced map $f_t : M \to M$ (which is smooth, by virtue of (2)) is a diffeomorphism.

The advantage of this description is that it does away with some analysis. It tends to be easier to describe what we mean by a continuous map $K \to \mathrm{Diff}(M)$ when K is a simplex (which is equivalent to describing the simplicial set $\mathrm{Sing}_\bullet \mathrm{Diff}(M)$) than in the case where K is a general space (which is equivalent to describing the topological space $\mathrm{Diff}(M)$).

It is even easier to describe the class of *smooth* maps from a simplex into $\mathrm{Diff}(M)$. These can be organized into another simplicial set $\mathrm{Sing}^{\mathrm{sm}}_{\bullet} \mathrm{Diff}(M)$, whose n-simplices are diffeomorphisms $f : M \times \Delta^n \to M \times \Delta^n$ which commute with the projection to Δ^n. There is no harm in restricting our attention to such simplices, by virtue of the following:

Proposition 1. *The inclusion* $\mathrm{Sing}^{\mathrm{sm}}_{\bullet} \mathrm{Diff}(M) \subseteq \mathrm{Sing}_{\bullet} \mathrm{Diff}(M)$ *is a homotopy equivalence of Kan complexes.*

Proof. By general nonsense, it suffices to show the following: given a map $f_0 : \partial \Delta^k \to \mathrm{Sing}^{\mathrm{sm}}_{\bullet} \mathrm{Diff}(M)$ and an extension of f_0 to $f : \Delta^k \to \mathrm{Sing}_{\bullet} \mathrm{Diff}(M)$, there exists another extension $f' : \Delta^k \to \mathrm{Sing}^{\mathrm{sm}}_{\bullet} \mathrm{Diff}(M)$ which is homotopic to f via a homotopy fixed on f_0.

Unwinding the definitions, we can view f_0 as a smooth map $M \times \partial \Delta^k \to M$ and f as an extension $M \times \Delta^k \to M$. Identify M with a smooth submanifold of $\mathbb{R}^n$ for $n \gg 0$, and let N be a tubular neighborhood of M in $\mathbb{R}^n$ equipped with a smooth projection $\pi : N \to M$.

Since f_0 is smooth, it can be extended to a smooth map $f_1 : M \times U_0 \to M$ where U_0 is an open neighborhood of $\partial \Delta^k$. Shrinking U_0 if necessary, we may assume that $f_1 | M \times \{t\}$ is a diffeomorphism for each $t \in U_0$. Choose an open covering of $\Delta^k - U_0$ by small open subsets $\{U_i \subseteq \Delta^k\}_{1 \leq i \leq n}$, choose a point t_i in each U_i, and let $f_i : M \times \Delta^k \to M$ be given by the formula $f_i(m,t) = f(m, t_i)$. Let $\{\phi_i : \Delta^k \to [0,1]\}_{0 \leq i \leq n}$ be a smooth partition of unity subordinate to the covering $\{U_i\}_{0 \leq i \leq n}$. We now define f' by the formula $f'(m,t) = \pi(\sum_{0 \leq i \leq n} \phi_i(t) f_i(m,t))$. If the open covering is fine enough, then f' will be a smooth extension of f_0 which is a diffeomorphism for each $t \in \Delta^k$, and the functions

$$h_s(m,t) = \pi(s f(m,t) + (1-s) f'(m,t))$$

will give a homotopy from f to f' which is fixed on $M \times \partial \Delta^k$. □

Remark 2. The definitions of $\mathrm{Sing}_{\bullet} \mathrm{Diff}(M)$ and $\mathrm{Sing}^{\mathrm{sm}}_{\bullet} \mathrm{Diff}(M)$ extend easily to the case when M is not compact. In this case, one can also define a topology on $\mathrm{Diff}(M)$, but the discussion becomes more technical.

It is convenient to study topological groups G by means of their classifying spaces. In our context, there is a convenient model for these classifying spaces.

Notation 3. Let V be a finite dimensional real vector space, and M a smooth m-manifold. We let $\mathrm{Emb}_{\mathrm{sm}}(M,V)$ denote the simplicial set of *embeddings* of M into V: that is, the simplicial set whose n-simplices are smooth embeddings $M \times \Delta^n \to V \times \Delta^n$ which commute with the projection to n. We let $\mathrm{Sub}^m_{\mathrm{sm}}(V)$ denote the simplicial set of *submanifolds* of V, whose n-simplices are given by smooth submanifolds $X \subseteq V \times \Delta^n$ such that the projection $X \to \Delta^n$ is a smooth fiber bundle of relative dimension m.

If V is infinite dimensional, we let $\mathrm{Emb}_{\mathrm{sm}}(M,V)$ and $\mathrm{Sub}^m_{\mathrm{sm}}(V)$ denote the direct limits of $\mathrm{Emb}_{\mathrm{sm}}(M,V_0)$ and $\mathrm{Sub}_{\mathrm{sm}}(V_0)$, as V_0 ranges over all finite dimensional subspaces of V.

