Uniqueness of Triangulations (Lecture 5)

February 13, 2009

Our goal in this lecture is to prove the following result:

Theorem 1. *Let M be a smooth manifold, and suppose we are given a pair of PD homeomorphisms $f : K \to M$ and $g : L \to M$. Then there exist PD homeomorphisms $f' : K \to M$, $g' : L \to M$ which are arbitrarily good approximations to f and g (in the C^1-sense) such that ${f'}^{-1} \circ g' : L \to K$ is a PL homeomorphism. In particular, there is a PL homeomorphism between L and K.*

For simplicity, we will assume that M is compact (so that the polyhedra K and L are finite). We will need three lemmas, the first of which is a more refined version of the result of Lecture 3:

Lemma 2. *Let $f : K \to \mathbb{R}^n$ be a PD map and $K_0 \subseteq K$ a finite subpolyhedron. Then there exists another PD map $f' : K \to \mathbb{R}^n$ which is piecewise linear on K_0 and agrees with f outside a compact set. Moreover, we can arrange that f' is arbitrarily good approximation to f (in the C^1-sense), and that f' coincides with f on any subpolyhedron $L \subseteq K$ such that $f|L$ is piecewise linear.*

Proof. We apply the same argument as in Lecture 3: choose a PL map $\chi : K \to [0,1]$ such that χ is supported in a compact subpolyhedron $K_1 \subseteq K$ with $K_0 \subseteq \chi^{-1}\{1\}$. Let S_0 be a triangulation of K_1 such that $L \cap K_1$ is a union of simplices of S_0 and $f|K_1$ is smooth on each simplex of S_0. In lecture 3, we saw that for an appropriate subdivision S of S_0, if we define $f'(x) = \begin{cases} f(x) & \text{if } x \notin K_1 \\ \chi(x) L^S_f(x) + (1-\chi(x)) f(x) & \text{if } x \in K_1. \end{cases}$ then f' is a good approximation to f which is PL on K_0 and coincides with f outside of K_1. It also coincides with f on $L \cap K_1$, since the linearization construction will not change the values of f on any simplex where f is already linear. □

Lemma 3. *Let K be a finite polyhedron, K_0 a finite subpolyhedron, and let $f : K \to M$ be a PD map. Let $f'_0 : K_0 \to M$ be another map. If f'_0 is sufficiently close to $f|K_0$, then f'_0 can be extended to a PD map $f' : K \to \mathbb{R}^n$. Moreover, we can arrange that f' is an arbitrarily close approximation to f (in the C^1-sense) provided that f'_0 is a sufficiently good approximation to $f|K_0$ (in the C^1-sense).*

Proof. Working simplex by simplex in a sufficiently fine triangulation, we can reduce to the case where $K = \Delta^k$, $K_0 = \partial\,\Delta^k$, and $M = \mathbb{R}^n$. Let $C \subseteq K$ be a piecewise linear collar of the boundary $\partial\,\Delta^k$, so that $C \simeq [0,1] \times \partial\,\Delta^k$. Let $\pi_1 : C \to [0,1]$ and $\pi_2 : C \to \partial\,\Delta^k$ denote the two projection maps. We define f' by the formula

$$f'(x) = \begin{cases} f(x) & \text{if } x \notin C \\ (1-\pi_1(x))(f'_0(\pi_2(x)) - f(\pi_2(x))) + f(x) & \text{if } x \in C. \end{cases}$$

Then f' is a PD extension of f which coincides with f'_0 on K_0 Moreover, the difference $f' - f$ (and its first derivatives) are easily bounded in terms of the difference $f'_0 - f|K_0$ (and its first derivatives). □

Lemma 4. *Let K be a polyhedron, M a smooth manifold, and $f : K \to M$ a PD homeomorphism. Fix a smooth chart $\mathbb{R}^n \hookrightarrow M$, and let $B \subseteq \mathbb{R}^n$ be an open ball. Then there exist arbitrarily close approximations $f' : K \to M$ to f (in the C^1-sense) such that the restriction of f' to ${f'}^{-1}(B)$ is a PL homeomorphism.*

Proof. Let B' be an open ball in $\mathbb{R}^n$ containing the closure of B, let $L \subseteq K$ be the inverse image of $\mathbb{R}^n \subseteq M$, and let $L_0 \subseteq L$ be a finite polyhedron containing the inverse image $f^{-1}(B')$. Applying Lemma 2, we conclude that there exist arbitrarily close approximations f'_0 to $f|L$ such that $f'_0|L_0$ is PL and f'_0 agrees with f outside a compact subset of L. Provided that f'_0 is sufficiently close to $f|L$, we deduce that ${f'_0}^{-1}(B) \subseteq f^{-1}(B') \subseteq L_0$, so that the restriction of f'_0 to ${f'_0}^{-1}(B)$ is PL. We conclude by defining

$$f'(x) = \begin{cases} f(x) & \text{if } x \notin L \\ f'_0(x) & \text{if } x \in L. \end{cases}$$

□

We now return to the proof of Theorem 1. Since K is compact, there exists a finite collection of closed subpolyhedra $\{K_i \subseteq K\}_{1 \leq i \leq m}$ with the following property: the image $f(K_i)$ is contained in a smooth chart $\mathbb{R}^n \simeq U_i \subseteq M$. We will prove the following claim by induction on i:

($*$) There exist arbitrarily good approximations f_i and g_i to f and g, respectively, such that $f_i|(K_1 \cup \ldots \cup K_i)$ is compatible with g_i.