Remark 4. There is a canonical (free) action of $\mathrm{Sing}^{\mathrm{sm}}_{\bullet} \mathrm{Diff}_{\mathrm{sm}}(M)$ on $\mathrm{Emb}_{\mathrm{sm}}(M,V)$, and the quotient $\mathrm{Emb}_{\mathrm{sm}}(M,V)/\mathrm{Sing}^{\mathrm{sm}}_{\bullet} \mathrm{Diff}(M)$ can be identified with the union of those components of $\mathrm{Sub}^m_{\mathrm{sm}}(V)$ spanned by submanifolds of V which are diffeomorphic to M.

Remark 5. If V is infinite dimensional, then the simplicial set $\mathrm{Emb}_{\mathrm{sm}}(M,V)$ is a contractible Kan complex. In other words, every smooth embedding $M \times \partial \Delta^n \to V_0 \times \partial \Delta^n$ can be extended to a smooth embedding $M \times \Delta^n \to V_1 \times \Delta^n$ for some $V_0 \subseteq V_1$. This follows from general position arguments.

Combining these remarks, we obtain the following:

Proposition 6. *Let V be infinite dimensional. Then the simplicial set* $\mathrm{Sub}^m_{\mathrm{sm}}(V)$ *is homotopy equivalent to a disjoint union* $\coprod_M B(\mathrm{Sing}^{\mathrm{sm}}_{\bullet} \mathrm{Diff}(M))$*, where M ranges over all diffeomorphism types of smooth m-manifolds.*

We note that all of the above constructions make sense also in the piecewise linear category. Namely, we have the following definitions:

(1) If M is a piecewise linear m-manifold, we can define a simplicial group $\mathrm{Homeo}_{PL}(M)_\bullet$, whose n-simplices are PL homeomorphisms from $M \times \Delta^n$ to itself that commute with the projection to Δ^n.

(2) If V is a finite dimensional vector space, we let $\mathrm{Emb}_{PL}(M, V)$ be the simplicial set whose n-simplices are PL embeddings $M \times \Delta^n \to V \times \Delta^n$ which commute with the projection to Δ^n. These simplicial sets are acted on freely by $\mathrm{Homeo}_{PL}(M)_\bullet$.

(3) If V is infinite dimensional, we set $\mathrm{Emb}_{PL}(M, V) = \varinjlim_{V_0} \mathrm{Emb}_{PL}(M, V)$, where the colimit is taken over all finite dimensional subspaces $V_0 \subseteq V$. As before, general position arguments guarantee that $\mathrm{Emb}_{PL}(M, V)$ is a contractible Kan complex, so that the quotient $\mathrm{Emb}_{PL}(M, V)/\mathrm{Homeo}_{PL}(M)_\bullet$ is a classifying space for $\mathrm{Homeo}_{PL}(M)_\bullet$.

(4) If V is a finite dimensional vector space, we let $\mathrm{Sub}^m_{PL}(V)$ denote the simplicial set whose n-simplices are subpolyhedra $X \subseteq V \times \Delta^n$ which are PL homeomorphic to $M \times \Delta^n$, for some PL m-manifold M. If V is infinite dimensional set $\mathrm{Sub}^m_{PL}(V) = \varinjlim_{V_0 \subset V} \mathrm{Sub}^m_{PL}(V_0)$.

We have the following analogue of Proposition 7:

Proposition 7. *Let V be infinite dimensional. Then the simplicial set $\mathrm{Sub}^m_{PL}(V)$ is homotopy equivalent to a disjoint union $\coprod_M B(\mathrm{Homeo}_{PL}(M))$, where M ranges over all diffeomorphism types of PL m-manifolds.*

Fix an infinite dimensional vector space V. We define $\mathrm{Man}^m_{\mathrm{sm}} = \mathrm{Sub}^m_{\mathrm{sm}}(V)$, and $\mathrm{Man}^m_{PL} = \mathrm{Sub}^m_{PL}(V)$. We can think of $\mathrm{Man}^m_{\mathrm{sm}}$ and Man^m_{PL} as *classifying spaces* for smooth and PL m-manifolds, respectively. We wish to compare these classifying spaces. To this end, we introduce the following definition:

Definition 8. We define a simplicial set Man^m_{PD} as follows. The n-simplices of Man^m_{PD} are triples (K, M, f) where $K \subseteq V \times \Delta^n$ is an n-simplex of Man^m_{PL}, $M \subseteq V \times \Delta^n$ is an n-simplex of $\mathrm{Man}^m_{\mathrm{sm}}$, and $f : K \to M$ is a PD homeomorphism which commutes with the projection to Δ^n.

By construction, we have forgetful maps

$$\mathrm{Man}^m_{PL} \xleftarrow{\theta'} \mathrm{Man}^m_{PD} \xrightarrow{\theta} \mathrm{Man}^m_{\mathrm{sm}} .$$

In the next lecture, we will sketch the following more refined version of Whitehead's results on the existence and uniqueness of triangulations:

Theorem 9. *The map θ is a trivial Kan fibration.*

It follows that θ admits a section s. Composing s with θ', we obtain a map of classifying spaces $\mathrm{Man}^m_{\mathrm{sm}} \to \mathrm{Man}^m_{PL}$: this is a fancy way of saying that every family of smooth manifolds admits a family of triangulations.

We will eventually sketch the proof of the following "converse":

Theorem 10. *If $m \leq 3$, then the map θ' is a trivial Kan fibration. In particular, the Kan complexes Man^m_{PL} and $\mathrm{Man}^m_{\mathrm{sm}}$ are homotopy equivalent to one another.*