Taking $i = m$, we will be able to deduce that f_m is compatible with g_m and the proof of Theorem 1 will be complete. The base case for the induction is obvious: if $i = 0$, we can take $f_i = f$ and $g_i = g$. It will therefore suffice to carry out the inductive step.

Assume that f_i and g_i have already been constructed. Let $K(i) = K_1 \cup \ldots \cup K_i$. Since $f_i|K(i)$ is compatible with g_i, we deduce that $g_i^{-1} f_i K(i)$ is a subpolyhedron of L, which we will denote by $L(i)$. Moreover, the composition $g_i^{-1} \circ f_i$ is a PL homeomorphism h from $K(i)$ to $L(i)$.

Applying Lemma 4, we can find a map f'_i which approximates f_i such that the f'_i induces a PL homeomorphism between an open neighborhood V of K_{i+1} and an open ball $B \subseteq U_{i+1}$. The composition $f'_i \circ h^{-1} : L(i) \to M$ is a close approximation to $g_i|L(i)$. Applying Lemma 3, we can extend $f'_i \circ h^{-1}$ to a PD map $g'_i : L \to M$, which we can assume is an arbitrarily close approximation to g_i (and therefore a PD homeomorphism). By construction, $f'_i|K(i)$ is compatible with g'_i.

Let $W \subseteq L$ be the inverse image ${g'_i}^{-1}(B)$. Since h is PL and the homeomorphism $V \simeq B$ is PL, we deduce that the homeomorphism $k : W \simeq B$ obtained by restricting g'_i is piecewise linear on $L(i) \cap W$. Let $B' \subset B$ be a slightly smaller ball which still contains the image $f_i(K_{i+1})$. It follows from Lemma 2 that k admits arbitrarily close approximations k' such that k' is PL on ${k'}^{-1}B'$, k' agrees with k outside a compact set, and k' agrees with k on $L(i) \cap W$. We now set $f_{i+1} = f'_i$ and define g_{i+1} by the formula

$$g_{i+1}(x) = \begin{cases} k'(x) & \text{if } x \in W \\ g'_i(x) & \text{if } x \notin W. \end{cases}$$

Since f_{i+1} and g_{i+1} are both PL on the inverse image of B', we deduce that $f_{i+1}|K_{i+1}$ is compatible with g_{i+1}. The compatibility of $f_{i+1}|K(i)$ with g_{i+1} follows from the compatibility of $f_{i+1}|K(i)$ with g'_i (since $g_{i+1} = g'_i$ on $L(i)$). This completes the proof of Theorem 1

The results of Whitehead can be summarized as follows: every smooth manifold M admits a Whitehead compatible triangulation, which yields a piecewise linear manifold K. Moreover, this piecewise linear manifold is unique up to piecewise linear homeomorphism. Our next goal in this course is to obtain a more refined uniqueness result: roughly speaking, we would like to know not only that K is unique up to PL homeomorphism but in some sense up to a contractible space of choices. Another way of articulating this idea is to say that the existence and uniqueness results for Whitehead triangulations are true not only for individual manifolds, but for parametrized families of manifolds. Many of the results of the last few lectures have parametrized analogues, which can be proven using exactly the same arguments. We will conclude this lecture with an example. First, we need to introduce a bit of terminology:

Definition 5. Let $f : K \to L$ be a PL map of polyhedra. We will say that f is a *submersion* (of dimension n) if for every point $x \in K$, there exist open neighborhoods $U \subseteq K$ of x and $V \subseteq L$ of $f(x)$ and a PL homeomorphism $U \simeq V \times \mathbb{R}^n$ (such that f is given by projection onto the first factor).

Example 6. A polyhedron K is a piecewise linear manifold if and only if the unique map $K \to *$ is a submersion.

There is an analogous notion of submersion in the smooth category, which is probably more familiar: a map of smooth manifolds $M \to N$ is a submersion if its differential is surjective at every point. By the implicit function theorem, this is equivalent to the assertion that every point $x \in M$ has a neighborhood diffeomorphic to $V \times \mathbb{R}^n$, where V is an open subset of N.

The main result of lecture 3 admits the following relative version:

Theorem 7. *Suppose given a commutative diagram*

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "q"] & M \arrow[d, "p"] \\
L \arrow[r] & N
\end{tikzcd}$$

where K and L are polyhedra, M and N are smooth manifolds, and the horizontal maps are PD homeomorphisms. Assume that p is a submersion of smooth manifolds. Then q is a submersion of PL manifolds.

If $L = N = *$, then the theorem reduces to the assertion that for any Whitehead compatible triangulation of a smooth manifold, the underlying polyhedron is a PL manifold. In the general case, we can use essentially the same argument. The assertion is local, so we can assume that M has the form $N \times \mathbb{R}^n$. We can then apply the "linearization" construction to the composite map

$$K \to M \to \mathbb{R}^n,$$

to approximate f arbitrarily well by maps $K \to L \times \mathbb{R}^n$ which are piecewise linear in a neighborhood of any given point in $x \in K$. Any sufficiently good approxmation will be a PL homeomorphism in a neighborhood of x, so that q is a submersion.